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ON BOUNDARY VALUE PROBLEMS OF PERIODIC TYPE FOR ORDINARY ODD ORDER DIFFERENTIAL EQUATIONS

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1. Statement of the Existence and Uniqueness Theorems

Boundary value problems of periodic type for n-th order ordinary differential equations and systems have been attracting attention of specialists for a long time and are being studied in many research works (see e.g. [1–7] and the references mentioned there in). However they still remain insufficiently investigated for essentially nonlinear differential equations, i.e. equations which can't be interpreted as small perturbations of linear equations. In the present paper an attempt is made to make up this defficiency, to a certain extent, for odd order ordinary differential equations.

Suppose that n is a natural integer, $0 < \omega < +\infty$, R is the set of real numbers, $a_{ik}, b_{ik} \in R$ $(i, k = 1, \dots, 2n + 1)$ and $f : [0, \omega] \times R \to R$ is a continuous function. Consider the problem of finding $2n + 1$ times continuously differentiable function u which satisfies the differential equation

$$u^{(2n+1)} = f(t, u) \tag{1.1}$$

on $[0, \omega]$ and the boundary conditions

$$\sum_{i=1}^{2n+1} [a_{ik}u^{(k-1)}(0) + b_{ik}u^{(k-1)}(\omega)] = 0 \qquad (k = 1, \dots, 2n + 1). \tag{1.2}$$

The special cases of (1.2) are e.g. the periodic boundary conditions

$$u^{(i-1)}(0) = u^{(i-1)}(\omega) \qquad (i = 1, \dots, 2n + 1) \tag{1.3}$$

and the Vallée–Poussin type boundary conditions

$$u^{(i-1)}(0) = 0 \ (i = 1, \dots, m), \qquad u^{(k-1)}(\omega) = 0 \ (k = 1, \dots, 2n + 1 - m), \tag{1.4}$$

where $m \in \{1, \dots, 2n + 1\}$.

For each $\sigma \in \{-1; 1\}$ let

$$\psi_\sigma(y_1, \dots, y_{2n+1}; z_1, \dots, z_{2n+1}) = \sigma \sum_{k=1}^{n} (-1)^{k-1}(z_{2n+2-k}z_k - y_{2n+2-k}y_k) + \\ + (-1)^n \frac{\sigma}{2}(z_{n+1}^2 - y_{n+1}^2).$$

We shall prove the following theorems.

Theorem 1.1. *Let* $\sigma \in \{-1; 1\}$,

$$\sum_{i=1}^{2n+1} \left| \sum_{k=1}^{2n+1} (a_{ik}y_k + b_{ik}z_k) \right| > 0 \quad \textit{when } \psi_\sigma(y_1, \dots, y_{2n+1}; z_1, \dots, z_{2n+1}) > 0 \tag{1.5}$$

and let the inequality

$$f(t, x) \operatorname{sign}(\sigma x) \geqq h(x) \tag{1.6}$$

be fulfilled on the set $[0, \omega] \times R$ *where* $h : R \to R$ *is a continuous function and*

$$\liminf_{|x| \to +\infty} h(x) > 0. \tag{1.7}$$

Then the problem (1.1), (1.2) *has at least one solution.*

Theorem 1.2. *Suppose that* $\sigma \in \{-1; 1\}$, *the condition* (1.5) *is satisfied and*

$$[f(t, x) - f(t, y)] \operatorname{sign}[\sigma(x - y)] > 0 \qquad \text{for } x \neq y. \tag{1.8}$$

Then the problem (1.1), (1.2) *has at most one solution.*

Theorem 1.3. *Suppose that* $\sigma \in \{-1; 1\}$, *the condition* (1.5) *is satisfied and the inequalities* (1.6) *and* (1.8) *hold on the set* $[0, \omega] \times R$ *where* $h : R \to R$ *is a continuous function satisfying* (1.7). *Then the problem* (1.1), (1.2) *has exactly one solution.*

Corollary 1.1. Let the inequality (1.6) hold on the set $[0, \omega] \times R$ where $\sigma \in \{-1; 1\}$, $h : R \to R$ is a continuous function satisfying the condition (1.7). Then the problem (1.1), (1.3) has at least one solution. If, in addition, the condition (1.8) is fulfilled, then this solution is unique.

Corollary 1.2. Let $\sigma \in \{-1; 1\}$, $m = n + \frac{1}{2}[1 - (-1)^{n + \frac{\sigma-1}{2}}]$, and let the inequality (1.6) hold on the set $[0, \omega] \times R$ where $h : R \to R$ is a continuous function satisfying (1.7). Then the problem (1.1), (1.4) has at least one solution. If, in addition, the condition (1.8) is fulfilled, then this solution is unique.

As an example consider the differential equation

$$u^{(2n+1)} = \sigma \frac{|u|^\lambda}{1 + |u|^r} \operatorname{sign} u + c, \tag{1.9}$$

where $\sigma \in \{-1; 1\}$, $\lambda > 0$, $r \geqq 0$ and $c \in R$. The function

$$f(t, x) = \sigma \frac{|x|^\lambda}{1 + |x|^r} \operatorname{sign} x + c$$

evidently, satisfies the inequality (1.6) with

$$h(x) = \frac{|x|^\lambda}{1 + |x|^r} - |c|,$$

and (1.7) is valid if and only if

$$\text{(1.10)} \qquad \text{either } \lambda > r, \text{ or } \lambda = r \text{ and } |c| < 1.$$

By Theorem I.3, if (1.5) and (1.10) hold, then the problem (1.9), (1.2) is uniquely solvable.

In particular, the condition (1.10) is sufficient for the unique solvability of the problem (1.9), (1.3). On the other hand, if $\lambda = r$ and $|c| \geqq 1$ then the problem (1.9), (1.3) has no solutions since an arbitrary solution of the equation (1.9) satisfies the inequality

$$|u^{(2n+1)}(t)| > 1 - \frac{|u(t)|^\lambda}{1 + |u(t)|^\lambda} > 0 \qquad \text{for } 0 \leqq t \leqq \omega.$$

Hence, if $\lambda = r$ then the condition $|c| < 1$ is necessary for the unique solvability of the problem (1.9), (1.3). This example shows that the condition (1.7) is essential in theorems I.1 and I.3 and cannot be omitted.

In the paper of A. V. Kibenko and A. Kipnis [6] it is stated that the problem

$$u''' = f(t, u); \qquad u^{(i-1)}(0) = u^{(i-1)}(\omega) \qquad (i = 1, 2, 3)$$

is uniquely solvable if f has a continuous partial derivative with respect to the second variable satisfying the inequality $a \leqq \sigma \dfrac{\partial f(t, x)}{\partial x} \leqq b$ where $0 < a < b < +\infty$. It is clear that this result is a consequence of Corollary 1.1.

2. Some Auxiliary Results

Consider the differential equation

$$\text{(2.1)} \qquad u^{(2n+1)} = p(t)\, u + q(t, u, u', \ldots, u^{(2n)}),$$

where $p : [0, \omega] \to R$ and $q : [0, \omega] \times R^{2n} \to R$ are continuous functions.

Lemma 2.1. *Suppose that* $\sigma \in \{-1; 1\}$, *the condition* (1.5) *is satisfied,*

$$\text{(2.2)} \qquad \sigma p(t) \geqq 0 \quad \textit{for } 0 \leqq t \leqq \omega,\ p(t) \not\equiv 0$$

and

(2.3) $\sup \{| q(t, x_1, \dots, x_{2n+1}) | : 0 \leqq t \leqq \omega, \quad (x_1, \dots, x_{2n+1}) \in R^{2n+1}\} < +\infty.$

Then the problem (2.1), (1.2) *has at least one solution.*

Proof. By one of the theorems of Conti [2] for the proof of Lemma 2.1 it is sufficient to show that the linear differential equation

$$u^{(2n+1)} = p(t)\, u \tag{2.4}$$

under the boundary conditions (1.2) has only zero solution.

Let u be an arbitrary solution of the problem (2.4), (1.2). Then

$$\psi_\sigma(u(0), \dots, u^{(2n)}(0); u(\omega), \dots, u^{(2n)}(\omega)) = \sigma \int_0^\omega u^{(2n+1)}(t)\, u(t)\, dt =$$
$$= \sigma \int_0^\omega p(t)\, u^2(t)\, dt.$$

On the other hand, by (1.2) and (1.5) we have

$$\psi_\sigma(u(0), \dots, u^{(2n)}(0); \quad u(\omega), \dots, u^{(2n)}(\omega)) \leqq 0.$$

Thus

$$\sigma \int_0^\omega p(t)\, u^2(t)\, dt \leqq 0.$$

Consequently, in view of (2.2), $u(t) \equiv 0$. This completes the proof.

In the sequel we have to consider boundary value problems of periodic type for differential inequalities

$$g(u(t)) \leqq u^{(2n+1)}(t) \operatorname{sign}(\sigma u(t)) \leqq \bar{g}(u(t)) \qquad \text{for } 0 \leqq t \leqq \omega \tag{2.5}$$

where $g : R \to R$ and $\bar{g} : R \to R$ are continuous functions. Under a solution of (2.5) we understand a $2n + 1$ times continuously differentiable function $u : [0, \omega] \to R$ satisfying this inequality in all points of the segment $[0, \omega]$.

Lemma 2.2. *Suppose that* $\sigma \in \{-1; 1\}$, *the condition* (1.5) *is satisfied and there exist numbers* $\delta > 0$ *and* $r_0 > 0$ *such that*

$$g(x) > \delta \qquad \textit{for } | x | > r_0. \tag{2.6}$$

Then any solution u *of the problem* (2.5), (1.2) *satisfies the inequality*

$$| u(t) | \leqq r^* \qquad \textit{for } 0 \leqq t \leqq \omega \tag{2.7}$$

with

$$r^* = 2r_1\, \omega^{2n+1} + r_0(4n + 2)^{2n+2} \left(1 + \frac{r_1}{\delta}\right) \tag{2.8}$$

and

$$r_1 = \max \{| g(x) | + | \bar{g}(x) | : | x | \leqq r_0 \}.$$

Proof. By (1.2) and (1.5)

$$\sigma \int_0^\omega u^{(2n+1)}(t)\, u(t)\, dt = \psi_\sigma(u(0), \dots, u^{(2n)}(0); u(\omega), \dots, u^{(2n)}(\omega)) \leqq 0.$$

Hence by multiplying the inequality (2.5) by $|u(t)|$ and integrating it on [0, ω] we obtain

$$\int_0^\omega g(u(t))\,|u(t)|\, dt \leqq \sigma \int_0^\omega u^{(2n+1)}(t)\, u(t)\, dt \leqq 0. \tag{2.9}$$

Set

$$I = \{t \in [0, \omega] : |u(t)| \leqq r_0\}$$

then in view of (2.5), (2.6) and (2.9) we get

$$|u^{(2n+1)}(t)| \leqq |g(u(t))| + |\bar{g}(u(t))| \leqq r_1 \qquad \text{for } t \in I, \tag{2.10}$$

$$\sigma u^{(2n+1)}(t)\, u(t) > 0 \qquad \text{for } t \in [0, \omega] \setminus I, \tag{2.11}$$

$$|u(t)| < \frac{1}{\delta} g(u(t))\,|u(t)| \qquad \text{for } t \in [0, \omega] \setminus I, \tag{2.12}$$

$$\int_{[0,\omega]\setminus I} g(u(t))\,|u(t)|\, dt \leqq \int_I g(u(t))\,|u(t)|\, dt \leqq r_1 r_0 \omega \tag{2.13}$$

and

$$\sigma \int_{[0,\omega]\setminus I} u^{(2n+1)}(t)\, u(t)\, dt \leqq \sigma \int_I u^{(2n+1)}(t)\, u(t)\, dt. \tag{2.14}$$

It follows from (2.10), (2.11) and (2.14) that

$$\begin{aligned} \int_0^\omega |u^{(2n+1)}(t)|\, dt &= \int_{[0,\omega]\setminus I} |u^{(2n+1)}(t)|\, dt + \int_I |u^{(2n+1)}(t)|\, dt \leqq \\ &\leqq \frac{1}{r_0} \sigma \int_{[0,\omega]\setminus I} u^{(2n+1)}(t)\, u(t)\, dt + r_1 \omega \leqq \\ \leqq \frac{1}{r_0} \int_I u^{(2n+1)}(t)\, u(t)\, dt + r_1 \omega &\leqq \int_I |u^{(2n+1)}(t)|\, dt + r_1 \omega \leqq 2 r_1 \omega \end{aligned} \tag{2.15}$$

and from (2.12), (2.13) we have

$$\begin{aligned} \int_0^\omega |u(t)|\, dt &\leqq \int_I |u(t)|\, dt + \int_{[0,\omega]\setminus I} |u(t)|\, dt \leqq \\ &\leqq r_0 \omega + \frac{1}{\delta} \int_{[0,\omega]\setminus I} g(u(t))\,|u(t)|\, dt \leqq C_1 \omega, \end{aligned} \tag{2.16}$$

where

$$C_1 = r_0 \left(1 + \frac{r_1}{\delta}\right).$$

Let

$$a_i = \frac{\omega}{4n+2} i \qquad (i = 1, \dots, 4n+2),$$

and let the numbers $t_i \in [a_{2i-1}, a_{2i}]$ $(i = 1, \ldots, 2n+1)$ be chosen so that

$$|u(t_i)| = \min\{|u(t)| : a_{2i-1} \leqq t \leqq a_{2i}\} \qquad (i = 1, \ldots, 2n+1).$$

Obviously

$$t_{i+1} - t_i \geqq a_{2i+1} - a_{2i} = \frac{\omega}{4n+2} \qquad (i = 1, \ldots, 2n+1). \tag{2.17}$$

On the other hand, in view of (2.16) we have

$$|u(t_i)| \leqq \frac{4n+2}{\omega} \int_{a_{2i-1}}^{a_{2i}} |u(t)|\, dt \leqq (4n+2)\, C_1 \qquad (i = 1, \ldots, n). \tag{2.18}$$

Let u_0 be the Lagrange interpolating polynomial which is equal to $u(t_i)$ $(i = 1, \ldots, 2n+1)$ in the points of interpolation t_i $(i=1, \ldots, 2n+1)$, i. e.

$$u_0(t) = \sum_{i=1}^{2n+1} \frac{(t-t_1)\ldots(t-t_{i-1})(t-t_{i+1})\ldots(t-t_{2n+1})}{(t_i-t_1)\ldots(t_i-t_{i-1})(t_i-t_{i+1})\ldots(t_i-t_{2n+1})}\, u(t_i).$$

Let

$$v(t) = u(t) - u_0(t).$$

Then

$$v(t_i) = 0 \qquad (i = 1, \ldots, 2n+1) \tag{2.19}$$

and

$$v^{(2n+1)}(t) = u^{(2n+1)}(t). \tag{2.20}$$

By (2.19) and the Roll theorem there exist points $s_k \in [0, \omega]$ $(k = 1, \ldots, 2n+1)$ such that

$$v^{(k-1)}(s_k) = 0 \qquad (k = 1, \ldots, 2n+1). \tag{2.21}$$

Because of (2.15), (2.20) and (2.21) we have

$$|v^{(2n)}(t)| \leqq \int_0^\omega |u^{(2n+1)}(t)|\, dt \leqq 2r_1\omega$$

and

$$|v^{(k-1)}(t)| \leqq 2r_1\omega^{2n+2-k} \qquad (k = 1, \ldots, 2n+1).$$

Hence

$$|u(t)| \equiv |v(t) + u_0(t)| \leqq 2r_1\omega^{2n+1} + |u_0(t)| \qquad \text{for } 0 \leqq t \leqq \omega.$$

On the other hand, in view of (2.17) and (2.18)

$$|u_0(t)| \leqq (2n+1)\left(\frac{\omega}{\dfrac{\omega}{4n+2}}\right)^{2n} (4n+2)\, C_1 \leqq (4n+2)^{2n+2} C_1.$$

Consequently, the estimate (2.7) is true. This completes the proof.

3. Proof of the Existence and Uniqueness Theorems

Consider the differential equation

$$u^{(2n+1)} = f(t, u, u', \dots, u^{(2n)}) \tag{3.1}$$

with the continuous right-hand side $f: [0, \omega] \times R^{2n+1} \to R$. Instead of Theorem 1.1 we shall prove the following more general one.

Theorem 3.1. *Suppose that $\sigma \in \{-1; 1\}$ the condition* (1.5) *holds and the inequality*

$$h(x_1) \leqq f(t, x_1, \dots, x_{2n+1}) \operatorname{sign}(\sigma x_1) \leqq \bar{h}(x_1) \tag{3.2}$$

is fulfilled on the set $[0, \omega] \times R^{2n+1}$ where the functions $h: R \to R$ and $\bar{h}: R \to R$ are continuous and satisfy the condition (1.7). *Then the problem* (3.1), (1.2) *has at least one solution.*

Proof. Due to the condition (1.7) we can choose $\delta > 0$ and $r_0 > 1$ such that

$$h(x) > \delta \qquad \text{for } |x| \geqq r_0. \tag{3.3}$$

Let

$$r_1 = \max\{|h(x)| + |\bar{h}(x)| : 0 \leqq x \leqq \omega\}$$

and let the number r^* be defined by the equality (2.8).

Put

$$\chi(x) = \begin{cases} x & \text{for } |x| \leqq r^*, \\ r^* \operatorname{sign} x & \text{for } |x| > r^*, \end{cases}$$

$$q(t, x_1, \dots, x_{2n+1}) = f(t, \chi(x_1), x_2, \dots, x_{2n+1}) - \sigma\chi(x_1) \tag{3.4}$$

and consider the differential equation

$$u^{(2n+1)} = \sigma u + q(t, u, u', \dots, u^{(2n)}). \tag{3.5}$$

By the conditions (3.2), (3.3) and (3.4),

$$\sigma x_1 + q(t, x_1, \dots, x_{2n+1}) = f(t, x_1, \dots, x_{2n+1}) \qquad \text{for } |x_1| \leqq r^* \tag{3.6}$$

and

$$|q(t, x_1, \dots, x_{2n+1})| = |\chi(x_1) + f(t, \chi(x_1), x_2, \dots, x_{2n+1})| \leqq$$
$$\leqq r^* + |h(\chi(x_1))| + |\bar{h}(\chi(x_1))| \leqq C_0$$

where

$$C_0 = r^* + \max\{|h(x)| + |\bar{h}(x)| : |x| \leqq r^*\}.$$

Consequently the condition (2.3) is satisfied.

According to Lemma 2.1, the problem (3.5), (1.2) is solvable. Let u be its arbitrary solution.

Then

$$u^{(2n+1)}(t)\,\mathrm{sign}\,(\sigma u(t)) = [\sigma u(t) - \sigma\chi(u(t)) +$$
$$+ f(t, \chi(u(t)), u'(t), \ldots, u^{(2n)}(t))\,\mathrm{sign}\,(\sigma u(t)) = |\,u(t) - \chi(u(t))\,| +$$
$$+ f(t, \varkappa(u(t)), u'(t), \ldots, u^{(2n)}(t))\,\mathrm{sign}\,(\sigma\chi(u(t))).$$

This by the condition (3.2) implies the inequality (2.5) where

$$g(x) = |\,x - \chi(x)\,| + h(\chi(x)), \qquad \bar{g}(x) = |\,x - \chi(x)\,| + \bar{h}(\chi(x)).$$

Since $r_0 \leqq r^*$ we have

$$\max\,\{|\,g(x)\,| + |\,\bar{g}(x)\,| : |\,x\,| \leqq r_0\} = \max\,\{|\,h(x)\,| + |\,\bar{h}(x)\,| : |\,x\,| \leqq r_0\} = r_1.$$

On the other hand, in view of (3.3) it is clear that the condition (2.6) holds.

By Lemma 2.2 the function u satisfies (2.7). But it follows from (2.7) and (3.6) that u is a solution of the equation (3.1). This completes the proof.

If $f(t, x_1, \ldots, x_{2n+1}) \equiv f(t, x_1)$ then (1.6) implies (3.2), where

$$\bar{h}(x) = \max\,\{|\,f(t, x)\,| : 0 \leqq t \leqq \omega\}.$$

Thus theorem 1.1 is a consequence of Theorem 3.1.

Proof of Theorem 1.2. Let u_1 and u_2 be arbitrary solutions of the problem (1.1), (1.2). Set

$$v(t) = u_1(t) - u_2(t).$$

Then

$$v^{(2n+1)}(t) = f(t, u_1) - f(t, u_2).$$

According to (1.2) and (1.5), by multiplying both sides of this equality by $\sigma v(t)$ and integrating on $[0, \omega]$ we obtain

$$\int_0^{\omega} [f(t, u_1(t)) - f(t, u_2(t))]\, v(t)\, \mathrm{d}t = \sigma \int_0^{\omega} v^{(2n+1)}(t)\, v(t)\, \mathrm{d}t =$$
$$= \psi_\sigma(u(0), \ldots, u^{(2n)}(0); u(\omega), \ldots, u^{(2n)}(\omega)) \leqq 0. \tag{3.7}$$

In view of the condition (1.8)

$$\sigma[f(t, u_1(t)) - f(t, u_2(t))]\, v(t) \geqq 0 \qquad \text{for } 0 \leqq t \leqq \omega.$$

Besides, the left-hand side of this inequality is equal to zero only at those points, where $v(t) = 0$. Hence (3.7) implies

$$v(t) = 0, \qquad 0 \leqq t \leqq \omega.$$

Therefore the problem (1.1), (1.2) can't have two different solutions. This completes the proof. Theorem 1.3 immediately follows from Theorems 1.1 and 1.2.

In order to verify the validity of Corollaries 1.1 and 1.2 it is sufficient to note that

$$\sum_{i=1}^{2n+1} | y_i - z_i | > 0 \qquad \text{for } \psi_\sigma(y_1, \dots, y_{2n+1}; z_1, \dots, z_{2n+1}) > 0,$$

and, if $m = n + \frac{1}{2}\left[1 - (-1)^{n+\frac{\sigma-1}{2}}\right]$ then

$$\sum_{i=1}^{m} | y_i | + \sum_{i=1}^{n} | z_i | > 0 \qquad \text{for } \psi_\sigma(y_1, \dots, y_{2n+1}; z_1, \dots, z_{2n+1}) > 0.$$

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