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ON x-OPERATORS IN AN ARBITRARY SEMIGROUP

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Abstract. In this paper there are defined one-sided and two-sided partial x-operators in an arbitrary semigroup, and the theorem on existence of x-extension (left, right) of a partial x-operator (left right) is proved.

Key words: general closure operator, closure operator, modification of a closure operator, semigroup, left (right) partial x-operator in a semigroup, partial x-operator in a semigroup, x-extension (left, right) of a partial x-operator (left, right).

This note presents an extension of some results of the paper [1] on nonabelian semigroups.

We introduce the notions of left, right (and two-sided) partial x-operators in an arbitrary semigroup and investigate their properties (s. sections 2, 3). In section 4 we receive the theorem on existence of x-extensions of partial (left, right) x-operators.

Following [1] we accept the convention: if I, P are sets and $\{A_i\}_{i\in I} \subset 2^P$, then for $I = \emptyset$

$$\bigcup_{i\in I} A_i = \emptyset, \qquad \bigcap_{i\in I} A_i = P.$$

1. We recall after [1] some definitions and theorems on general closure operators.

1.1 **Definition.** Let P be a set and

$$z : 2^P \ni A \mapsto A_z \in 2^P.$$

The mapping z is a general closure operator in P iff for all $A, B \in 2^P$ it holds:

(i) $A \subset A_z$,

(ii) $A \subset B \Rightarrow A_z \subset B_z$.

If moreover

(iii) $A_z = A_{zz}$ for $A \in 2^P$, then z is called closure operator in P.

1.2 **Definition.** For general closure operators z_1, z_2 in P we put

$$z_1 \leqq z_2 :\Leftrightarrow A_{z_1} \subset A_{z_2} \qquad \text{for every } A \subset P.$$

We say that z_2 is coarser than z_1.

1.3. **Corollary.** The relation $\leqq$ is a partial order in the set of all general closure operators in P.

1.4. **Definition.** Let z be a general closure operator in P. The modification of z is the least (in the sense of $\leqq$) closure operator in P coarser than z.

1.5. **Definition.** Let P be a set, z a general closure operator in P. Using transfinite induction for an ordinal ξ we define the general closure operator z_ξ as follows: if $M \subset P$, then

$$M_{z_1} := M_z$$

$$M_{z_\xi} := \begin{cases} (M_{z_\eta})_z & \text{for } \xi = \eta + 1 > 1 \\ \bigcup\limits_{0<\eta<\xi} M_z & \text{for a limit ordinal } \xi. \end{cases}$$

1.6. **Theorem.** *There exists an ordinal $\xi > 0$ such, that z_ξ is the modification of z.*

1.7. **Definition.** Let z be a general closure operator in P and $p \in P$. A set $U \subset P$ is said to be a z-neighbourhood of p provided it fulfils a condition $p \notin (P - U)_z$.

1.7. **Theorem.** *Suppose that $p \in P$, $M \subset P$ and z is a general closure operator in P. Then it holds: $p \in M_z \Leftrightarrow U \cap M \neq \emptyset$ for every z-neighbourhood U of p.*

2. In this section $S = (S; .)$ will denote an arbitrary semigroup.

2.1. **Definition.** Let $\mathscr{I} \subset 2^S$ and $y : \mathscr{I} \to 2^S$. A mapping y is said to be a left (right) partial x-operator in S iff:

(a) $A \subset A_y$, for every $A \in \mathscr{I}$,

(b) $A \subset B_y \Rightarrow A_y \subset B_y$, for $A, B \in \mathscr{I}$,

(c) $a . A \subset B_y \Rightarrow a . A_y \subset B_y$, for $a \in S$, $A, B \in \mathscr{I}$,
($A . a \subset B_y \Rightarrow A_y . a \subset B_y$, for $a \in S$, $A, B \in \mathscr{I}$).

A mapping y is a partial x-operator in S iff it is a left partial x-operator in S and a right partial x-operator in S and moreover it fulfils the condition

(d) $a . A . b \subset B_y \Rightarrow a . A_y . b \subset B_y$, for $a, b \in S$, $A, B \in \mathscr{I}$. A partial (right, left) x-operator in S with property

(e) $\mathscr{I} = 2^S$

is said to be an x-operator (right, left) in S.

2.2. **Corollary.** If S is abelian, then every partial right (left) x-operator in S is a partial x-operator in S. Evidently it holds

2.3. **Corollary.** If y is an x-operator in S (right, left) then y is a closure operator in S. We shall prove now

2.4. **Corollary.** If y is a right and a left x-operator in S, then it is an x-operator in S.

Proof. By supposition and (2.3) y is a closure operator in S. Let now $a, b \in S$, $A, B \subset S$, $a . A . b \subset B_y$. Then $(a . A . b)_y \subset B_{yy} = B_y$. Since $a . A . b \subset (a . A . b)_y$ and y is a left x-operator in S we obtain $a . (A . b)_y \subset (a . A . b)_y$.

Analogously from $Ab \subset (A \,.\, b)_y$ we have $A_y b \subset (Ab)_y$ and consequently $a \,.\, (A_y \,.\, b) \subset a \,.\, (A \,.\, b)_y$, which completes the proof. Evidently we have also:

2.5. **Corollary.** If S is a semigroup with the identity element, then the condition (d) of (2.1) implies (c).

The following example shows that $y\colon \mathcal{J} \to 2^S$ being a left and a right partial x-operator in S must not be a partial x-operator in S.

2.6. **Example.** Let $X = \{a, b, c\}$, $0 \notin X \times X$. Consider the semigroup $(S, .)$ where:

$$S := X \times X \cup \{0\},$$

$$(x, y) \,.\, (z, t) := \begin{cases} (x, t), & \text{when } y = z \\ 0, & \text{when } y \neq z \end{cases} \qquad \text{for } x, y, z, t \in X$$

and $0 \,.\, s = s \,.\, 0 = 0$ for $s \in S$.

It is a special case of the Brandt-semigroup. Moreover let

$$A = \{(a, a), (b, c)\},$$
$$\mathcal{J} = \{A\},$$
$$A_y = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), 0\}.$$

It is elear, that

$$A \,.\, s \subset A_y \Rightarrow A_y \,.\, s \subset A_y, \qquad \text{for } s \in S,$$

and

$$s \,.\, A \subset A_y \Rightarrow s \,.\, A_y \subset A_y, \qquad \text{for } s \in S.$$

Thus y is a left and a right partial x-operator in S. But $(c, b) \,.\, A \,.\, (a, c) = (c, b) \,.\, \{(a, c), 0\} = \{0\} \subset A_y$ and $(c, b) \,.\, A_y \,.\, (a, c) = (c, b) \,.\, \{(a, c), (b, c), 0\} = \{(c, c), 0\} \not\subset A_y$ and the condition (d) of 2.1 is not fulfilled.

2.7. **Definition.** Let x be a closure operator in S. We say that the operation „." in the semigroup $(S, .)$ is right (left) weakly continuous iff for each $a, b \in S$ and x-neighbourhood V of $a \,.\, b$ there exists an x-neighborhoodu of b(an x-neighbourhood of a) such that $a \,.\, U \subset V(Ub \subset V)$.

2.8. **Theorem.** *Let x be a closure operator in S. Then the following statements are equivalent:*

(a) *x is a right x-operator in $(S, .)$,*

(b) *the operation „." is left weakly continuous,*

(c) $(\bigcup_{i \in I} A_i)_x \,.\, A \subset (\bigcup_{i \in I} A_i \,.\, A)_x$ *for each $A, A_i \subset S$ $(i \in I)$,*

(d) $[(\bigcup_{i \in I} A_i)_x \,.\, A]_x = [\bigcup_{i \in I} (A_i \,.\, A)_x]_x$, *for each $A, A_i \subset S$ $(i \in I)$.*

The proof is like that of the theorem 2.4 in [1] (see [1], p. 480).

It is evident that we have also the dual.

2.9. **Theorem.** *Let x be a closure operator in S. Then the following statements are equivalent:*

(a′) *x is a left x-operator in* $(S, .)$,

(b′) *the operation „.“ is right weakly continuous,*

(c′) $A . (\bigcup_{i\in I} A_i)_x \subset (\bigcup_{i\in I} A . A_i)_x$ *for each* $A, A_i \subset S, (i \in I)$,

(d′) $[A . (\bigcup_{i\in I} A_i)_x]_x = [\bigcup_{i\in I} (A . A_i)_x]_x$.

From 2.8 and 2.9 it follows

2.10. **Theorem.** *If x is a right (left) x-operator in S, then*

1° $A \subset S, a \in S \Rightarrow A_x . a \subset (A . a)_x$, $(A \subset S, a \in S \Rightarrow a . A_x \subset (a . A)_x)$,

2° $A, B \subset S \Rightarrow A_x . B \subset (A . B)_x$, $(A, B \subset S \Rightarrow A . B_x \subset (A . B)_x)$,

3° $A, B \subset S \Rightarrow (A_x . B)_x = (A . B)_x$, $(A, B \subset S \Rightarrow (A . B_x)_x = (A . B)_x)$.

Moreover if x is an x-operator in $(S, .)$, *then*

4° $A, B \subset S \Rightarrow (A_x . B_x)_x = (A . B)_x$.

We shall prove the 4° only, since the statements 1°, 2°, 3° follow immediately from 2.8 and 2.9. Suppose that x is an x-operator in $(S, .)$ and $A, B \subset S$. By 2.1 x is a right and a left x-operator in $(S, .)$ and from 2° we have $A_x . B \subset (A . B)_x$. Then $a . B \subset (A . B)_x$ for each $a \in A_x$, hence according to (c) of 2.1 there is $a . A_x \subset \subset (A . B)_x$ for $a \in A_x$, which leads to $A_x . B_x \subset (A . B)_x$. Using (b) of 2.1 we conclude that

$$(A_x . B_x)_x \subset (A . B)_x.$$

On the other hand there is

$$A \subset A_x, \qquad B \subset B_x,$$

then

$$A . B \subset A_x . B_x$$

and

$$(A . B)_x \subset (A_x . B_x)_x$$

and the 4° is proved.

It can be easily verified, that we have

2.11. **Theorem.** *If x is a closure operator in S and the following statement*

$$a \in S, A \subset S \Rightarrow A_y . a \subset (A . a)_y,$$
$$(a \in S, A \subset S \Rightarrow a . A_y \subset (a . A)_y)$$

holds, then x is a right (left) x-operator in $(S, .)$. As in [1] (see [1], lemma 3.1, p. 484) we obtain

2.12. **Theorem.** *Let x be a general closure operator in S with property*

$$a \in S, A \subset S \Rightarrow a . A_x \subset (a . A)_x \; (a \in S, A \subset S \Rightarrow (A_x . a) \subset (A . a)_x).$$

Then the modification of x is a left (right) x-operator in the semigroup S.

3. $S = (S, .)$ is an arbitrary semigroup.

As in [1] (see [1], def. 2.3, p. 479) for a given closure operator x in S we can introduce the operation „$\circ$" in 2^S as follows $A \circ B = (A \,.\, B)_x$ for each $A, B \in 2^S$. $I(S)$ will denote the image of 2^S in the mapping x. For a mapping

$$y : 2^S \supset \mathscr{I} \to 2^S,$$

we can introduce the sets

$$E(y) := \{s \in S: \bigwedge_{A \in \mathscr{I}} s \,.\, A_y \subset A_y\},$$
$$(y)E := \{s \in S: \bigwedge_{A \in \mathscr{I}} A_y \,.\, s \subset A_y\}.$$

The theorems 2.8, 2.9, 2.10 of [1] (see [1], p. 481) hold for such defined sets $E(y)$, $(y)E$. If y is a partial right (left) x-operator in S then for the sets $E(y)$, $(y)E$ the theorem 2.11 of [1] holds. The theorem 2.12 of [1] takes now the form: *Let x be a left (right) x-operator in S. Then the following statements are equivalent:*

(a) *the semigroup $(I(S), \circ)$ contains a left identity element (a right identity element).*

(b) $\bigwedge_{s \in S} s \in (E(x) \,.\, s)_x$, $\quad (\bigwedge_{s \in S} s \in (s \,.\, (x)E)_x)$.

If $(\mathscr{I}(S), \circ)$ *contains the identity element $\mathscr{I}$, then $\mathscr{I} = E(x) = (x)E$.*

The theorems 2.15, 2.16, 2.17 from [1] hold for each x-operator in the sense of definition 2.1 of this note.

4. For the sequel we assume, that $S = (S, .)$ is an arbitrary semigroup.

4.1. **Definition.** For $A \subset S$, $s \in S$ we introduce the sets:

$$A \,/\, s := \{x \in S: x \,.\, s \in A\},$$
$$A \,/\, s := \{x \in S: s \,.\, x \in A\}.$$

4.2. **Definition.** Let y: $2^S \supset \mathscr{I} \to 2^S$. We define now the mappings:

$$z_i: 2^S \to 2^S, \qquad v_i: 2^S \to 2^S, \qquad i = 1, 2, 3$$

as follows: for $A \subset S$ we put

$$A_{z_1} := A \cup \bigcup\{B_y: B \in \mathscr{I}, B \subset A\} \cup \bigcup\{s \,.\, B_y: B \in \mathscr{I}, s \in S, s \,.\, B \subset A\},$$
$$A_{z_2} := A \cup \bigcup\{B_y: B \in a\mathscr{I}, B \subset A\} \cup \bigcup\{B_y \,.\, s : B \in \mathscr{I}, s \in S, B \,.\, s \subset A\},$$
$$A_{z_3} := A \cup \bigcup\{B_y: B \in \mathscr{I}, B \subset A\} \cup \bigcup\{s \,.\, B_y : B \in \mathscr{I}, s \in S, s \,.\, B \subset A\} \cup$$
$$\cup \bigcup\{B_y \,.\, s: B \in \mathscr{I}, s \in S, B \,.\, s \subset A\} \cup$$
$$\cup \bigcup\{s_1 \,.\, B_y \,.\, s_2 : B \in \mathscr{I}, s_1, s_2 \in S, s_1 \,.\, B \,.\, s_2 \subset A\},$$
$$A_{v_1} = \bigcap\{B_y : B \in \mathscr{I}, B_y \supset A\} \cap \bigcap\{B_{y!} / s : s \in S: B \in \mathscr{I}, B_y \supset A \,.\, s\},$$
$$A_{v_2} = \bigcap\{B_y : B \in \mathscr{I}, B_y \supset A\} \cap \bigcap\{B_y / s : s \in S, B \in \mathscr{I}, B_y \supset s \,.\, A\},$$

$$A_{v_3} = \bigcap\{B_y : B \in \mathcal{I}, A \subset B_y\} \cap \bigcap\{B_y / s : B \in \mathcal{I}, B_y \supset A . s\} \cap$$
$$\cap \bigcap\{B_y \setminus s : B \in \mathcal{I}, B_y . s \supset A\} \cap$$
$$\cap \bigcap\{(B_y \setminus s_1) / s_2 : B \in \mathcal{I}, s_1 . A . s_2 \subset B_y\}.$$

4.3. **Corollary.** Let S be abelian semigroup. Then $A_{z_1} = A_{z_2} = A_{z_3}$, $A_{v_1} = A_{v_2} = A_{v_3}$ for each $A \subset S$.

Using definition 4.2, corollary 2.4 and theorem 2.12 we conclude that it holds

4.4. **Theorem.** *The mappings* z_i $(i = 1, 2, 3)$ *are generalized closure operators in* S *with following properties:*

(a) $a . A_{z_1} \subset (a . A)_{z_1}$, for $a \in S, A \subset S$,
(b) $A_{z_2}a \subset (A . a)_{z_2}$, for $a \in S, A \subset S$,
(c) z_3 fulfils both above conditions (a), (b),
(d) $A_y \subset A_{z_i}$, for $A \in \mathcal{I}, i = 1, 2, 3$,
(e) the modification of the operator $z_3(z_1, z_2)$ is an x-operator (left, right) in S.

4.5. **Theorem.** *The mapping* $v_3(v_2, v_1)$ *is an x-operator (left, right) in S. If y is a partial x-operator (left, right) in S, then*

$$A_{v_i} = A_y, \quad \text{for } A \in \mathcal{I}, \quad i = 3, 2, 1.$$

We shall prove the case of v_3 only. First we shall verify, that v_3 fulfils the suppositions of theorem 2.11. By definition 4.2 for each $A \subset S$ there is $A \subset A_{v_3}$ and $A_{v_3} \subset A_{v_3v_3}$. Moreover,

$$(*) \quad A_{v_3v_3} = \bigcap\{B_y : B \in \mathcal{I}, A_{v_3} \subset B_y\} \cap \bigcap\{B_y / s : B \in \mathcal{I}, s \in S, A_{v_3} . s \subset B_y\} \cap$$
$$\cap \bigcap\{B_y \setminus s : B \in \mathcal{I}, s \in S, s . A_{v_3} \subset B_y\} \cap$$
$$\cap \bigcap\{(B_y \setminus s_1)/s_2 : B \in \mathcal{I}, s_1, s_2 \in S, s_1 . A_{v_3} . s_2 \subset B_y\}.$$

Consider $B \in \mathcal{I}$ such, that $A \subset B_y$. By definition 4.2 we have $A_{v_3} \subset B_y$ and consequently $A_{v_3v_3} \subset B_y$. Thus

$$A_{v_3v_3} \subset \bigcap\{B_y : B \in \mathcal{I}, A \subset B_y\}.$$

Let now $B \in \mathcal{I}$, $s_1, s_2 \in S$ be such, that $s_1 . A . s_2 \subset B_y$. Then $A_{v_3} \subset (B_y \setminus s_1)/s_2$ and $s_1 . A_{v_3} . s_2 \subset B_y$, so that

$$A_{v_3v_3} \subset \bigcap\{(B_y \setminus s_1)/s_2 : B \in \mathcal{I}, s_1, s_2 \in S, s_1 . A . s_2 \subset B_y\}.$$

In the same way we show

$$A_{v_3v_3} \subset \bigcap\{B_y / s : B \in \mathcal{I}, s \in S, A . s \subset B_y\}$$

and

$$A_{v_3v_3} \subset \bigcap\{B_y / s : B \in \mathcal{I}, s \in S, s . A \subset B_y\}.$$

These facts together imply that

$$A_{v_3v_3} \subset A_{v_3}$$

and consequently

$$A_{v_3 v_3} = A_{v_3}.$$

Suppose that

$$A \subset C \subset S.$$

Since $A \subset C$, by definition 4.2 we obtain

$$A_{v_3} \subset \bigcap\{B_y : B \in \mathscr{I}, A \subset B_y\} \subset \bigcap\{B_y : B \in \mathscr{I}, C \subset B_y\},$$
$$A_{v_3} \subset \bigcap\{B_y / s : B \in \mathscr{I}, s \in S, A . s \subset B_y\} \subset$$
$$\subset \bigcap\{B_y / s : B \in \mathscr{I}, s \in S, C . s \subset B_y\},$$

and similarly A_{v_3} is contained in the next two factors of C_{v_3}; thus $A_{v_3} \subset C_{v_3}$. Hence v_3 is a closure operator in S. Let further $a \in S$, $A \subset S$. By definition 4.2 we have

$$(**) \qquad (A . a)_{v_3} = \bigcap\{B_y : B \in \mathscr{I}, A . a \subset B_y\} \cap$$
$$\cap \bigcap\{B_y / s : B \in \mathscr{I}, s \in S, A . a . s \subset B_y\} \cap \bigcap\{B_y \setminus s : B \in \mathscr{I}, s \in S, s . A . a_2 \subset B_y\} \cap$$
$$\cap \bigcap\{B_y \setminus s_1) / s_2 : B \in \mathscr{I}, s_1, s_2 \in S, s_1 . A . a . s_2 \subset B_y\}.$$

Consider $B \in \mathscr{I}$ such that $A . a \subset B_y$. From definition 4.2 there is $A_{v_3} \subset B_y / a$ and hence $A_{v_3} . a \subset B_y$, according to definition 4.1. This implies that

$$A_{v_3} . a \subset \bigcap\{B_y : B \in \mathscr{I}, A . a \subset B_y\}.$$

Let now $B \in \mathscr{I}$, $s_1, s_2 \in S$ be such that $s_1 . A . a . s_2 \subset B_y$.
Hence by def. 4.2 $A_{v_3} \subset (B_y \setminus s_1) / as_2$ and by def. 4.1

$$(A_{v_3} a) . s_2 \subset (B_y \setminus s_1).$$

Consequently $A_{v_3} . a \subset (B_y \setminus s_1) / s_2$, according to def. 4.1, and

$$A_{v_3} . a \subset \bigcap\{(B_y \setminus s_1) / s_2 : B \in \mathscr{I}, s_1, s_2 \in S, s_1 . A . a . s_2 \subset B_y\}.$$

In the same way we show that

$$A_{v_3} . a \subset \bigcap\{B_y / s : B \in \mathscr{I}, s \in S, A . a . s \subset B_y\},$$
$$A_{v_3} . a \subset \bigcap\{B_y \setminus s : B \in \mathscr{I}, s \in S, s . A . a \subset B_y\}.$$

In consequence we obtain inclusion

$$A_{v_3} . a \subset (A . a)_{v_3}.$$

The proof of inclusion

$$a . A_{v_3} \subset (a . A)_{v_3}$$

is analogous.
This completes the proof of the first thesis of theorem 4.2 (see theorem 2.11, corollary 2.4).

We come now to prove the second thesis of our theorem. Let y be a partial x-operator in S and $A \in \mathscr{I}$; from $A \subset A_y$ we deduce that

$A_y \in \{B_y: B \in \mathscr{I}, A \subset B\}$ and since $A_{v_3} \subset \bigcap\{B_y: B \in \mathscr{I}, A \subset B_y\}$ we obtain $A_{v_3} \subset A_y$. It remained to verify the inverse inclusion

$$A_y \subset A_{v_3}.$$

Let $B \in \mathscr{I}$ and $A \subset B_y$. Since y is a partial x-operator in S we have $A_y \subset B_y$ and consequently $A_y \subset \bigcap\{B_y: B \in \mathscr{I}, A \subset B_y\}$. Anologously we can prove that A_y is contained in the next factors of the intersection defining A_{v_3}.

In the theorem 3.3.2 of [1] the hypothesis that y is a partial x-operator unfortunately was omitted.

The following example shows, that the implication

$$A \in \mathscr{I} \Rightarrow A_v \subset A_y$$

is not true without this hypothesis.

4.6. **Example.** Let $S = \{0, 1, 2\}$ and the operation "." be given by the table

.	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Take $\mathscr{I} = \{\{1\}\}$ and $\{1\}_y = \{2\}$. Then $\{1\}_v = \{1\}$ and $\{1\}_v \not\subset \{1\}_y$.

4.7. **Definition.** Let

$$y: 2^S \supset \mathscr{I} \to 2^S.$$

A mapping w: $2^S \to 2^S$ with property $A_w = A_y$, for each $A \in \mathscr{I}$ is called an extension of y on 2^S.

A mapping w: $2^S \to 2^S$ is said to be an x-extension of y (left x-extension, right x-extension) provided it has the properties:

a) w is an extension of y on 2^S,

b) w is an x-operator (left, right) in S.

4.8. **Definition.** We denote by u_i the modification of z_i ($i = 1, 2, 3$) (see def. 4.2). As in [1] we can prove

4.9. **Theorem.** *Let* $y: 2^S \supset \mathscr{I} \to 2^S$.

The following statements are equivalent:

1° *y is a partial x-operator (right, left) in S,*

2° $A_y = A_{z_3} = A_{z_3 z_3}$ *for* $A \in \mathscr{I}$ ($A_y = A_{z_i} = A_{z_i z_i}$, *for* $A \in \mathscr{I}$, $i = 1, 2$),

3° $u_3(u_2, u_1)$ *is an x-extension (right, left) of y,* 4° *$v_3(v_1, v_2)$ is an x-extension (right, left) of y,*

5° *there exists an x-extension (right, left) of y.*

If 1° holds then $u_3(u_2, u_1)$ is the finest x-operator (right, left) in S, which is an x-extension (right, left) of y and $v_3(v_1, v_2)$ is the coarsest x-operator (right, left) in S, which is an x-extension (right, left) of y.

Proof. We consider the case of z_3, u_3, v_3 only; the remained cases are analogous. Notice that 2° ⇒ 3° by theorem 4.4, and 1° ⇒ 4° by theorem 4.5.
Evidently 4° ⇒ 5° and 5° ⇒ 1°.
It suffices to show 1° ⇒ 2°.
Let y be a partial x-operator in S and $A \in \mathscr{I}$.
According to theorem 4.4 there is $A_y \subset A_{z_3}$ and $A_{z_3} \subset A_{z_3 z_3}$, so that

$$A_y \subset A_{z_3}.$$

Obviously $A \subset A_y$.
If $B \in \mathscr{I}$ and $B \subset A$, then $B_y \subset A_y$ and $\bigcup\{B_y : B \in \mathscr{I}, B \subset A\} \subset A_y$. Let $s \in S$, $B \in \mathscr{I}$, $s \,.\, B \subset A$. Since $s \,.\, B \subset A \subset A_y$ then $s \,.\, B_y \subset A_y$ and

$$\bigcup\{s \,.\, B_y : B \in \mathscr{I}, s \in S, s \,.\, B \subset A\} \subset A_y.$$

In the same way we conclude, that

$$\bigcup\{B_y \,.\, s : B \in \mathscr{I}, s \in S, B \,.\, s \subset A\} \subset A_y,$$
$$\bigcup\{s_1 \,.\, B_y \,.\, s_2 : B \in \mathscr{I}, s_1, s_2 \in S, s_1 \,.\, B \,.\, s_2 \subset A\} \subset A_y,$$

whence

$$A_{z_3} \subset A_y$$

and

$$A_{z_3} = A_y.$$

To complete the proof it suffices to examine the inclusion

$$A_{z_3 z_3} \subset A_{z_3}.$$

If, for example, $s_1, s_2 \in S$, $B \in \mathscr{I}$, $s_1 \,.\, B \,.\, s_2 \subset A_{z_3}$, then by def. 2.1 $s_1 \,.\, B_y \,.\, s_2 \subset A_{z_3}$ and

$$\bigcup\{s_1 \,.\, B_y \,.\, s_2 : s_1, s_2 \in S, B \in \mathscr{I}, s_1 \,.\, B \,.\, s_2 \subset A_{z_3}\} \subset A_{z_3}.$$

Analogously

$$\bigcup\{s \,.\, B_y : s \in S, B \in \mathscr{I}, s \,.\, B \subset A_{z_3}\} \subset A_{z_3},$$
$$\bigcup\{B_y \,.\, s : s \in S, B \in \mathscr{I}, B \,.\, s \subset A_{z_3}\} \subset A_{z_3}$$

and consequently

$$A_{z_3 z_3} \subset A_{z_3}.$$

The proof of the second part of theorem 4.9 is analogous to that in [1] (see th. 3.3.4, p. 485).

Using theorem 4.9 we obtain

4.10. **Theorem.** *Let M be nonempty subset of S.*
The following statements are equivalent:
(a) *there exists an x-operator (right, left) in S, say y, such that $\emptyset_y = M$,*
(b) *M is an ideal (right, left) in S.*

Proof. We consider the case of left ideal. The other cases are analogous. Let y be a left x-operator such that $\emptyset_y = M$.
Then by theorem 2.10 we have

$$S . M = S . \emptyset_y \subset (S . \emptyset)_y = \emptyset_y = M$$

and M is a left ideal of S.
Conversely let M be a left ideal of S and $\mathscr{I} = \{\emptyset\}$. Put y: $\mathscr{I} \to 2^S$ as follows: $\emptyset_y := M$. Evidently y is a partial x-operator in S and by theorem 4.9 there exists a left x-extension, say w, of y.
Then $\emptyset_w = \emptyset_y = M$.

REFERENCES

[1] L. Skula, *On extensions of partial x-operators,* Czechoslovak Mathematical Journal, 26 (191) 1976, Praha, (p. 477–505).

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