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ASYMPTOTIC NATURE OF SOLUTIONS OF THE EQUATION $\dot{z} = f(t, z)$ WITH A COMPLEX VALUED FUNCTION f

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1. INTRODUCTION

In this paper we consider a differential equation

$$\dot{z} = G(t, z)\,[h(z) + g(t, z)], \tag{1}$$

where G is a real-valued function and h, g are complex-valued functions, t or z being a real or complex variable, respectively. The function h is assumed to be holomorphic in a simply connected region Ω containing zero, and to satisfy the condition $h(z) = 0 \Leftrightarrow z = 0$. The right hand side of (1) will be supposed to be in a certain meaning "close" to $h(z)$. Several authors investigated the asymptotic behaviour of the solutions of the equation (1), or of the equation (or of the corresponding system of two real equations) which is convertible into the equation of the form (1), on the condition that $h'(0) \neq 0$: [1–2], [4–8], [10–12]. However, there are only few papers, such as [3], [10], which are devoted to the differential equation convertible into the equation (1), where $h'(0) = 0$, $h''(0) \neq 0$. The aim of the present paper is to study the asymptotic nature of the solutions of (1) under the condition $h^{(n)}(0) \neq 0$, $h^{(j)}(0) = 0$ for $j = 1, \ldots, n - 1$, where $n \geqq 2$ is an integer. The technique of the proofs of the results is based on the Liapunov function method with the "Liapunov-like" function $W(z)$ defined in [9]. Although $W(z)$ does not satisfy all the conditions usually required for Liapunov functions, it is very helpful for the investigation of the asymptotic character of the solutions of (1).

Throughout the paper we use the following notation:

C	– Set of all complex numbers
N	– Set of all positive integers
I	– Interval $[t_0, \infty)$
$\bar{b}$	– Conjugate of a complex number b

Re b	– Real part of a complex number b
$C(\Gamma)$	– Class of all continuous real-valued functions defined on the set Γ
$\tilde{C}(\Gamma)$	– Class of all continuous complex-valued functions defined on the set Γ
Cl Γ	– Closure of a set $\Gamma \subset C$
Int Γ	– Interior of a Jordan curve $z = z(t)$, $t \in [\alpha, \beta]$ whose points z form a set $\Gamma \subset C$
k, $W(z)$	– see [9, pp. 66–67]
$\lambda_+, \lambda_-, \mathcal{T}^+, \mathcal{T}^-, \varphi$	– see [9, pp. 73–74].

Let $\mathcal{S}^+ \in \mathcal{T}^+/\varphi$ and $\mathcal{S}^- \in \mathcal{T}^-/\varphi$ be fixed. Then $\mathcal{S}^+ = \{\mathring{K}(\lambda) : 0 < \lambda < \lambda_+\}$, $\mathcal{S}^- = \{\mathring{K}(\lambda) : \lambda_- < \lambda < \infty\}$, where $\mathring{K}(\lambda)$ are the geometric images of Jordan curves such that: $0 \in \mathring{K}(\lambda)$, the equality $W(z) = \lambda$ holds for $z \in \mathring{K}(\lambda) - \{0\}$, and $\mathring{K}(\lambda_1) - \{0\} \subset \operatorname{Int} \mathring{K}(\lambda_2)$ for $0 < \lambda_1 < \lambda_2 < \lambda_+$ or $\mathring{K}(\lambda_2) - \{0\} \subset \operatorname{Int} \mathring{K}(\lambda_1)$ for $\lambda_- < \lambda_1 < \lambda_2 < \infty$. Define

$$K(\lambda_1, \lambda_2) = \bigcup_{\lambda_1 < \mu < \lambda_2} \mathring{K}(\mu) - \{0\} \qquad \text{for } 0 \leqq \lambda_1 < \lambda_2 \leqq \lambda_+$$

and

$$K(\lambda_1, \lambda_2) = \bigcup_{\lambda_2 < \mu < \lambda_1} \mathring{K}(\mu) - \{0\} \qquad \text{for } \lambda_- \leqq \lambda_2 < \lambda_1 \leqq \infty.$$

2. MAIN RESULTS

Assume $G \in C(I \times (\Omega - \{0\}))$, $g \in \tilde{C}(I \times (\Omega - \{0\}))$.

Theorem 1. *Let* $\delta \geqq 0$, $\vartheta \leqq \lambda_+$. *Suppose that*

(i) *for any* $\tau \geqq t_0$, *the initial value problem* (1), $z(\tau) = 0$, *possesses the unique solution* $z \equiv 0$;

(ii) *there exists a function* $E(t) \in C[t_0, \infty]$ *such that*

$$\sup_{t_0 \leqq s \leqq t < \infty} \int_s^t E(\xi)\, d\xi = \varkappa < \infty, \tag{2}$$

$$\delta\, e^{\varkappa} < \vartheta \tag{3}$$

and

$$G(t, z) \operatorname{Re}\left\{kh^{(n)}(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq E(t) \tag{4}$$

holds for $t \geqq t_0$, $z \in K(\delta, \vartheta)$.

If a solution $z(t)$ *of* (1) *satisfies*

$$z(t_1) \in \operatorname{Cl} \bar{K}(0, \gamma),$$

where $t_1 \geqq t_0$ *and* $0 < \gamma\, e^{\varkappa} < \vartheta$, *then*

$$z(t) \in \mathrm{Cl}\, K(0, \beta) \qquad \text{for } t \geqq t_1,$$

where $\beta = e^{\varkappa} \max [\gamma, \delta]$.

Proof. Put $\mathcal{M} = \{t \geqq t_1 : z(t) \in K(\delta, \vartheta)\}$. For $t \in \mathcal{M}$ we have

$$\dot{W}(z) = G(t, z)\, W(z) \operatorname{Re}\left\{k h^{(n)}(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\}, \tag{5}$$

where $z = z(t)$. Using (4) we get

$$\dot{W}(z(t)) \leqq E(t)\, W(z(t)) \qquad \text{for } t \in \mathcal{M}. \tag{6}$$

Suppose that there is a $t^* > t_1$ such that $z(t^*) \in K(\beta, \vartheta)$ and $z(t) \in K(0, \vartheta)$ for $t \in [t_1, t^*]$. Choose γ_1 so that $\beta < \gamma_1 e^{\varkappa} < W(z(t^*))$. Clearly $\delta < \gamma_1 < W(z(t^*))$, $\gamma_1 > \gamma$. Define $t_2 = \sup \{t \in [t_1, t^*] : z(t) \in \mathrm{Cl}\, K(\gamma_1)\}$. The inequality (6) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\{W(z(t)) \exp [-\int_{t_2}^{t} E(s)\, \mathrm{d}s]\} \leqq 0, \qquad t \in \mathcal{M}.$$

Integrating this inequality over $[t_2, t^*]$, we obtain

$$W(z(t^*)) \exp [-\int_{t_2}^{t^*} E(s)\, \mathrm{d}s] - W(z(t_2)) \leqq 0.$$

Using (2) and $W(z(t_2)) = \gamma_1$, we get

$$W(z(t^*)) \leqq \gamma_1 \exp [\int_{t_2}^{t^*} E(s)\, \mathrm{d}s] \leqq \gamma_1\, e^{\varkappa} < W(z(t^*)).$$

This contradiction implies

$$z(t) \in \mathrm{Cl}\, K(0, \beta) \qquad \text{for } t \geqq t_1.$$

Theorem 2. *Let* $\delta_j \geqq 0$, $\vartheta \leqq \lambda_+$, $s_j \in I$ *for* $j \in N$. *Suppose that the hypothesis* (i) *of Theorem* 1 *is fulfilled and there are functions* $E_j(t) \in C[t_0, \infty)$ *such that:*

(i) *for* $j \in N$, *the following conditions are satisfied:*

$$\int_{t_0}^{\infty} E_j(s)\, \mathrm{d}s = -\infty \qquad \text{whenever } j \geqq 2, \tag{7}$$

$$\sup_{s_j \leqq s \leqq t < \infty} \int_{s}^{t} E_j(\xi)\, \mathrm{d}\xi = \varkappa_j < \infty, \tag{8}$$

$$\delta_j\, e^{\varkappa_j} < \vartheta; \tag{9}$$

(ii) *the inequality*

$$G(t, z) \operatorname{Re}\left\{k h^{(n)}(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq E_j(t) \tag{10}$$

holds for $t \geqq s_j$, $z \in K(\delta_j, \vartheta)$, $j \in N$.

Denote

$$\delta = \inf_{j \in N} [\delta_j e^{\varkappa j}].$$

If a solution $z(t)$ *of* (1) *satisfies*

(11) $$z(t_1) \in K(0, \vartheta e^{-\varkappa_1}) \cup \{0\},$$

where $t_1 \geqq s_1$, *then to any* ε, $\delta < \varepsilon < \lambda_+$, *there is a* $T = T(\varepsilon, t_1) > 0$, *independent of the solution* $z(t)$, *such that*

$$z(t) \in K(0, \varepsilon) \cup \{0\}$$

for $t \geqq t_1 + T$.

Proof. Put $\mathcal{M}_j = \{t \geqq s_j : z(t) \in K(\delta_j, \vartheta)\}$. For $t \in \mathcal{M}_j$ we get (5), where $z = z(t)$. This relation together with (10) yields

(12) $$\dot{W}(z(t)) \leqq E_j(t)\, W(z(t)) \qquad \text{for } t \in \mathcal{M}_j.$$

By Theorem 1 we have $z(t) \in K(0, \vartheta) \cup \{0\}$ for $t \geqq t_1$. Let ε, $\delta < \varepsilon < \lambda_+$ be given. Without restriction we may suppose that $\varepsilon < \vartheta$. Choose a fixed integer $j \geqq 2$ such that

$$\delta_j e^{\varkappa j} < \varepsilon.$$

Put $\sigma = \max [s_j, t_1]$. Let $T = T(\varepsilon, t_1) > | s_j - s_1 |$ be such a number that

$$\int_\sigma^t E_j(s)\, ds < \ln \frac{\varepsilon}{2\vartheta}$$

for $t \geqq t_1 + T$. Clearly $t_1 + T > \sigma$.

We claim that $z(t) \in K(0, \varepsilon) \cup \{0\}$ for $t \geqq t_1 + T$. If it is not the case, there exists a $t^* \geqq t_1 + T$ for which

(13) $$z(t^*) \notin K(0, \varepsilon) \cup \{0\}.$$

Using Theorem 1 we obtain

$$z(t) \in K(\varepsilon e^{-\varkappa j}, \vartheta) \cup [\bar{K}(\varepsilon e^{-\varkappa j}) - \{0\}] \subset K(\delta_j, \vartheta)$$

for $t \in [\sigma, t^*]$. The inequality (12) is equivalent to

$$\frac{d}{dt} \{W(z(t)) \exp [-\int_\sigma^t E_j(s)\, ds]\} \leqq 0, \qquad t \in \mathcal{M}_j.$$

Integration over $[\sigma, t^*]$ yields

$$W(z(t^*)) \exp [-\int_\sigma^{t^*} E_j(s)\, ds] - W(z(\sigma)) \leqq 0.$$

Hence

$$W(z(t^*)) \leqq W(z(\sigma)) \exp [\int_\sigma^{t^*} E_j(s)\, ds] \leqq \vartheta \frac{\varepsilon}{2\vartheta} = \frac{\varepsilon}{2} < \varepsilon,$$

which contradicts (13) and proves $z(t) \in K(\varepsilon) \cup \{0\}$ for $t \geqq t_1 + T$.

Theorem 3. *Let the assumptions of Theorem 2 be fulfilled except* (7) *is replaced by*

$$(14) \qquad \int_s^{s+t} E_j(\xi)\,d\xi \to -\infty \qquad \textit{as } t \to \infty$$

uniformly for $s \in [s_j, \infty)$ *whenever* $j \geqq 2$.

If a solution $z(t)$ *of* (1) *satisfies* (11), *where* $t_1 \geqq s_1$, *then to any* ε, $\delta < \varepsilon < \lambda_+$, *there exists a* $T = T(\varepsilon) > 0$ *independent of* t_1 *and of the solution* $z(t)$ *such that*

$$z(t) \in K(0, \varepsilon) \cup \{0\}$$

for $t \geqq t_1 + T$.

Proof. Because of (14) for $j \geqq 2$ there exists a $T = T(\varepsilon, j) > |s_j - s_1| + s_j - t_0$ such that $t - t_1 \geqq T$ implies

$$\int_\sigma^{\sigma+(t-t_1-s_j+t_0)} E_j(\xi)\,d\xi < \ln\frac{\varepsilon}{2\vartheta} - \varkappa_j.$$

The statement follows now from the proof of Theorem 2, since

$$\int_\sigma^t E_j(\xi)\,d\xi = \int_\sigma^{\sigma+(t-t_1-s_j+t_0)} E_j(\xi)\,d\xi + \int_{\sigma+t-t_1-s_j+t_0}^t E_j(\xi)\,d\xi <$$

$$< \ln\frac{\varepsilon}{2\vartheta} - \varkappa_j + \varkappa_j = \ln\frac{\varepsilon}{2\vartheta}$$

and j depends only on ε.

Remark 1. Let $0 \leqq \delta_j < \vartheta \leqq \lambda_+$, $\sigma_j \geqq t_0$ and $\Theta_j < 0$ for $j \in N$. Denote

$$\delta = \liminf_{j\to\infty} \delta_j.$$

Assume that $g_1(t, z)$, $g_2(t, z) \in C(I \times (\Omega - \{0\}))$ and define $g(t, z) = g_1(t, z) + g_2(t, z)$. Suppose there are nonnegative functions $F_j(t) \in C[t_0, \infty)$ such that

$$(15) \qquad \lim_{t\to\infty} \int_t^{t+1} F_j(s)\,ds = 0 \qquad \text{for } j \in N$$

and the conditions

$$G(t, z)\,\mathrm{Re}\left[k g_1(t, z) \frac{h^{(n)}(0)}{h(z)}\right] \leqq F_j(t),$$

$$G(t, z)\,\mathrm{Re}\left\{k h^{(n)}(0)\left[1 + \frac{g_2(t, z)}{h(z)}\right]\right\} \leqq \Theta_j$$

hold for $t \geqq \sigma_j$, $z \in K(\delta_j, \vartheta)$, $j \in N$.

Then to any $\varepsilon > 0$ there exists a sequence $\{s_j\}$, $s_j \geqq \sigma_j$, such that the hypotheses (i), (ii) of Theorem 2 are fulfilled for $E_j(t) = \Theta_j + F_j(t)$, where $\varkappa_1 < \varepsilon$ and

$$\liminf_{j\to\infty} [\delta_j e^{\varkappa_j}] = \delta.$$

Moreover,

$$\int_s^{s+t} E_j(\xi)\,d\xi \to -\infty \qquad as\ t \to \infty$$

uniformly for $s \in [s_j, \infty)$.

Remark 2. Notice that the condition (15) is satisfied if $\lim_{t\to\infty} F_j(t) = 0$ for $j \in N$ or $\int_{t_0}^{\infty} F_j^{\alpha}(s)\,ds < \infty$ for $j \in N$, where $\alpha \geqq 1$.

Theorem 4. *Let* $0 < \gamma < \lambda_+$. *Suppose that the hypothesis* (i) *of Theorem* 1 *is fulfilled. Assume*

$$G(t, z)\,\mathrm{Re}\left\{kh^{(n)}(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} < 0 \tag{16}$$

for $t \geqq t_0$, $z \in \hat{K}(\gamma) - \{0\}$.

If a solution $z(t)$ *of* (1) *satisfies*

$$z(t_1) \in \mathrm{Cl}\,K(0, \gamma),$$

where $t_1 \geqq t_0$, *then* $z(t) \in K(0, \gamma) \cup \{0\}$ *for* $t > t_1$.

Proof. Put $\mathcal{M} = \{t \geqq t_1 : z(t) \in K(0, \lambda_+)\}$. For $t \in \mathcal{M}$ we get (5), where $z = z(t)$. If there is a $t_2 \geqq t_1$ such that $z(t_2) \in \hat{K}(\gamma) - \{0\}$, then (16) implies

$$\dot{W}(z(t_2)) < 0. \tag{17}$$

Suppose that there exists a $t^* > t_1$ for which $z(t^*) \notin K(0, \gamma) \cup \{0\}$. Define $t_3 = \inf\{t^* > t_1 : z(t^*) \notin K(0, \gamma) \cup \{0\}\}$. In view of (17) we have $t_3 > t_1$. Furthermore $z(t_3) \in \hat{K}(\gamma) - \{0\}$, and $z(t) \in K(0, \gamma)$ holds for $t \in (t_1, t_3)$. On the other hand, the condition (17) assures the existence of a $t_4 \in (t_1, t_3)$ such that $W(z(t_4)) > \gamma$. Thus our supposition is false and $z(t) \in K(0, \gamma) \cup \{0\}$ for $t > t_1$.

Theorem 5. *Assume* $\delta \geqq 0$, $\vartheta \leqq \lambda_+$. *Suppose that*

(i) *for any* $\tau \geqq t_0$, *the initial value problem* (1), $z(\tau) = 0$, *possesses the unique solution* $z \equiv 0$;

(ii) *there is an* $E(t) \in C[t_0, \infty)$ *such that the conditions* (2), (3) *are fulfilled and*

$$-G(t, z)\,\mathrm{Re}\left\{kh^{(n)}(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq E(t) \tag{18}$$

holds for $t \geqq t_0$, $z \in K(\delta, \vartheta)$.

If a solution $z(t)$ *of* (1) *satisfies* $z(t_1) \in \hat{K}(\gamma)$, *where* $t_1 \geqq t_0$ *and* $\delta\, e^{\varkappa} < \gamma < \vartheta$, *then* $z(t) \notin K(0, \gamma\, e^{-\varkappa})$ *for all* $t \geqq t_1$ *for which* $z(t)$ *is defined.*

Proof. In view of (5) and (18) we get

$$\dot{W}(z(t)) \geqq -E(t)\, W(z(t)) \tag{19}$$

for $t \in \mathcal{M} = \{t \geqq t_1 : z(t) \in K(\delta, \vartheta)\}$. Suppose there is a $t^* \geqq t_1$ such that $z(t^*) \in$

$\in K(\delta, \gamma\, e^{-\varkappa})$. Define $\sigma = \sup\{t \in [t_1, t^*] : z(t) \in \mathring{K}(\gamma)\}$. Without loss of generality it may be assumed that $z(t) \in K(\delta, \vartheta)$ for $t \in (\sigma, t^*]$.

The inequality (19) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\{W(z(t)) \exp [\int_\sigma^t E(s)\,\mathrm{d}s]\} \geqq 0.$$

Integration over $[\sigma, t^*]$ yields

$$W(z(t^*)) \exp [\int_\sigma^{t^*} E(s)\,\mathrm{d}s] - W(z(\sigma)) \geqq 0.$$

Using (2) and $W(z(\sigma)) = \gamma$ we obtain

$$W(z(t^*)) \geqq \gamma \exp [-\int_\sigma^{t^*} E(s)\,\mathrm{d}s] \geqq \gamma\, e^{-\varkappa} > W(z(t^*)).$$

This contradiction proves $z(t) \in K(0, \gamma\, e^{-\varkappa})$ for all $t \geqq t_1$ for which $z(t)$ is defined.

Theorem 6. *Let $\delta > 0$, $\vartheta_j \leqq \lambda_+$, $s_j \geqq t_0$ for $j \in N$. Assume that the hypothesis* (i) *of Theorem 5 is fulfilled and suppose there are functions $E_j(t) \in C[t_0, \infty)$ such that:*

(i) *for $j \in N$ the following conditions are satisfied:*

(7) $$\int_{t_0}^{\infty} E_j(s)\,\mathrm{d}s = -\infty \qquad \textit{whenever } j \geqq 2,$$

(8) $$\sup_{s_j \leqq s \leqq t < \infty} \int_s^t E_j(\xi)\,\mathrm{d}\xi = \varkappa_j < \infty,$$

(20) $$\delta\, e^{\varkappa_j} < \vartheta_j;$$

(ii) *the inequality*

(21) $$-G(t, z) \operatorname{Re}\left\{k h^{(n)}(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq E_j(t)$$

holds for $t \geqq s_j$, $z \in K(\delta, \vartheta_j)$, $j \in N$.

Denote

$$\vartheta = \sup_{j \in N} [\vartheta_j\, e^{-\varkappa_j}].$$

If a solution $z(t)$ of (1) *satisfies*

(22) $$z(t_1) \in K(\delta\, e^{\varkappa_1}, \lambda_+) \cup \{0\},$$

where $t_1 \geqq s_1$, then to any ε, $0 < \varepsilon < \vartheta$, there exists a $T = T(\varepsilon, t_1) > 0$, independent of $z(t)$, such that

$$z(t) \notin \mathrm{Cl}\, K(0, \varepsilon) - \{0\}$$

for all $t \geqq t_1 + T$ for which $z(t)$ is defined.

Proof. Because of (21) and (5) we obtain

$$\dot{W}(z(t)) \geqq -E_j(t)\, W(z(t)) \tag{23}$$

for $t \in \mathcal{M}_j = \{t \geqq s_j : z(t) \in K(\delta, \vartheta_j)\}$. From Theorem 5 it follows that

$$z(t) \notin \mathrm{Cl}\, K(0, \delta) - \{0\}$$

for all $t \geqq t_1$ for which $z(t)$ exists. Choose ε, $0 < \varepsilon < \vartheta$. Without loss of generality we may suppose that $\delta < \varepsilon$. Let $j \geqq 2$ be such a positive integer that

$$\varepsilon < \vartheta_j\, e^{-\varkappa j}.$$

Put $\sigma = \max\,[s_j, t_1]$. Choose $T = T(\varepsilon, t_1) > |\, s_j - s_1 \,|$ so that

$$\int_\sigma^t E_j(s)\, \mathrm{d}s < -\ln \frac{2\varepsilon}{\delta}$$

for $t \geqq t_1 + T$. Clearly $t_1 + T > \sigma$.

We claim that $z(t) \notin \mathrm{Cl}\, K(0, \varepsilon) - \{0\}$ holds for all $t \geqq t_1 + T$, for which $z(t)$ is defined. Suppose for the sake of argument that there is a $t^* \geqq t_1 + T$ such that

$$z(t^*) \in \mathrm{Cl}\, K(0, \varepsilon) - \{0\}. \tag{24}$$

Using Theorem 5, we get

$$z(t) \in \mathrm{Cl}\, K(\delta, \varepsilon\, e^{\varkappa j}) - \mathring{K}(\delta) \subset K(\delta, \vartheta_j)$$

for $t \in [\sigma, t^*]$. The inequality (23) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\{W(z(t)) \exp\,[\int_\sigma^t E_j(s)\, \mathrm{d}s]\} = 0.$$

Integrating over $[\sigma, t^*]$, we obtain

$$W(z(t^*)) \exp\,[\int_\sigma^{t^*} E_j(s)\, \mathrm{d}s] - W(z(\sigma)) \geqq 0.$$

Therefore

$$W(z(t^*)) = W(z(\sigma)) \exp\,[-\int_\sigma^{t^*} E_j(s)\, \mathrm{d}s] \geqq \delta \frac{2\varepsilon}{\delta} = 2\varepsilon > \varepsilon.$$

Since it contradicts (24), the proof is complete.

Theorem 7. *Let the assumptions of Theorem* 6 *be fulfilled except* (7) *is replaced by*

$$\int_s^{s+t} E_j(\xi)\, \mathrm{d}\xi \to -\infty \qquad \textit{as } t \to \infty$$

uniformly for $s \in [s_j, \infty)$ *whenever* $j \geqq 2$.

If a solution $z(t)$ of (1) *satisfies* (22), *where $t_1 \geqq s_1$, then to any ε, $0 < \varepsilon < \vartheta$, there is a $T = T(\varepsilon) > 0$, independent of t_1 and $z(t)$, such that*

$$z(t) \notin \operatorname{Cl} K(0, \varepsilon) - \{0\}$$

for all $t \geqq t_1 + T$ for which $z(t)$ is defined.

Proof. The proof is essentially the same as that of Theorem 6. In view of the proof of Theorem 3, T can be chosen independently of t_1.

Remark 3. Let $0 \leqq \delta < \vartheta_j \leqq \lambda_+$, $\sigma_j \geqq t_0$ and $\Theta_j < 0$ for $j \in N$. Denote

$$\vartheta = \limsup_{j \to \infty} \vartheta_j .$$

Assume that $g_1(t, z)$, $g_2(t, z) \in \tilde{C}(I \times (\Omega - \{0\}))$ and define $g(t, z) = g_1(t, z) + g_2(t, z)$. Suppose there are nonnegative functions $F_j(t) \in C[t_0, \infty)$ such that (15) and the conditions

$$-G(t, z) \operatorname{Re}\left[k g_1(t, z) \frac{h^{(n)}(0)}{h(z)} \right] \leqq F_j(t),$$

$$-G(t, z) \operatorname{Re}\left\{ k h^{(n)}(0) \left[1 + \frac{g_2(t, z)}{h(z)} \right] \right\} \leqq \Theta_j$$

hold for $t \geqq \sigma_j$, $z \in K(\delta, \vartheta_j)$, $j \in N$.

Then to any $\varepsilon > 0$ there exists a sequence $\{s_j\}$, $s_j \geqq \sigma_j$, such that the hypotheses (i), (ii) of Theorem 6 are fulfilled for $E_j(t) = \Theta_j + F_j(t)$, where $\varkappa_1 < \varepsilon$ and

$$\limsup_{j \to \infty} [\vartheta_j e^{-\varkappa_j}] = \vartheta .$$

Moreover,

$$\int_s^{s+t} E_j(\xi) \, d\xi \to -\infty \qquad \text{as } t \to \infty$$

uniformly for $s \in [s_j, \infty)$.

Theorem 8. *Let $0 < \gamma < \lambda_+$. Suppose that the hypothesis* (i) *of Theorem* 5 *is fulfilled. Assume*

$$G(t, z) \operatorname{Re}\left\{ k h^{(n)}(0) \left[1 + \frac{g(t, z)}{h(z)} \right] \right\} > 0$$

for $t \geqq t_0$, $z \in \hat{K}(\gamma) - \{0\}$.

If a solution $z(t)$ of (1) *satisfies*

$$z(t_1) \notin K(0, \gamma),$$

where $t_1 \geqq t_0$, then

$$z(t) \notin \operatorname{Cl} K(0, \gamma) - \{0\}$$

for all $t > t_1$ for which $z(t)$ is defined.

Proof. The proof is analogous to that of Theorem 4.

Theorems 1–8 describe the behaviour of the solutions of (1) on certain subsets of $K(0, \lambda_+) \cup \{0\}$. Analogously we can derive corresponding results (Theorems 1'–8') describing the asymptotic behaviour of the solutions of (1) on subsets of $K(\infty, \lambda_-) \cup \{0\}$. For clearness we formulate here the first of these theorems.

Theorem 1'. *Let $\delta \leqq \infty$, $\vartheta \geqq \lambda_-$. Suppose that*

(i) *for any $\tau \geqq t_0$, the initial value problem* (1), $z(\tau) = 0$, *possesses the unique solution $z \equiv 0$;*

(ii) *there exists a function $E(t) \in C[t_0, \infty)$ such that the conditions* (2) *and*

$$\vartheta e^{\varkappa} < \delta \tag{25}$$

are fulfilled, and (18) *holds for $t \geqq t_0$, $z \in K(\delta, \vartheta)$.*

If a solution $z(t)$ of (1) *satisfies*

$$z(t_1) \in \text{Cl}\, K(\infty, \gamma),$$

where $t_1 \geqq t_0$ and $\vartheta < \gamma e^{-\varkappa} < \infty$, then

$$z(t) \in \text{Cl}\, K(\infty, \beta) \qquad \textit{for } t \geqq t_1,$$

where $\beta = e^{-\varkappa} \min[\gamma, \delta]$.

3. AN EXAMPLE

Suppose $q(t, z) \in \tilde{C}(I \times C)$ and consider an equation

$$\dot{z} = z^2 q(t, z), \tag{26}$$

where $q(t, z)$ satisfies locally a Lipschitz condition with respect to z. Putting $G(t, z) \equiv 1$, $h(z) = b(z - a) z^2$, $g(t, z) = [q(t, z) + b(a - z)] z^2$, where $a, b \in C$, $a \neq 0 \neq b$, we can write (26) in the form

$$\dot{z} = G(t, z) [h(z) + g(t, z)]. \tag{1}$$

From [9, Example 2] we have $h'(z) = b(3z - 2a) z$, $h''(z) = 2b(3z - a)$, $n = 2$, $W(z) = |a|\,|z|\,|z - a|^{-1} \exp\{\text{Re}[-az^{-1}]\}$, $\lambda_+ = \lambda_- = |a|$, $k = a/2$. For $t \geqq t_0$, $z \notin \{0, a\}$, we get

$$G(t, z)\, \text{Re}\left\{kh^{(n)}(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} =$$

$$= \text{Re}\left\{\frac{a}{2}(-2ab)\left[1 + \frac{z^2[q(t, z) + (a - z) b]}{bz^2(z - a)}\right]\right\} \leqq$$

$$\leqq -\text{Re}(a^2 b) + \frac{|a|^2\, |q(t, z) + (a - z) b|}{|z - a|}.$$

Supposing that there is an $H(t) \in C(I)$ such that $| q(t, z) + (a - z) b | \leqq H(t) | z - a |$ for $t \geqq t_0$, $z \in C$, we obtain

$$G(t, z) \operatorname{Re} \left\{ k h^{(n)}(0) \left[1 + \frac{g(t, z)}{h(z)} \right] \right\} \leqq -\operatorname{Re}(a^2 b) + |a|^2 H(t).$$

Applying Theorem 1 and Theorem 2, we get the following assertion: *Let* $a, b \in C$, $H(t) \in C(I)$ *exist such that* $b \neq 0$,

(27) $$| q(t, z) + (a - z) b | \leqq H(t) | z - a | \qquad \textit{for } t \geqq t_0, z \in C,$$

and the function

(28) $$|a|^2 \int_s^t H(\xi) \, d\xi - \operatorname{Re}(a^2 b)(t - s) \qquad \textit{is upper bounded on } t_0 \leqq s \leqq t < \infty.$$

Then every solution $z(t)$ *of* (26) *satisfying*

(29) $$| z(t_1) | \, | z(t_1) - a |^{-1} \exp \{ \operatorname{Re} [-a z^{-1}(t_1)] \} = \omega < e^{-\varkappa},$$

where $t_1 \geqq t_0$ *and*

(30) $$\varkappa = \sup_{t_0 \leqq s \leqq t < \infty} \{ |a|^2 \int_s^t H(\xi) \, d\xi - \operatorname{Re}(a^2 b)(t - s) \},$$

is defined for all $t \geqq t_1$, *and*

$$| z(t) | \, | z(t) - a |^{-1} \exp \{ \operatorname{Re} [-a z^{-1}(t)] \} \leqq \omega \, e^{\varkappa} \qquad (< 1)$$

holds for $t \geqq t_1$. *If, in addition,*

(31) $$\lim_{t \to \infty} [|a|^2 \int_{t_0}^t H(\xi) \, d\xi - \operatorname{Re}(a^2 b) \, t] = -\infty,$$

then every solution $z(t)$ *of* (26) *satisfying* (29), *where* $t_1 \geqq t_0$ *and* $\varkappa$ *is defined by* (30), *fulfils the condition*

$$\lim_{t \to \infty} z(t) = 0.$$

Analogously, applying Theorems 5′ and 6′, we get the statement: *If there exist* $a, b \in C$, $H(t) \in C(I)$ *such that* $b \neq 0$ *and the conditions* (27), (28) *are fulfilled, then each solution* $z(t)$ *of* (26), *for which*

(32) $$| z(t_1) | \, | z(t_1) - a |^{-1} \exp \{ \operatorname{Re} [-a z^{-1}(t_1)] \} = \omega > 1,$$
$$\operatorname{Re} [\bar{a} z(t_1)] < 0,$$

where $t_1 \geqq t_0$ *and* $\varkappa$ *is defined by* (30), *satisfies the condition*

$$| z(t) | \, | z(t) - a |^{-1} \exp \{ \operatorname{Re} [-a z^{-1}(t)] \} \leqq \omega \, e^{\varkappa}$$

or all $t \geqq t_1$ *for which* $\operatorname{Re} [\bar{a} z(t)] < 0$. *If, in addition, the condition* (31) *is fulfilled,* *then to any solution $z(t)$ of* (26) *satisfying* (32), *where* $t_1 \geqq t_0$, *and to any* ε, $e^{\varkappa} < \varepsilon < \infty$ *there is a* $T > 0$ *such that*

$$|z(t)|\,|z(t) - a|^{-1} \exp\{\mathrm{Re}[-az^{-1}(t)]\} < \varepsilon$$

for all $t \geqq t_1 + T$ *for which* $\mathrm{Re}[\bar{a}z(t)] < 0$.

Notice that the conditions (28), (29) imply $a \neq 0$, $\mathrm{Re}(a^2b) \geqq 0$ and the condition (31) implies $\mathrm{Re}(a^2b) > 0$. Similarly, the conditions (32), (28) imply $a \neq 0$, $\mathrm{Re}(a^2b) \geqq 0$. It can be easily verified that the conditions (28) and (31) are fulfilled if $\mathrm{Re}(a^2b) > 0$ and there holds $\int_{t_0}^{\infty} H(s)\,ds < \infty$ or $\limsup_{t\to\infty} H(t) < |a|^{-2}\,\mathrm{Re}(a^2b)$.

Application of the rest of Theorems 1–8 and Theorems 1′–8′ yields further results describing the asymptotic behaviour of the solutions of (26).

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