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ON URABE'S APPLICATION OF NEWTON'S METHOD TO NONLINEAR BOUNDARY VALUE PROBLEMS*)

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1. Introduction

In this paper we shall consider following boundary value problem

$$\frac{dx}{dt} = g(x, t), \tag{1.1}$$

$$f(x) = 0, \tag{1.2}$$

where x and $g(x, t)$ are n dimensional vectors and $f(x)$ is an operator from $C(I)$ into R^n, $C(I)$ is the space of all real n vector functions continuous on $I = [a, b]$.

We shall show that the results of Urabe [13, 14], which he calls application of Newton's method, can be obtained as an application of Contraction mapping theorem. In section 2, we begin with certain properties of square matrices and state Contraction mapping theorem in complete generalized norm spaces. This theorem is a particular case of more general result contributed by Schröder [12] also see [5, 7, 8, 11]. Since it is well recognized that working with generalized norm spaces have qualitative as well as quantitative advantages [1–4, 7, 8, 11], our theorem 4.1 is more general and informative than the results obtained in [10, 13, 14]. In section 5, we shall show that the solution obtained in theorem 4.1 is infact isolated. In section 6, we shall consider a perturbed problem of (1.1), (1.2) and, as an application of our theorem 4.1, show that the perturbation method produces an approximate solution within the error $0(\lambda^2)$.

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More general results than those obtained in [17, 18] for the least square problems can be obtained as an application to our theorem 4.1 and this we shall take up some where else. We also remark that merely the existence and uniqueness of solutions of (1.1), (1.2) (even for more abstract problems) has been discussed under weaker conditions e.g. see [6,9 and references therein], however the results obtained here have practical advantage and should be called Picard's iterative methods.

2. Fixed Point Theorem

Throughout, we consider the inequalities between two vectors in R^n component-wise where-as between two $n \times n$ matrices element-wise. The generalized norm (vector norm) space B is a linear space with norm $\| \cdot \|$ which is a mapping into R^n_+ satisfying the properties of usual norm component-wise e.g. see [7, 11].

The following well known properties of matrices will be used frequently without further mention:

(a) For any square matrix A, $\lim_{m\to\infty} A^m = 0$ if and only if $\varrho(A) < 1$ where $\varrho(A)$ denotes the spectral radius of A.

(b) For any square matrix A, $(E - A)^{-1}$ exists and $(E - A)^{-1} = \sum_{m=0}^{\infty} A^m$ if $\varrho(A) < 1$ also if $A \geqq 0$ then $(E - A)^{-1} \geqq 0$. If $0 \leqq A \leqq B$ and $\varrho(B) < 1$ then $\varrho(A) < 1$.

(c) For all square matrices A and B, $\varrho(A + B) \leqq \varrho(A) + \varrho(B)$.

We shall need the following particular case of more general Contraction mapping theorem proved in [8, 12] also see [5].

Theorem 2.1. Let B be a complete generalized norm space, and let for $r \in R^n_+$, $r > 0$; $S(x_0, r) = \{x \in B : \| x - x_0 \| \leqq r\}$. Let T map $S(x_0, r)$ into B and

(c_1) for all $x, y \in S(x_0, r)$

$$\| Tx - Ty \| \leqq K_0 \| x - y \|,$$

where $K_0 \geqq 0$ is an $n \times n$ matrix with $\varrho(K_0) < 1$

(c_2) $$r_0 = (E - K_0)^{-1} \| Tx_0 - x_0 \| \leqq r.$$

Then,

(1) T has a fixed point x^* in $S(x_0, r_0)$

(2) x^* is the unique fixed point of T in $S(x_0, r)$,

(3) the sequence $\{x_m\}$ defined by

$$x_{m+1} = Tx_m, \qquad m = 0, 1, \ldots$$

converges to x^*, with

$$\| x^* - x_m \| \leqq K_0^m r_0$$

(4) for any $x \in \bar{S}(x_0, r_0)$

$$x^* = \lim_{m \to \infty} T^m x$$

(5) any sequence $\{\bar{x}_m\}$ such that $\bar{x}_m \in \bar{S}(x_m, K_0^m r_0)$, $m = 0, 1, \ldots$ converges to x^*.

Remark. Most of his [15, 16] component-wise study can be deduced from theorem 2.1 and this we shall take up in our later papers.

3. Linear Problems

Here, we consider the differential system

$$\frac{dx}{dt} = A(t)\,x(t) + \varphi(t) \tag{3.1}$$

together with

$$L[x] = \mathrm{e}, \tag{3.2}$$

where $A(t)$ is an $n \times n$ continuous matrix on I, $\varphi(t)$ is an $n \times 1$ continuous vector on I, L is a linear operator mapping $C(I)$ into R^n i.e. $\mathrm{e} \in R^n$.

In what follows, we shall denote $Y(t)$ as the fundamental matrix solution of the homogeneous system

$$\frac{dy}{dt} = A(t)\,y(t), \tag{3.3}$$

such that $Y(a) = E$ (unit matrix). $G = L[Y(t)]$ represents the $n \times n$ matrix whose column vectors are $L[y^{(i)}(t)]$, $1 \leqq i \leqq n$ where $y^{(i)}(t)$ is the i-th column vector of $Y(t)$.

Lemma 3.1. *If the matrix G is non-singular then,* (3.1), (3,2) *has a unique solution $x(t)$ and can be represented as*

$$x(t) = H_1[\varphi(t)] + H_2[\mathrm{e}], \tag{3.4}$$

where H_1 is the linear operator mapping $C(I)$ into $C^{(1)}(I)$ such that

$$H_1[\varphi(t)] = Y(t)\int_a^t Y^{-1}(s)\,\varphi(s)\,ds - Y(t)G^{-1}L[Y(t)\int_a^t Y^{-1}(s)\,\varphi(s)\,ds]$$

and H_2 is the linear operator mapping R^n into $C^{(1)}(I)$ such that

$$H_2[\mathrm{e}] = Y(t)\,G^{-1}\mathrm{e}.$$

Proof. Any solution of (3.1) can be expressed as

$$x(t) = Y(t)\,c + Y(t)\int_a^t Y^{-1}(s)\,\varphi(s)\,ds, \tag{3.5}$$

where c is a constant vector. The solution (3.5) satisfies (3.2) if and only if

$$L[Y(t)]\,c + L[Y(t)\int_a^t Y^{-1}(s)\,\varphi(s)\,\mathrm{d}s] = \mathrm{e}. \tag{3.6}$$

Since $\det G \neq 0$, from (3.6) we get

$$c = G^{-1}\mathrm{e} - G^{-1}L[Y(t)\int_a^t Y^{-1}(s)\,\varphi(s)\,\mathrm{d}s]. \tag{3.7}$$

Substituting (3.7) in (3.5) the result (3.4) follows.

4. Existence and Uniqueness

In what follows, we consider the generalized norm space B as $C(I)$ and for $x(t) \in C(I)$, $\| x \| = (\max_{t\in I} | x_1(t) |, \max_{t\in I} | x_2(t) |, \dots, \max_{t\in I} | x_n(t) |)$. In (1.1), (1.2) the function $g(x, t)$ is assumed to be continuously differentiable with respect to x in $R^n \times I$ and $g_x(x, t)$ represents the Jacobian matrix of $g(x, t)$ with respect to x; $f(x)$ is continuously differentiable in $C(I)$ and $f_x(x)$ denotes the linear operator mapping $C(I)$ in R^n.

Definition 4.1. *A function $\bar{x}(t) \in C^{(1)}(I)$ is called an approximate solution of* (1.1), (1.2) *if there exist δ_1 and δ_2 nonnegative vectors such that* $\left\| \frac{\mathrm{d}\bar{x}}{\mathrm{d}t} - g(\bar{x}, t) \right\| \leqq \delta_1$ *and* $\| f(\bar{x}) \| \leqq \delta_2$ *i.e. there exists a function $\eta(t)$ and a constant vector* e' *such that* $\frac{\mathrm{d}\bar{x}}{\mathrm{d}t} = g(\bar{x}(t), t) + \eta(t)$ *and* $f(\bar{x}) = \mathrm{e}'$ *with* $\| \eta(t) \| \leqq \delta_1$ *and* $\| \mathrm{e}' \| \leqq \delta_2$.

Theorem 4.1. *With respect to* (1.1), (1.2) *we assume that there exists an approximate solution $\bar{x}(t)$ and*

(i) *there exists an $n \times n$ continuous matrix $A(t)$, $t \in I$ and L a linear operator mapping $C(I)$ into R^n such that if $Y(t)$ is the fundamental matrix solution of $y' = A(t)\,y$, then $G = L[Y(t)]$ is nonsingular,*

(ii) *there exist $n \times n$ nonnegative matrices M_1 and M_2 such that $\| H_1 \| \leqq M_1$, $\| H_2 \| \leqq M_2$, where H_1 and H_2 are linear operators defined in lemma* 3.1,

(iii) *there exist $n \times n$ nonnegative matrices M_3 and M_4, and a positive vector r such that for all $x(t) \in S(\bar{x}, r) = \{z(t) \in C(I) : \| z - \bar{x} \| \leqq r\}$, $\| g_x(x, t) - A(t) \| \leqq$ $\leqq M_3$ and $\| f_x(x) \pm L \| \leqq M_4$,*

(iv) $K_0 = M_1 M_3 + M_2 M_4$, $\varrho(K_0) < 1$ and $(E - K_0)^{-1}\,(M_1\delta_1 + M_2\delta_2) \leqq r$.

Then,

(1) *there exists a solution $x^*(t)$ of* (1.1), (1.2) *in $S(\bar{x}, r_0)$,*

(2) *$x^*(t)$ is the unique solution of* (1.1), (1.2) *in $S(\bar{x}, r)$,*

(3) *the sequence* $\{x_m(t)\}$ *defined by*

$$x_{m+1}(t) = H_1[g(x_m(t), t) - A(t)\,x_m(t)] + H_2[L[x_m] \pm f(x_m)],$$

$$x_0(t) = \bar{x}(t); \qquad m = 0, 1, \ldots \tag{4.1}$$

converges to $x^*(t)$, *with*

$$\| x^* - x_m \| \leqq K_0^m r_0,$$

(4) *for* $x_0(t) = x(t) \in \bar{S}(\bar{x}, r_0)$ *the iterative process* (4.1) *converges to* $x^*(t)$,

(5) *any sequence* $\{\bar{x}_m(t)\}$ *such that* $\bar{x}_m(t) \in \bar{S}(x_m, K_0^m r_0)$, $m = 0, 1, \ldots$ *converges to* $x^*(t)$,

where $r_0 = (E - K_0)^{-1} \| x_1 - \bar{x} \|$.

Proof. We note that the approximate solution $\bar{x}(t)$ can be expressed as

$$\bar{x}(t) = H_1[g(\bar{x}(t), t) + \eta(t) - A(t)\,\bar{x}(t)] + H_2[L[\bar{x}] \pm f(\bar{x}) \mp e']. \tag{4.2}$$

Next, we define an operator $T : \bar{S}(\bar{x}, r) \to B$ as follows

$$Tx(t) = H_1[g(x(t), t) - A(t)\,x(t)] + H_2[L[x] \pm f(x)]. \tag{4.3}$$

Obviously any fixed point of (4.3) is a solution of (1.1), (1.2).

For all $x(t), y(t) \in \bar{S}(\bar{x}, r)$, we find from (4.3)

$$Tx(t) - Ty(t) =$$

$$= H_1[g(x(t), t) - g(y(t), t) - A(t)(x(t) - y(t))] + H_2[L[x - y] \pm (f(x) - f(y))] =$$

$$= H_1[\int_0^1 [g_x(x(t) + \Theta_1(y(t) - x(t)), t) - A(t)]\,(x(t) - y(t))\,d\Theta_1] +$$

$$+ H_2[\int_0^1 [L \pm f_x(x + \Theta_2(y - x))]\,[x - y]\,d\Theta_2]$$

and hence, from (ii) and (iii) and the fact that $x(t) + \Theta_i(y(t) - x(t)) \in \bar{S}(\bar{x}, r)$, $i = 1, 2$ we obtain

$$\| Tx - Ty \| \leqq (M_1 M_3 + M_2 M_4) \| x - y \| = K_0 \| x - y \|.$$

Also, from (4.2) and (4.3), we get

$$T\bar{x}(t) - \bar{x}(t) = Tx_0(t) - x_0(t) = H_1[-\eta(t)] + H_2[\pm e']$$

and hence from (ii) and the definition 4.1 it follows that

$$\| Tx_0 - x_0 \| \leqq M_1 \delta_1 + M_2 \delta_2 \tag{4.4}$$

of from (iv)

$$r_0 = (E - K_0)^{-1} \| Tx_0 - x_0 \| \leqq r.$$

Thus, the conditions of theorem 2.1 are satisfied and the conclusions (1)–(5) follow.

Remark. From the conclusion (3) and (4.4), we obtain

$$\| x^* - \bar{x} \| \leqq r_0 = (E - K_0)^{-1} \| x_1 - x_0 \| \leqq \leqq (E - K_0)^{-1} (M_1\delta_1 + M_2\delta_2). \tag{4.5}$$

Also, since

$$\frac{dx^*}{dt} - \frac{d\bar{x}}{dt} = g(x^*, t) - g(\bar{x}, t) - \eta(t) =$$
$$= \int_0^1 [g_x(x^* + \Theta(\bar{x} - x^*), t)](x^* - \bar{x})\, d\Theta - \eta(t).$$

we obtain

$$\left\| \frac{dx^*}{dt} - \frac{d\bar{x}}{dt} \right\| \leqq M_5 \| x^* - \bar{x} \| + \| \eta(t) \| \leqq$$
$$\leqq M_5(E - K_0)^{-1} (M_1\delta_1 + M_2\delta_2) + \delta_1,$$

where M_s is an $n \times n$ nonnegative matrix such that $\| g_x(x, t) \| \leqq M_s$ for all $(x, t) \in \in S(\bar{x}, r) \times I$.

5. Isolated Solution

Definition 5.1. *Any solution* $\hat{x}(t) \in C^{(1)}(I)$ *of* (1.1), (1.2) *will be called isolated if* $f_x(\hat{x})[Y(t)]$ *is nonsingular, where* $Y(t)$ *is the fundamental matrix solution of* $\frac{dy}{dt} = g_x(\hat{x}, t)\, y$.

Theorem 5.1. *Let* $\hat{x}(t)$ *be an isolated solution of* (1.1), (1.2). *Then, there is no other solution in a sufficiently small neighborhood of* $\hat{x}(t)$.

Proof. Let $Y(t)$ be as in definition 5.1, for this $Y(t)$ there exists M_1^* and M_2^*, $n \times n$ nonnegative matrices such that $\| H_1 \| \leqq M_1^*$ and $\| H_2 \| \leqq M_2^*$, where H_1 and H_2 are defined in lemma 3.1. Since $g_x(x, t)$ and $f_x(x)$ are continuous, there exists a positive vector r_1 such that for all $x(t) \in S(\hat{x}, r_1)$, $\| g_x(x, t) - g_x(\hat{x}, t) \| \leqq \leqq M_6$, $\| f_x(x) - f_x(\hat{x}) \| \leqq M_7$, where M_6 and M_7 are $n \times n$ positive matrices such that $\varrho(M_1^* M_6 + M_2^* M_7) < 1$.

Let $\hat{x}^*(t)$ be any other solution of (1.1), (1.2) in $S(\bar{x}, r_1)$. Then, for $x(t) = \hat{x}(t) - - \hat{x}^*(t)$, we find

$$\frac{dx}{dt} = g(\hat{x}(t), t) - g(\hat{x}^*(t), t) = \int_0^1 [g_x(\hat{x}(t) + \Theta_1 x(t), t)]\, x(t)\, d\Theta_1 \tag{5.1}$$

and

$$0 = f(\hat{x}) - f(\hat{x}^*) = \int_0^1 [f_x(\hat{x} + \Theta_2 x)][x]\, d\Theta_2. \tag{5.2}$$

From lemma 3.1, the system (5.1), (5.2) can be written as

$$x(t) = H_1[\int_0^1 [g_x(\hat{x}(t) + \Theta_1 x(t), t) - g_x(\hat{x}(t), t)]\, x(t)\, d\Theta_1] + \tag{5.3}$$

$$+ H_2[\int_0^1 - [f_x(\hat{x} + \Theta_2 x) - f_x(\hat{x})]\, [x]\, d\Theta_2].$$

Since $\hat{x}(t) + \Theta_i x(t) \in \bar{S}(\hat{x}, r_1)$, $i = 1, 2$; equation (5.3) provides

$$\| x \| \leqq (M_1^* M_6 + M_2^* M_7) \| x \|$$

and from $\varrho(M_1^* M_6 + M_2^* M_7) < 1$, we get

$$\| x \| \leqq 0$$

which is a contradiction and hence $\hat{x}(t) \equiv \hat{x}^*(t)$.

Theorem 5.2. *The solution* $x^*(t)$ *of* (1.1), (1.2) *obtained in theorem* 4.1 *is isolated solution.*

Proof. If not, then there exists a nonzero vector p such that $f_x(x^*)\, [Y(t)]\, p =$ where $Y(t)$ is the fundamental matrix solution of $\frac{dy}{dt} = g_x(x^*, t)\, y$.

We define $z(t) = Y(t)\, p$, then

$$\frac{dz}{dt} = g_x(x^*, t)\, z(t), \tag{5.4}$$

$$f_x(x^*)\, [z] = 0. \tag{5.5}$$

From lemma 3.1, the system (5.4), (5.5) is equivalent to

$$z(t) = H_1[g_x(x^*(t), t)\, z(t) - A(t)\, z(t)] + H_2[L[z] \pm f_x(x^*)\, [z]].$$

Thus, from (ii)–(iv) of theorem 4.1

$$\| z \| \leqq (M_1 M_3 + M_2 M_4) \| z \| = K_0 \| z \|$$

or

$$\| z \| \leqq 0$$

which implies $z(t) \equiv 0$ or $Y(t)\, p \equiv 0$. Since $Y(t)$ is nonsingular we find p This contradiction proves that $x^*(t)$ is isolated.

6. Application to the Perturbation Method

Here, we shall consider the boundary value problem

$$\frac{dx}{dt} = g(x, t) + \lambda h(x, t, \lambda), \tag{6.1}$$

$$(6.2) \qquad f(x) + \lambda\, \mathrm{d}(x, \lambda) = 0$$

as the perturbed problem of (1.1), (1.2). In (6.1) and (6.2) λ is a small parameter such that $\lambda \in \Lambda = \{\lambda : |\lambda| \leqq \varrho\}$, $\varrho > 0$; $h(x, t, \lambda)$ is continuously differentiable with respect to x in $R^n \times I \times \Lambda$ and $h_x(x, t, \lambda)$ represents the Jacobian matrix of $h(x, t, \lambda)$ with respect to x; $\mathrm{d}(x, \lambda)$ is continuously differentiable in $C(I) \times \Lambda$ and $\mathrm{d}_x(x, \lambda)$ denotes the linear operator mapping $C(I) \times \Lambda$ into R^n.

Let $\hat{x}(t)$ be an isolated solution of (1.1), (1.2) and for $\lambda \neq 0$ we seek the approximate solution $\bar{x}(t)$ of (6.1), (6.2) of the form $\bar{x} = \hat{x} - \lambda u$. We substitute this in (6.1), (6.2) and neglect the terms higher than order one in λ and obtain

$$(6.3) \qquad \frac{\mathrm{d}u}{\mathrm{d}t} = g_x(\hat{x}, t)\, u - h(\hat{x}, t, 0),$$

$$(6.4) \qquad f_x(\hat{x})\,[u] = \mathrm{d}(\hat{x}, 0).$$

Since $\hat{x}(t)$ is isolated $G = f_x(\hat{x})\,[Y(t)]$ is nonsingular and from lemma 3.1, (6.3) (6.4) is equivalent to

$$(6.5) \qquad u(t) = H_1[-h(\hat{x}(t), t, 0)] + H_2[\mathrm{d}(\hat{x}, 0)]$$

and hence

$$x(t) = \hat{x}(t) - \lambda(H_1[-h(\hat{x}(t), t, 0)] + H_2[\mathrm{d}(\hat{x}, 0)]).$$

Next, for this approximate solution $\bar{x}(t)$ of (6.1), (6.2) we shall show that the conditions of theorem 4.1 are satisfied. For this, we take $A(t) = g_x(\hat{x}, t)$, $L = f_x(\hat{x})$ so that condition (i) is satisfied. As in the proof of theorem 5.1, we have M_1^* and M_2^* such that $\| H_1 \| \leqq M_1^*$, $\| H_2 \| \leqq M_2^*$ so condition (ii) is also satisfied.

Let δ_3 and δ_4 be nonnegative vectors such that $\| h(\hat{x}(t), t, 0) \| \leqq \delta_3$, $\| d(x, 0) \| \leqq \leqq \delta_4$. Then, from (6.5) it follows that

$$\| u(t) \| \leqq M_1^* \delta_3 + M_2^* \delta_4 = \delta_5 .$$

Next, let r_1 be the positive vector as in theorem 5.1, we choose r_2 a positive vector and λ so that

$$(6.6) \qquad r_2 + |\lambda|\, \delta_5 \leqq r_1 .$$

Let $x(t) \in S(\bar{x}, r_2)$, then we find

$$\| x - \hat{x} \| \leqq \| x - \bar{x} \| + \| \bar{x} - \hat{x} \| \leqq r_2 + |\lambda|\, \delta_5 \leqq r_1$$

and hence $S(\bar{x}, r_2) \subseteq S(\hat{x}, r_1)$. As in the proof of theorem 5.1 for all $x(t) \in S(\bar{x}, r_2)$, $\| g_x(x, t) - g_x(\hat{x}, t) \| \leqq M_6$, $\| f_x(x) - f_x(\hat{x}) \| \leqq M_7$. Further, $h_x(x, t, \lambda)$ and $\mathrm{d}_x(x, \lambda)$ are continuous, there exists $n \times n$ nonnegative matrices M_8 and M_9 such that for all $x(t) \in S(\hat{x}, r_1)$, $t \in I$ and $\lambda \in \Lambda$, $\| h_x(x, t, \lambda) \| \leqq M_8$ and $\| \mathrm{d}_x(x, \lambda) \| \leqq \leqq M_9$. Thus, for all $x(t) \in S(\bar{x}, r_2)$, $t \in I$, $\lambda \in \Lambda$ we have

$$\| g_x(x, t) + \lambda h_x(x, t, \lambda) - g_x(\hat{x}, t) \| \leqq M_6 + |\lambda|\, M_8$$

and

$$\| f_x(x) + \lambda \, d_x(x, \lambda) - f_x(\hat{x}) \| \leqq M_7 + |\lambda| M_9$$

so condition (iii) is also satisfied in $\bar{S}(\bar{x}, r_2)$. In condition (iv) we need $\varrho(K_0) < 1$, i.e.

$$\varrho(M_1^* M_6 + |\lambda| M_1^* M_8 + M_2^* M_7 + |\lambda| M_2^* M_9) = \varrho(K_0^\lambda) < 1.$$

(6.7)

In theorem 5.1, $\varrho(M_1^* M_6 + M_2^* M_7) < 1$, and hence (6.7) is satisfied provided

(6.8) $$|\lambda| < \frac{1 - \varrho(M_1^* M_6 + M_2^* M_7)}{\varrho(M_1^* M_8 + M_2^* M_9)}.$$

Next, we assume that for all $x(t) \in \bar{S}(\hat{x}, r_1)$, $t \in I$ and $\lambda \in \Lambda$, the following holds

$$\| h(x, t, \lambda) - h(x, t, 0) \| \leqq |\lambda| \delta_6$$

and

$$\| d(x, \lambda) - d(x, 0) \| \leqq |\lambda| \delta_7,$$

where δ_6 and δ_7 are nonnegative vectors.

An easy computation shows that

$$\frac{d\bar{x}}{dt} - g(\bar{x}, t) - \lambda h(\bar{x}, t, \lambda) = \lambda \int_0^1 [g_x(\hat{x} - \Theta_1 \lambda u, t) - g_x(\hat{x}, t)] \, u \, d\Theta_1 -$$

$$- \lambda [h(\hat{x} - \lambda u, t, \lambda) - h(\hat{x} - \lambda u, t, 0)] + \lambda^2 \int_0^1 h_x(\hat{x} - \Theta_2 \lambda u, t, 0) \, u \, d\Theta_2 .$$

Since $\hat{x} - \Theta_i u \in \bar{S}(\hat{x}, r_1)$, we find

(6.9) $$\left\| \frac{d\bar{x}}{dt} - g(\bar{x}, t) - \lambda h(\bar{x}, t, \lambda) \right\| \leqq |\lambda| M_6 \delta_5 + |\lambda|^2 \delta_6 + |\lambda|^2 M_8 \delta_5 .$$

Similarly, we obtain

$$\| f(\bar{x}) + \lambda \, d(\bar{x}, \lambda) \| = \| - \lambda \int_0^1 [f_x(\hat{x} - \Theta_1 \lambda u) - f_x(\hat{x})] \, u \, d\Theta_1 +$$

$$+ \lambda [d(\hat{x} - \lambda u, 0) - d(\hat{x} - \lambda u, 0) | - \lambda^2 \int_0^1 dx(\hat{x} - \Theta_2 \lambda u, 0) \, u \, d\Theta_2 \| \leqq$$

(6.10) $$\leqq |\lambda| M_7 \delta_5 + |\lambda|^2 \delta_7 + |\lambda|^2 M_8 \delta_5 .$$

If (6.8) is satisfied, we have $\varrho(K_0^\lambda) < 1$ and hence $(E - K_0^\lambda)^{-1}$ exists and nonnegative. Thus, the second part of condition (iv) i.e. $(E - K_0)^{-1}(M_1 \delta_1 + M_2 \delta_2) \leqq$ $\leqq r$ is satisfied provided

(6.11) $$r_0^* = |\lambda| (E - K_0^\lambda)^{-1} (K_0^\lambda \delta_5 + |\lambda| (M_1^* \delta_6 + M_2^* \delta_7)) \leqq r_2 .$$

Thus, we see that if $|\lambda| < \varrho$ and if (6.6), (6.8) and (6.11) are satisfied (which is always the case if $|\lambda|$ is sufficiently small) the conditions of theorem 4.1 for the system (6.1), (6.2) with this approximate solution $\bar{x}(t)$ are satisfied and hence, all the (1)–(5) conclusions of theorem 4.1 for this problem also follow.

If we further assume that for all $x(t) \in \bar{S}(\hat{x}, r_1)$ and $t \in I$, $\| g_x(x, t) - g_x(\hat{x}, t) \| \leqq \leqq C_1 \| x - \hat{x} \|$ and $\| f_x(x) - f_x(\hat{x}) \| \leqq C_2 \| x - \hat{x} \|$ where C_1 and C_2 are constant 3rd order tensor with nonnegative components, then the right side of (6.9) can be replaced by $| \lambda |^2 \left(\frac{1}{2} C_1 \delta_5 + \delta_5 \,.\, \delta_6 + M_8 \delta_5 \right)$ and of (6.10) by $| \lambda |^2 \left(\frac{1}{2} C_2 \delta_5 \,.\, \delta_5 + \delta_7 + M_9 \delta_5 \right)$.

With this replacement (6.11) takes the form

$$\dot{r}_0^{**} = | \lambda |^2 (E - K_0^{\lambda})^{-1} \left(\frac{1}{2} C_1 \delta_5 \,.\, \delta_5 + \delta_6 + M_8 \delta_5 + \frac{1}{2} C_2 \delta_5 \,.\, \delta_5 + \delta_7 + M_9 \delta_5 \right) \leqq r_2 .$$

Hence, if $x^*(t)$ is the solution of (6.1), (6.2), we find from inequality (4.5) that

$$\| x^*(t) - \bar{x}(t) \| \leqq r_0^{**}$$

i.e. the perturbation method produces an approximate solution within the error $0(\lambda^2)$.

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