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CYCLICITY OF A COMPLETE 3-EQUIPARTITE GRAPH

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Abstract. The cyclicity of a graph G is the minimal number of circuits in G which cover all edges of G. In the paper the cyclicity of a complete 3-equipartite graph is studied; this is a graph whose vertex set can be partitioned into three classes of equal cardinalities so that two vertices are adjacent if and only if they belong to different classes of this partition.

Key words: cyclicity, complete 3-equipartite graph.

In [1] J. Akiyama, G. Exoo and F. Harary suggested the problem to determine the cyclicity of a complete tripartite graph. We shall determine it for a special type of such a graph – a complete 3-equipartite graph.

A complete tripartite graph is a graph G for which there exists a partition $\mathscr{T} = \{A, B, C\}$ of its vertex set $V(G)$ such that two vertices of G are adjacent if and only if they belong to different classes of this partition. If $|A| = |B| = |C| = m$, such a graph will be called 3-equipartite and denoted by $K(m, m, m)$.

The cyclicity $o(G)$ of a finite undirected graph G is the minimum number of circuits in G which cover all edges of G.

We shall prove two theorems.

Theorem 1. *Let m be a positive integer with the property that there exist orthogonal Latin squares of the order m. Then for a complete 3-equipartite graph $K(m, m, m)$ there is*

$$o(K(m, m, m)) = m.$$

Proof. The number of edges of $K(m, m, m)$ is $3m^2$ and any circuit in this graph has at most $3m$ edges. Hence $o(K(m, m, m)) \geqq 3m^2/3m = m$. Therefore it suffices to construct m circuits which cover all edges of $K(m, m, m)$.

Let $\mathbf{R}$, $\mathbf{S}$ be two orthogonal Latin squares of the order m. The number in the i-th row and the j-th column in $\mathbf{R}$ (or in $\mathbf{S}$) will be denoted by r_{ij} (or s_{ij} respectively). Consider a multigraph M whose vertex set $V(M) = U \cup V$, where $U \cap V = \emptyset$,

$U = \{u_1, \ldots, u_m\}$, $V = \{v_1, \ldots, v_m\}$ and in which each vertex of U is joined with each vertex of V by two edges (one red edge and one blue edge) and no two vertices of U and no two vertices of V are joined by an edge.

For each $k = 1, \ldots, m$ we construct a circuit F_k in the following way. A red edge joining vertices u_i, v_j belongs to F_k if and only if $r_{ij} = k$. Further if a red edge joining u_i, v_j belongs to F_k, then also the blue edge joining v_j, u_{i+1} belongs to F_k, where $i + 1$ is taken modulo m. In this way we obtain the circuits $F_1, \ldots, F_m$ which cover all edges of M. To each red edge we assign a label equal to s_{ij}, where u_i, v_j are the end vertices of this edge. Thus we obtain a labelling of red edges of M such that two different red edges belonging to the same one of the circuits $F_1, \ldots, F_m$ have always different labels. Also no two edges of the same label have a common end vertex.

To each F_i we assign a circuit F_i' in $K(m, m, m)$ in the following way. Denote $A = \{a_1, \ldots, a_m\}$, $B = \{b_1, \ldots, b_m\}$, $C = \{c_1, \ldots, c_m\}$. If F_i contains a blue edge joining u_i, v_j, then F_i' contains the edge a_ib_j. If F_i contains a red edge joining u_i, v_j, then F_i' contains the edges a_ic_k, c_kb_j, where k is the label of the mentioned edge. In this way we obtain the circuits $F_1', \ldots, F_m'$ which cover all edges of $K(m, m, m)$.

It is well-known that if m is odd or is divisible by 4, then there exists a pair of orthogonal Latin squares of the order m. Thus we have a corollary.

Corollary. *Let m be a positive integer such that $m \not\equiv 2 \pmod 4$. Then for the cyclicity of a complete 3-equipartite graph $K(m, m, m)$ there is*

$$o(K(m, m, m)) = m.$$

Theorem 2. *For a complete 3-equipartite graph $K(m, m, m)$, where m is an arbitrary positive integer, there is*

$$m \leqq o(K(m, m, m)) \leqq m + 1.$$

Proof. For m odd the assertion follows from Corollary. For $m = 2$ there is $o(K(2, 2, 2)) = 3 = m + 1$, because after deleting edges of a Hamiltonian circuit from $K(2, 2, 2)$ we obtain a graph consisting of two disjoint triangles. Therefore we may suppose that m is even and $m \geqq 4$. Consider a complete graph K_m with m vertices. In [1] it was proved that $o(K_m) = m/2$. There exist $m/2$ circuits in K_m which cover all edges of K_m; moreover all of them are Hamiltonian. Let the vertex set of K_m be $\{u_1, \ldots, u_m\}$, let the mentioned circuits be $F_1, \ldots, F_{m/2}$. In the graph $K(m, m, m)$ let $A = \{a_1, \ldots, a_m\}$, $B = \{b_1, \ldots, b_m\}$, $C = \{c_1, \ldots, c_m\}$. To each circuit F_i we assign two circuits F_i', F_i'' in $K(m, m, m)$ in the following way. If F_i contains an edge u_ju_k, then F_i' contains the edges a_jb_k, b_kc_{i+j}, $c_{i+j}a_k$ and F_i'' contains the edges b_ja_k, a_kc_{j-i+1}, $c_{j-i+1}b_j$, where $i + j$ and $j - i + 1$ are taken modulo m. It is easy to prove that each F_i' and each F_i'' are Hamiltonian circuits in $K(m, m, m)$. The circuits $F_1', \ldots, F_{m/2}'$, $F_1'', \ldots, F_{m/2}''$ cover all edges of $K(m, m, m)$ except the edges $a_1b_1, \ldots, a_mb_m$. Evidently there exists a circuit F_0 which contains all the edges $a_1b_1, \ldots, a_mb_m$. Then $F_1', \ldots, F_{m/2}', F_1'', \ldots, F_{m/2}''$, F_0 cover all edges of $K(m, m, m)$ and their number is $m + 1$; this implies $o(K(m, m, m)) \leqq m + 1$. The proof that $o(K(m, m, m)) \geqq m$ is the same as in the proof of Theorem 1.

Conjecture. *For a complete* 3-*equipartite graph* $K(m, m, m)$, *where* $m \neq 2$, *there is*

$$o(K(m, m, m)) = m.$$

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