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REGULARITY IN ARITHMETICAL VARIETIES

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An algebra $\mathfrak{A}$ is *regular* if any two congruences on $\mathfrak{A}$ coincide whenever they have a congruence class in common. A variety $\mathscr{V}$ is *regular* if it contains only regular algebras. Denote by Con($\mathfrak{A}$) the lattice of all congruences on $\mathfrak{A}$. A variety $\mathscr{V}$ is *distributive* (*modular*) if Con($\mathfrak{A}$) is distributive (modular) for each $\mathfrak{A} \in \mathscr{V}$. A variety $\mathscr{V}$ is *permutable* if $\Theta_1 . \Theta_2 = \Theta_2 . \Theta_1$ for each $\Theta_1, \Theta_2 \in \text{Con}(\mathfrak{A})$ and every $\mathfrak{A} \in \mathscr{V}$. By A. F. Pixley [7], permutable and distributive varieties are called *arithmetical.*

As it was mentioned in [9, Theorem 2.7], J. Hageman proved that regularity of $\mathscr{V}$ implies modularity of $\mathscr{V}$ (see also [1]). Since modularity is a weaker condition then permutability as well as distributivity, it is interesting to ask about mutual connections of permutability, distributivity and regularity. These three conditions are mutually independent (for independence of permutability and regularity, see e.g. [2], [10], [11], other relationships are well known). The aim of this paper is twofold:

– to give a Mal'cev characterization of regular arithmetical varieties,

– to give conditions under which an arithmetical variety is regular.

To the first topic, we can give an immediate answer applying results of [2] and [8]:

Theorem 1. *For a variety $\mathscr{V}$, the following conditions are equivalent:*

(1) *$\mathscr{V}$ is permutable, distributive and regular;*

(2) *there exist a $(3 + n)$-ary polynomial p and ternary polynomials q, $t_1, \ldots, t_n$ such that*

$$q(x, x, z) = q(x, z, x) = q(z, x, x) = x, \qquad t_i(x, x, z) = z \qquad \text{for } i = 1, \ldots, n,$$
$$x = p(x, y, z, z, \ldots, z), \qquad y = p(x, y, z, t_1(x, y, z), \ldots, t_n(x, y, z)).$$

The sketch of the proof. Applying the Theorem of [2] (we can simplify it using diagonal relations instead of tolerances as it was done by J. Duda), we have that $\mathscr{V}$ is regular and permutable if and only if there exist n-ary p and ternary $t_1, \ldots, t_n$ fulfilling their part of (2). By [8], permutable variety $\mathscr{V}$ is distributive if and only if there exists q with $q(x, x, z) = q(x, z, x) = q(z, x, x) = x$, whence (2) is completed.

Although (2) of Theorem 1 is not too complicated, we can give also another Mal'cev characterization of regularity, permutability and regularity containing no separated polynomial for one of these three conditions. Such a condition can be more convenient for the second topic of our paper as we shall see later.

Theorem 2. *For a variety $\mathscr{V}$, the following conditions are equivalent:*

(1) *$\mathscr{V}$ is permutable, distributive and regular,*

(2) *there exist $2k$-ary polynomial r, $(3+n)$-ary polynomials $p_1, \ldots, p_{2k}$ and ternary polynomials $t_1, \ldots, t_n$ such that*

$$t_j(x, x, z) = z \qquad \text{for } j = 1, \ldots, n,$$

$$p_i(x, y, x, \ldots, x) = p_i(x, y, x, t_1(x, y, x), \ldots, t_n(x, y, x)) \qquad \text{for } i = 1, \ldots, k,$$

$$p_i(x, y, y, \ldots, y) = p_i(x, y, y, t_1(x, y, y), \ldots, t_n(x, y, y)) \qquad \text{for } i = k+1, \ldots, 2k,$$

$$x = r(p_1(x, y, z, z, \ldots, z), \ldots, p_{2k}(x, y, z, z, \ldots, z)),$$

$$y = r(p_1(x, y, z, t_1(x, y, z), \ldots, t_n(x, y, z)), \ldots, p_{2k}(x, y, z, t_1(x, y, z), \ldots, t_n(x, y, z))).$$

Before the proof, we must introduce some concepts and formulate two lemmas.

Let $\mathfrak{A} = (A, F)$ be an algebra. Denote by $\triangle$ the so called *diagonal* of $\mathfrak{A}$, i.e. $\triangle = \{\langle x, x\rangle; x \in A\}$. A binary relation D on A will be called a *diagonal relation* if

(a) $\triangle \subseteq D$,

(b) D is a subalgebra of $\mathfrak{A} \times \mathfrak{A}$, i.e. D satisfies the *substitution property* with respect to F.

Clearly the set of all diagonal relations on $\mathfrak{A}$ forms a complete lattice with respect to the set inclusion, where $\triangle$ is the least and $A \times A$ the greatest element. Denote by $\vee$ the operation join in this lattice. Clearly the operation meet coincides with set intersection. Since this lattice is complete, there exists the least diagonal relation containing a set $M \subseteq A \times A$; denote it by $D(M)$.

Lemma 1. *Let R, S be two diagonal relations on $\mathfrak{A}$. Then $\langle x, y\rangle \in R \vee S$ if and only if there exists a $2k$-ary polynomial r over $\mathfrak{A}$ such that*

$$x = r(a_1, \ldots, a_{2k}),$$
$$y = r(b_1, \ldots, b_{2k}),$$

where $\langle a_i, b_i\rangle \in R$ for $i = 1, \ldots, k$ and $\langle a_i, b_i\rangle \in S$ for $i = k+1, \ldots, 2k$.

Proof. It is clear that the set of all pairs $\langle x, y\rangle$ for which there exists a polynomial r and elements a_i, b_i $(i = 1, \ldots, 2k)$ such as in this Lemma forms a diagonal relation on $\mathfrak{A}$ containing R and S. Moreover, every diagonal relation on $\mathfrak{A}$ containing R and S must contain also all these pairs, hence it is $R \vee S$.

Lemma 2. *Let $\mathfrak{A} = (A, F)$ be an algebra and $M \subseteq A \times A$. Then $\langle x, y\rangle \in D(M)$ if and only if there exists an n-ary algebraic function f over $\mathfrak{A}$ and elements*

$x_1, \ldots, x_n, y_1, \ldots, y_n \in A$ *such that*

$$x = f(x_1, \ldots, x_n), \qquad y = f(y_1, \ldots, y_n),$$

where $\langle x_i, y_i \rangle \in M$ *for* $i = 1, \ldots, n$.

The proof is straightforward and hence omitted. If $M = \{\langle a, b \rangle\}$, denote $D(a, b)$ instead of $D(\{\langle a, b \rangle\})$.

Proof of Theorem 2. (1) $\Rightarrow$ (2): Let $\mathscr{V}$ be regular, permutable and distributive and let $F_3(x, y, z)$ be a free algebra of $\mathscr{V}$ generated by the free generators x, y, z. Let $\Theta = \Theta(x, y)$ and $C = [z]_\Theta$, i.e. it is the congruence class of Θ containing z. Denote by $\Theta\{C\}$ the least congruence on $F_3(x, y, z)$ containing a congruence class C. Since $\mathscr{V}$ is regular, we have $\Theta\{C\} = \Theta(x, y)$. Clearly $\Theta\{C\} = \Theta(M)$ for $M = \{z\} \times [z]_\Theta$, where $\Theta(M)$ is the least congruence collapsing M. Since $\mathscr{V}$ is permutable, the Theorem of Werner [12] implies $\Theta(M) = D(M)$, thus $\Theta(x, y) = D(M)$. Hence

(i) $$\langle x, y \rangle \in D(M).$$

By this Theorem of Werner, permutability of $\mathscr{V}$ implies that for each $\mathfrak{A} \in \mathscr{V}$, the lattice of all diagonal relations on $\mathfrak{A}$ and $\mathrm{Con}(\mathfrak{A})$ coincide, thus $\vee$ is also join in $\mathrm{Con}(\mathfrak{A})$. Hence

$$\langle x, y \rangle \in \Theta(x, z) \vee \Theta(z, y)$$

implies also

(ii) $$\langle x, y \rangle \in D(x, z) \vee D(z, y),$$

thus (i) and (ii) give

$$\langle x, y \rangle \in (D(M) \cap D(x, z)) \vee (D(M) \cap D(z, y))$$

with respect to the distributivity of $\mathscr{V}$. By Lemma 1, there exists a $2k$-ary polynomial r such that

$$x = r(a_1, \ldots, a_{2k}),$$
$$y = r(b_1, \ldots, b_{2k}),$$

where

$$\langle a_i, b_i \rangle \in D(M) \cap D(x, z) \quad \text{for } i = 1, \ldots, k,$$
$$\langle a_i, b_i \rangle \in D(M) \cap D(z, y) \quad \text{for } i = k + 1, \ldots, 2k, \text{ i.e.}$$
$$\langle a_i, b_i \rangle \in D(M) \quad \text{for } i = 1, \ldots, 2k,$$

and

$$\langle a_i, b_i \rangle \in D(x, z) \quad \text{for } i = 1, \ldots, k,$$
$$\langle a_i, b_i \rangle \in D(z, y) \quad \text{for } i = k + 1, \ldots, 2k.$$

By Lemma 2, there exist $(3 + n)$-ary polynomials $p_1, \ldots, p_{2k}$ such that

$$a_i = p_i(x, y, z, z, \ldots, z),$$
$$b_i = p_i(x, y, z, v_1, \ldots, v_n),$$

where $v_1, \dots, v_n \in [z]_\Theta$. Since $v_i \in F_3(x, y, z)$, there exist ternary polynomials $t_1, \dots, t_n$ with

$$v_i = t_i(x, y, z), \qquad i = 1, \dots, n.$$

Then $v_i \in [z]_\Theta$ for $\Theta = \Theta(x, y)$ implies

$$t_i(x, x, z) = z \qquad \text{for } i = 1, \dots, n.$$

By the permutability, $D(x, z) = \Theta(x, z)$ and $D(z, y) = \Theta(z, y)$, thus $\langle a_i, b_i \rangle \in$ $\in D(x, z)$ implies

$$p_i(x, y, x, \dots, x) = p_i(x, y, x, t_1(x, y, x), \dots, t_n(x, y, x)) \qquad \text{for } i = 1, \dots, k,$$

and $\langle a_i, b_i \rangle \in D(z, y)$ implies

$$p_i(x, y, y, \dots, y) = p_i(x, y, y, t_1(x, y, y), \dots, t_n(x, y, y)) \qquad \text{for } i = k + 1, \dots, 2k.$$

(2) $\Rightarrow$ (1): We prove separately every of these three properties:

Regularity. By [11], it remains to prove that for each $\mathfrak{A} \in \mathscr{V}$, if $\Theta \in \mathrm{Con}(\mathfrak{A})$ has a one-element class, then $\Theta = \Delta$. Suppose $a, b, c \in A$, $\langle a, b \rangle \in \Theta \in \mathrm{Con}(\mathfrak{A})$ and let $\{c\}$ be a class of Θ. By (2), we have

$$t_j(a, b, c)\, \Theta\, t_j(a, a, c) = c,$$

thus $t_j(a, b, c) \in \{c\}$, i.e. $t_j(a, b, c) = c$. By (2),

$$\begin{aligned} a &= r(p_1(a, b, c, c, \dots, c), \dots, p_{2k}(a, b, c, c, \dots, c)) = \\ &= r(p_1(a, b, c, t_1(a, b, c), \dots, t_n(a, b, c)), \dots, \\ &\quad \dots, p_{2k}(a, b, c, t_1(a, b, c), \dots, t_n(a, b, c))) = b, \end{aligned}$$

thus $\Theta = \Delta$, and, by [11], the regularity is proved.

Permutability: Put

$$\begin{aligned} t(a, b, c) = r(&p_1(a, c, b, t_1(b, c, b), \dots, t_n(b, c, b)), \dots, \\ &\dots, p_{2k}(a, c, b, t_1(b, c, b_1, \dots, t_n(b, c, b))). \end{aligned}$$

Then clearly $t(a, c, c) = a$ and $t(a, a, c) = c$, thus t is a Mal'cev polynomial, i.e. $\mathscr{V}$ is permutable.

Distributivity: Let

$$q(a, b, c) =$$
$$r(\vec{\mathbf{p}}_i(a, c, b, t_1(a, c, b), \dots, t_n(a, c, b)), \vec{\mathbf{p}}_j(a, c, b, t_1(a, a, b), \dots, t_n(a, a, b))),$$

where $i = 1, \dots, k, j = k + 1, \dots, 2k$. Then clearly $q(a, c, a) = q(a, a, c)) =$ $= q(c, a, a) = a$ and, by [8], the existence of such q in a permutable variety secures the distributivity of $\mathscr{V}$.

Example 1. Let $\mathscr{V}$ be a *variety of Boolean algebras*. We can put $n = 2, k = 1$ and

$$t_1(x, y, z) = [((x \oplus y) \wedge (y \oplus z)) \vee (x \oplus z)] \oplus y,$$
$$t_2(x, y, z) = [((x \oplus y) \wedge (y \oplus z)) \vee (x \oplus z)] \oplus x,$$

$$p_1(x, y, z, v, w) = v,$$
$$p_2(x, y, z, v, w) = x \oplus w,$$
$$r(x_1, x_2) = x_1 \oplus x_2,$$

where $a \oplus b = (a' \wedge b) \vee (a \wedge b')$, the so called *symmetrical difference of* a, b.

Now, we are ready to give sufficient conditions for the second topic of our paper.

A *Pixley polynomial* is a ternary polynomial p over $\mathscr{V}$ such that

$$p(x, x, z) = z, \qquad p(x, y, x) = x, \qquad p(x, z, z) = x.$$

As it is proved in [7], a variety $\mathscr{V}$ is arithmetical if and only if there exists a Pixley polynomial in $\mathscr{V}$.

Theorem 3. *Let $\mathscr{V}$ be an arithmetical variety. If there exists a binary polynomial (denote it by $+$) such that*

$$x + (y + z) = (x + y) + z,$$
$$(x + x) + y = y + (x + x) = y,$$

then $\mathscr{V}$ is regular.

Proof. Let p be a Pixley polynomial in $\mathscr{V}$. Put $k = 1$, $n = 2$ and

$$t_1(x, y, z) = p(x, y, z),$$
$$t_2(x, y, z) = p(x, y, z) + (y + x),$$
$$p_1(x, y, z, v, w) = v,$$
$$p_2(x, y, z, v, w) = w + x,$$
$$r(x_1, x_2) = x_1 + x_2.$$

Then

$$t_1(x, x, z) = p(x, x, z) = z,$$
$$t_2(x, x, z) = p(x, x, z) + (x + x) = z,$$
$$p_1(x, y, x, t_1(x, y, x), t_2(x, y, x)) = t_1(x, y, x) = x = p_1(x, y, x, x, x),$$
$$p_2(x, y, y, t_1(x, y, y), t_2(x, y, y)) = t_2(x, y, y) + x =$$
$$= p(x, y, y) + (y + x) + x = x + y = p_2(x, y, y, y, y),$$
$$(rp_1(x, y, z, z, z), p_2(x, y, z, z, z)) = p_1(x, y, z, z, z) + p_2(x, y, z, z, z) =$$
$$= z + (z + x) = (z + z) + x = x,$$
$$r(p_1(x, y, z, t_1(x, y, z), t_2(x, y, z)), p_2(x, y, z, t_1(x, y, z), t_2(x, y, z))) =$$
$$= t_1(x, y, z) + (t_2(x, y, z) + x) =$$
$$= p(x, y, z) + ((p(x, y, z) + (y + x)) + x) = y.$$

By Theorem 2, $\mathscr{V}$ is regular.

Example 2. In the case of *Boolean algebras*, we can clearly put $a + b = a \oplus b$. We can give also another Mal'cev condition formulated in ternary polynomials:

Theorem 4. *Let $\mathscr{V}$ be an arithmetical variety and p be its Pixley polynomial. If among their Mal'cev polynomials exists a polynomial q with*

$$q(p(x, y, z), q(p(x, y, z), x, y), x) = y,$$

then $\mathscr{V}$ is regular.

Proof. We can put $n = 2$, $k = 2$ and

$$t_1(x, y, z) = p(x, y, z),$$
$$t_2(x, y, z) = q(p(x, y, z), x, y),$$
$$p_1(x, y, z, v, w) = v,$$
$$p_2(x, y, z, v, w) = x,$$
$$p_3(x, y, z, v, w) = w,$$
$$p_4(x, y, z, v, w) = y,$$
$$r(x_1, x_2, x_3, x_4) = q(x_1, x_3, x_2).$$

One can easy see that (2) of Theorem 2 is satisfied and hence $\mathscr{V}$ is regular.

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