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THE MINIMUM MODULUS OF A SUBHARMONIC FUNCTION

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1. Let $u(z)$, where $z = re^{i\Theta}$, be a subharmonic function, defined in the whole complex plane, and harmonic in a heighbourhood of $z = 0$. Hayman [2] proved the following result:

Lemma A. *Let $u(z)$ be subharmonic in the whole z-plane. For every $R > 0$, let $K(z, Re^{i\Theta})$ be the Poisson kernel defined by*

$$K(z, Re^{i\Theta}) = Re[(Re^{i\Theta} + z)/(Re^{i\Theta} - z)]$$

and let $G_R(z, t)$ be the Green's function for the disc $|z| < R$ with pole at t, that is,

$$G_R(z, t) = \log |(R^2 - z\bar{t})/R(z - t)|.$$

Then there exists a unique non-negative, additive set function $\mu(e)$ defined for all Borel sets e in the z-plane such that for any $R > 0$, $u(z)$ is represented by

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\Theta}) K(z, Re^{i\Theta}) \, d\Theta - \int_{|t|<R} G_R(z, t) \, d\mu(e_t), \tag{1.1}$$

for $|z| < R$.

Let $M(r) = \max_{|z|=r} [u(z)]$. Heins [4] defined the order ϱ of a subharmonic function $u(z)$ as

$$\limsup_{r \to \infty} \frac{\log M(r)}{\log r} = \varrho, \qquad (0 \leqq \varrho \leqq \infty). \tag{1.2}$$

Kennedy [5] proved the following representation theorem for subharmonic functions of finite order, which is a generalization of a result of Heins [4]:

Theorem A. *Let $u(z)$ be subharmonic in the whole z-plane and harmonic in a neighbourhood of $z = 0$, and of finite order ϱ. Let μ be the set function defined in Lemma A and let*

$$\mu^*(t) = \mu(|z| < t) \tag{1.3}$$

for all $t > 0$. Further let q be the least nonnegative integer such that

$$(1.4) \qquad \int_0^\infty t^{-(q+1)}\, d\mu^*(t) < \infty.$$

Then for all values of z,

$$(1.5) \qquad u(z) = Re\,[P(z)] + \lim_{R\to\infty} \int_{|\xi|<R} \log|E(z/\xi, q)|\, d\mu(e_\xi),$$

where $P(z)$ is a polynomial of degree ϱ at most and $E(x, q)$ is the Weierstrass factor defined by

$$E(x, q) = (1 - x)\exp\left[x + \frac{x^2}{2} + \dots + \frac{x^q}{q}\right].$$

It has been shown that such an integer q exists and $q \leqq \varrho \leqq q + 1$.

We define the type and lower type of $u(z)$ by

$$(1.6) \qquad \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{M(r)}{r^\varrho} = \begin{matrix}T\\ \tau\end{matrix}, \qquad \text{for } 0 < \varrho < \infty.$$

$u(z)$ is said to be of maximal, minimal or normal type according as $T = \infty$, $T = 0$ or $0 < T < \infty$.

In this paper, we shall obtain some estimates of the subharmonic function in the plane.

2. We write

$$(2.1) \qquad I(z) = \int_{|\xi|<\infty} \log|E(z/\xi, q)|\, d\mu(e_\xi),$$

where μ and q are same as defined in Theorem A. It can be easily seen that $I(z)$ is also subharmonic in the whole z-plane and $\mu^*(t)$ is a nonnegative, non-decreasing function of t. We can write (2.1) as

$$(2.2) \qquad I(z) = \int_0^\infty \log|E(z/t, q)|\, d\mu^*(t).$$

We now give an estimate of $I(z)$. We refer to the following result of Levin ([6], p. 11):

Lemma B. *For all $q > 0$ and all complex numbers z, we have*

$$(2.3) \qquad \log|E(z, q)| \leqq A_q \frac{|z|^{q+1}}{1 + |z|},$$

where $A_q = 3e(2 + \log q)$.

For $q = 0$, we have

$$(2.4) \qquad \log|E(z, 0)| \leqq \log(1 + |z|).$$

Now we prove

Theorem 1. *The function $I(z)$ defined by (2.2) satisfies the following inequality in the entire complex plane*

$$I(z) < k_q r^q \left[\int_0^r \frac{\mu^*(t)}{t^{q+1}} \, dt + r \int_r^\infty \frac{\mu^*(t)}{t^{q+2}} \, dt \right], \tag{2.5}$$

where $|z| = r$, $k_q = 3e(q + 1)(2 + \log q)$ *for* $q > 0$ *and* $k_0 = 1$.

Proof. First we consider the case when $q > 0$. From (2.3), we have for $|z| = r$,

$$\log |E(z/t, q)| < \frac{A_q r^{q+1}}{t^q(t + r)}.$$

Hence

$$\int_0^\infty \log |E(z/t, q)| \, d\mu^*(t) < A_q r^{q+1} \int_0^\infty \frac{d\mu^*(t)}{t^q(t + r)}.$$

It is known that $\mu^*(r) = 0(r^{q+1})$. Hence on integrating by parts the right hand side, we get

$$I(z) < A_q(q + 1) r^{q+1} \int_0^\infty \frac{\mu^*(t) \, dt}{t^{q+1}(t + r)} =$$

$$= A_q(q + 1) r^{q+1} \left\{ \int_0^r \frac{\mu^*(t) \, dt}{t^{q+1}(t + r)} + \int_r^\infty \frac{\mu^*(t) \, dt}{t^{q+1}(t + r)} \right\} <$$

$$< k_q r^q \left\{ \int_0^r \frac{\mu^*(t) \, dt}{t^{q+1}} + r \int_r^\infty \frac{\mu^*(t) \, dt}{t^{q+2}} \right\}.$$

Hence (2.5) follows for $q > 0$. Similarly, when $q = 0$, using (2.4) we get the corresponding inequality for $I(z)$.

We shall now obtain the estimate for the minimum of $u(z)$. We assume, without any loss of generality, that $u(0) = 0$. Hayman [3] proved the following result which is an analogue of the well known Bourtex–Carton Lemma for entire functions.

Lemma C. *Suppose that μ is a positive measure defined in the whole complex plane, vanishing outside a compact set and such that the measure 'n' of the whole plane satisfies the condition $0 < n < \infty$. Then we have*

$$\int \log |z - \xi| \, d\mu(e_\xi) \geqq n(\log \eta) \tag{2.6}$$

outside a set of circles the sum of whose radii is at most 32η.

The following result was proved by M. Essen et al. [1], but not in the present form. We give the proof for the sake of completeness. We shall be using Lemma C for this purpose.

Theorem 2. *Let $u(z)$ be subharmonic in the whole complex plane. Then for $|z| < R$,*

$$u(z) > -H(\eta)\, M(kR) \tag{2.7}$$

outside a set of circles the sum of whose radii is atmost $32\eta kR$ where $\eta > 0$, $k > 1$ and $H(\eta)$ denotes some function not depending on R.

Proof: For $1 < k_1 < k$, we have

$$u(z) = u_1(z) + u_2(z),$$

where

$$u_2(z) = \int_0^{R/k_1} \log \left| 1 - \frac{z}{t} \right| d\mu^*(t).$$

Now

$$u_2(z) = \int_0^{R/k_1} \log |z - t| \, d\mu^*(t) - \int_0^{R/k_1} (\log t)\, d\mu^*(t).$$

Using Lemma C and the fact that $(\log t)$ is an increasing function of t, we get

$$u_2(z) > \mu^*(R/k_1) \log (\eta R) - \mu^*(R/k_1) \log (R/k_1)$$

outside a set of circles the sum of whose radii is at most $32\eta R$. Now from (1.1), we get on putting $z = 0$,

$$\frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\Theta})\, d\Theta = u(0) + \int_{|\xi|<R} \frac{\mu(e_\xi)}{\xi} d\xi,$$

or

$$\frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\Theta})\, d\Theta = \int_0^R \frac{\mu^*(t)}{t} dt. \tag{2.8}$$

Let us write

$$N(r) = \int_0^r \frac{\mu^*(t)}{t} dt.$$

Then $N(r) \leqq M(r)$. Since $\mu^*(r)$ is non-decreasing, we have

$$\mu^*(R/k_1) \log k_1 = \mu^*(R/k_1) \int_{R/k_1}^R \frac{dt}{t} \leqq$$
$$\leqq \int_{R/k_1}^R \frac{\mu^*(t)}{t} dt \leqq N(R) \leqq M(R).$$

Hence

$$u_2(z) > -M(R) \frac{\log (1/\eta k_1)}{\log k_1} \tag{2.9}$$

outside the set of excluded circles. For sufficiently small values of η, we can choose k_2, $1 < k_1 < k_2 < k$, such that the circle $|z| = R/k_2$ avoids all the excluded circles. Hence (2.9) holds for $|z| = R/k_2$. We can choose the argument of $z/k_2 = Re^{i\theta}/k_2$ so that

$$u_1(z/k_2) = M_1(R/k_2),$$

where

$$M_1(r) = \max_{|z|=r} [u_1(z)].$$

Now

$$M_1(R/k_2) \leqq M(R/k_2) - u_2(Re^{i\theta}/k_2) < M(R) - \frac{M(R)\log(1/\eta k_1)}{\log k_1},$$

since $k_2 > 1$. By definition of $u_1(z)$ we have

$$\begin{aligned} u_1(z) &= \operatorname{Re}[P(z)] + \int_{R/k_1}^{\infty} \log |E(z/t, q)| \, d\mu^*(t) + \\ &+ \int_0^{R/k_1} \operatorname{Re}\left[\frac{z}{t} + \frac{z^2}{2t^2} + \dots + \frac{z^q}{qt^q}\right] d\mu^*(t) = \\ &= \operatorname{Re}[P(z)] + \operatorname{Re}\left[\sum_{m=1}^{q} \frac{z^m}{m} \int_0^{R/k_1} \frac{d\mu^*(t)}{t^m}\right] + \\ &+ \int_{R/k_1}^{\infty} \log |E(z/t, q)| \, d\mu^*(t). \end{aligned}$$

Hence $u_1(z)$ has no mass inside the circle $|z| = R/k_1$ and is therefore harmonic there. Applying the Caratheodory's inequality ([8], pp. 174–175), we have

$$u_1(z) > -\frac{2r}{(R/k_2) - r} M_1(R/k_2), \qquad |z| = r < R/k_2 < R/k_1. \tag{2.10}$$

Combining (2.9) and (2.10), we get

$$\begin{aligned} u(z) &> -\frac{2r}{(R/k_2) - r} M(R/k_2) - M(R) \frac{\log(1/\eta k_1)}{\log k_1} > \\ &> -\frac{2r}{(R/k_2) - r} M(R)\left[1 - \frac{\log(\eta k_1)}{\log k_1}\right] + M(R)\frac{\log(\eta k_1)}{\log k_1}. \end{aligned}$$

The coefficient of $-M(R)$ does not exceed

$$\frac{2k_2}{k - k_2} + \frac{(k + k_2)\log(1/\eta k_1)}{(k - k_2)\log k},$$

which is a suitable value of $H(\eta)$. Thus we have

$$u(z) > -H(\eta)\, M(R) \tag{2.11}$$

outside the set of excluded circles, or

$$u(z) > -H(\eta)\, M(kR) \tag{2.12}$$

outside a set of circles the sum of whose radii is atmost $32\eta kR$. Thus Theorem 2 is proved.

3. We now define the proximate order for a subharmonic function.

Definition. *A real valued function $\varrho(r)$ satisfying the following conditions is said to be a proximate order:*

(3.1) *$\varrho(r)$ is continuous, piecewise differentiable for $r > r_0 > 0$ except at isolated points where $\varrho'(r-0)$ and $\varrho'(r+0)$ exist.*

$$\lim_{r\to\infty} r\varrho'(r)\log r = 0. \tag{3.2}$$

$$\lim_{r\to\infty} \varrho(r) = \varrho, \qquad 0 < \varrho < \infty. \tag{3.3}$$

Further, if

$$\limsup_{r\to\infty} \frac{M(r)}{r^{\varrho(r)}} = \sigma_u, \qquad 0 < \sigma_u < \infty, \tag{3.4}$$

then $\varrho(r)$ is said to be a proximate order of the function $u(z)$. We call σ_u to be the type of $u(z)$ with respect to the proximate order $\varrho(r)$. The existence of proximate order for a subharmonic function $u(z)$ has been established in [7].

It is evident that every subharmonic function of finite order is of normal type with respect to its own proximate order. However, we can compare the growth of $M(r)$ with that of $r^{\varrho(r)}$ for any proximate order with the help of density function Δ of the mass distribution μ.

We define the Upper Density of the mass distribution μ as:

$$\limsup_{r\to\infty} \frac{\mu^*(r)}{r^{\varrho(r)}} = \Delta, \tag{3.5}$$

where $\varrho(r)$ is any proximate order satisfying the conditions (3.1) to (3.3).

Now for any $\varepsilon > 0$, we have for all $r > r_0(\varepsilon)$,

$$M(r) < (\sigma_u + \varepsilon)\, r^{\varrho(r)}.$$

From (2.8) we have

$$\mu^*(r) \leq \int_r^{er} \frac{\mu^*(t)}{t}\, dt \leq \int_0^{er} \frac{\mu^*(t)}{t}\, dt \leq M(er).$$

Hence we have $\mu^*(r) < (\sigma_u + \varepsilon)(er)^{\varrho(er)}$, $r > r_0(\varepsilon)$. Using inequality (2) of Levin ([6], p. 33), we have for $\eta > 0$,

$$\mu^*(r) < (\sigma_u + \varepsilon)\, e^{\varrho} r^{\varrho(r)}(1+\eta), \qquad \text{for } r > r_1(\varepsilon, \eta),$$

or,

$$\limsup_{r\to\infty} \frac{\mu^*(r)}{r^{\varrho(r)}} \leq e^{\varrho}\sigma_u. \tag{3.6}$$

For a subharmonic function of non-integral order, we can get a reverse estimate also. From Theorem A and (2.5), we have

$$M(r) < 0(r^q) + k_q r^q \left[\int_0^r \frac{\mu^*(t)}{t^{q+1}} dt + r \int_r^\infty \frac{\mu^*(t)}{t^{q+2}} dt \right].$$

By (3.5) we have

$$M(r) < 0(r^q) + k_q r^q (\Delta + \varepsilon) \left[\int_{r_0}^r \frac{t^{\varrho(t)}}{t^{q+1}} dt + r \int_r^\infty \frac{t^{\varrho(t)}}{t^{q+2}} dt \right].$$

Using (1.53) and (1.53′) of Levin [6], we have

$$M(r) < 0(r^q) + k_q r^q (\Delta + \varepsilon) \left[\frac{r^{\varrho(r)-q}}{\varrho - q} + \frac{r^{\varrho(r)-q}}{q + 1 - \varrho} + 0(r^{\varrho(r)-q}) \right],$$

since $q < \varrho < q + 1$. Hence we have

(3.7) $$\sigma_u = \limsup_{r\to\infty} \frac{M(r)}{r^{\varrho(r)}} \leqq A\Delta,$$

where A depends on ϱ and q only.

Combining (3.6) and (3.7), we get the following

Theorem 3. *Let $u(z)$ be subharmonic in the whole complex plane and of non-integral order ϱ. Let $\varrho(r)$ be a proximate order such that $\lim_{r\to\infty} \varrho(r) = \varrho$. Then $u(z)$ is of maximal, minimal or normal type with respect to $\varrho(r)$ according as Δ is infinite, zero or $0 < \Delta < \infty$.*

The corresponding result for functions of integral orders is not so straight forward. In this case, the knowledge of Δ alone is not sufficient to determine the growth of $M(r)$ with respect to $\varrho(r)$. We define a new constant δ_u for the subharmonic function $u(z)$ as follows. We represent $u(z)$ as

(3.8) $$u(z) = \mathrm{Re}\,[P(z)] + \int_{|\xi|<\infty} \log |E(z/\xi, q)|\, d\mu(e_\xi),$$

where $q \leqq \varrho$ and $P(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \ldots + \alpha_\varrho z^\varrho$.

We put

$$\delta_u(r) = \alpha_\varrho + \frac{1}{\varrho} \int_0^r \frac{d\mu^*(t)}{t^\varrho},$$

and

$$\delta_u = \lim_{r\to\infty} \frac{|\delta_u(r)|}{L(r)},$$

where

$$L(r) = r^{\varrho(r)-\varrho}$$

and we define

$$\nu_u = \max(\Delta, \delta_u).$$

Now we prove

Theorem 4. *Let $u(z)$ be a subharmonic function of integral order ϱ and let $\varrho(r)$ be a proximate order with* $\lim_{r\to\infty} \varrho(r) = \varrho$.

Then $u(z)$ is of maximal, minimal or normal type with respect to $\varrho(r)$ according as $v_u = \infty$, $v_u = 0$ or $0 < v_u < \infty$.

Proof: We define the subharmonic function

$$u_R(z) = \int_0^R \log |E(z/t, \varrho - 1)| \, d\mu^*(t) + \int_R^\infty \log |E(z/t, \varrho)| \, d\mu^*(t). \tag{3.9}$$

Then we can write

$$u(z) = \mathrm{Re}\,[P(z)] + u_R(z) + \int_0^R \mathrm{Re}\,(z^\varrho/\varrho t)\, d\mu^*(t),$$

or,

$$u(z) = \mathrm{Re}\,[P_{\varrho-1}(z)] + u_R(z) + \mathrm{Re}\,[\delta_u(R)\, z^\varrho], \tag{3.10}$$

where $P_{\varrho-1}(z)$ is a polynomial of degree atmost $\varrho - 1$. Let

$$M_R(r) = \max_{|z|=r} [u_R(z)]$$

and suppose that $\varrho > 1$. Then using (2.3) we get

$$M_R(R) < A_\varrho \left[R^\varrho \int_0^R \frac{d\mu^*(t)}{(t+R)\,t^\varrho} + R^{\varrho+1} \int_R^\infty \frac{d\mu^*(t)}{(t+R)\,t^\varrho} \right].$$

As in Theorem 1, integration by parts on the right hand side yields

$$M_R(R) < k_\varrho \left[R^{(\varrho-1)} \int_0^R \frac{\mu^*(t)}{t^\varrho}\, dt + R^{\varrho+1} \int_R^\infty \frac{d\mu^*(t)}{t^{\varrho+2}} \right]. \tag{3.11}$$

From (3.5) we have for any $\varepsilon > 0$ and $r > r_0(\varepsilon)$,

$$\mu^*(r) < (\Delta + \varepsilon)\, r^{\varrho(r)}.$$

Hence

$$M_R(R) < 0(R^{\varrho-1}) + k_\varrho R^{\varrho-1} \int_{r_0}^R (\Delta + \varepsilon)\, r^{\varrho(t)-\varrho}\, dt +$$
$$+ R^{\varrho+1} \int_R^\infty (\Delta + \varepsilon)\, t^{\varrho(t)-\varrho-2}\, dt.$$

Using again the relations (1.53) and (1.53′) of Levin, we have

$$M_R(R) < 2k_\varrho(\Delta + \varepsilon)\, R^{\varrho(R)} + 0(R^{\varrho(R)}) + 0(R^{(\varrho-1)}). \tag{3.12}$$

From (3.10) we have

$$M(R) < 0(R^{\varrho-1}) + 2k_\varrho(\Delta + \varepsilon)\, R^{\varrho(R)} + |\delta_u(R)|\, R^\varrho,$$

or,

$$\frac{M(R)}{R^{\varrho(R)}} < o(1) + 2k_\varrho(\Delta + \varepsilon) + \frac{|\delta_u(R)|}{L(R)},$$

or,

$$\sigma_u \leqq (1 + 2k_\varrho)\, \nu_u. \tag{3.13}$$

For the reverse inequality, we have from (3.6),

$$\Delta \leqq e^{\varrho}\sigma_u. \tag{3.14}$$

Let $\eta = (1/64)\, k$, $k > 1$. Then from Theorem 2, we have for $|z| \leqq r$,

$$u_R(z) > -H(\eta)\, M_R(R_1)$$

outside a set of circles the sum of whose radii is atmost $r/2$, $R_1 = kr$. We can find a number r_1 such that $r/2 < r_1 < r$ and the circle $|z| = r_1$ avoids all the excluded circles. Hence from the representation (3.10), we get

$$\mathrm{Re}\,[\delta_u(R)\, z^{\varrho}] < M(r_1) + H(\eta)\, M_R(R_1) + 0(R^{\varrho-1})$$

on the circumference of the circle $|z| = r_1$. We can choose the argument of z so that z^{ϱ} and $\delta_u(R)$ have arguments that cancel each other. Then we have

$$|\delta_u(R)| < \frac{M(r_1)}{r_1^{\varrho}} + \frac{H(\eta)\, M_R(R_1)}{r_1^{\varrho}} + 0(R^{-1}). \tag{3.15}$$

Dividing by $L(R)$ and proceeding to limits, we get

$$\delta_u \leqq c\sigma_u,$$

where c is some constant. Combining the above inequality with (3.14), we get

$$\sigma_u \geqq c_1\nu_u. \tag{3.16}$$

Theorem 4 now follows in view of (3.13) and (3.16).

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