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THE GROUP OF DIVISIBILITY OF $\hat{Z}$

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1. In [2] we investigated a notion of a topological group of divisibility of a GCD-domain in the following way: Let (K, T) be a topological field and let A be a GCD-domain in K with K as a quotient field. Suppose that the group $U(A)$ of units of A is closed in a multiplicative group $(K^*, ., T \mid K^*)$. Then the factor topological group $G(A) = K^*/U(A)$ is called a *topological group of divisibility* of A, in symbol $G(A) = (K, T, A)$, if $G(A)$ is a topological lattice. More generaly, for a topological lattice-ordered group G we set $G = (K, T, A)$ if K is a topological field with a topology T, A is a Bezout domain with the quotient field K, the group of units $U(A)$ of A is closed in K^*, and the topological factor group $K^*/U(A)$ is a tl-group which is tl-isomorphic to G. In this case ve say that G *has a representation.* Let us recall that a tl-group is a triple $(G, \leqq, F)$ where G is a group, $\leqq$ is a partial order, and F is a topology on the underlying set (G) of G such that $(G, \leqq)$ is an l-group, (G, F) is a topological group, and $(|G|, \leqq, F)$ is a topological lattice. Moreover, we say that two tl-groups are tl-isomorphic if there is a homeomorphism between them which is both a lattice and group isomorphism.

In [2] we have observed that there are tl-groups which have no representation. On the other hand, it is possible to construct examples of tl-groups with a representation. The example of a tl-group (G, F) we consider here is not a complete space and hence we may construct the completion $(\hat{G}, \hat{F})$ of (G, F). It is well known that $(\hat{G}, \hat{F})$ is a tl-group and the natural question arising here is whether $(\hat{G}, \hat{F})$ admits a representation. To tell the truth we cannot solve this question as stated here. On the other hand if we somewhat modify the notion of a representation we are able to answer affirmatively this question. To do it, we say that a tl-group (G, F) admits a *general representation* (K, T, A) (in symbol $(G, F) \sim = (K, T, A)$), if K is a ring (commutative) with possible zero divisors, A is a subring in K such that K is a total quotient ring of A, T is a ring topology on K such that $(U(K), T \mid U(K))$ is a topological group with $U(A)$ as a closed subgroup and the factor topological group $G(A) = U(K)/U(A)$ is a tl-group (with ordering defined by $(U(K)/U(A))_+ = A^*/U(A)$, where A^* is the set of regular elements of A, therefore, a po-group $G(A)$ is a *value group* of A in the sense of [4]) which is tl-isomorphic to G. In a sequel we use a method of non-standard analysis introduced by A. ROBINSON [5] and, especially, we employ a variant of nonstandard analysis introduced by E. ZAKON [7] since it requires only rudiments of first order logic.

2. The groups of divisibility we are dealing with are of the form $\boldsymbol{Z}^{(I)}$, where I is a subset of the set N of integers. Clearly, every such a group is a group of divisibility of a domain $A_I = \bigcap_{i \in I} R_{w_i} \subset \boldsymbol{Q}$, where $\boldsymbol{Q}$ is the field of rationals and w_i is the p_i-adic valuation on $\boldsymbol{Q}$. Let $\boldsymbol{F}$ be the topology on $\boldsymbol{Z}^{(I)}$ with a subbase of neighbourhoods of zero consisting of prime l-ideals

$$H_i = \{\alpha \in \boldsymbol{Z}^{(I)} : \alpha_i = 0\}, \qquad i \in I.$$

Then clearly $(\boldsymbol{Z}^{(I)}, \boldsymbol{F})$ is a tl-group (see [6]) and if card $I = \aleph_0$, then $\boldsymbol{F}$ is a non-discrete topology. If we denote by $\boldsymbol{T}_{w_i}$ the field topology on $\boldsymbol{Q}$ defined by w_i with a subbase of neighbourhoods of zero consisting of the sets $U_{w_i,a} = \{x \in \boldsymbol{Q} : w_i(x) > a\}$, $a \in N$, we obtain the following proposition.

Proposition 1. $(\boldsymbol{Z}^{(I)}, \boldsymbol{F}) = (\boldsymbol{Q}, \sup \{\boldsymbol{T}_{w_i} : i \in I\}, A_I)$.

Proof. At first we observe that $U(A)$ is closed in $\boldsymbol{Q}$, since $U(R_{w_i})$ is closed for every $i \in I$. Let

$$\varphi : G(A_I) = \boldsymbol{Q}^*/U(A_I) \to \boldsymbol{Z}^{(I)}$$

be defined such that $\varphi(w(x))(i) = \varphi(xU(A_I))(i) = w_i(x)$, $i \in I$. Clearly, φ is an o-isomorphism. By [2], Lemma 1, to prove the proposition it remains to show that φ is open and continuous. We have

$$\varphi^{-1}(H_i) = w(U(R_{w_i})) = U(R_{w_i})/U(A_i),$$

and it is an open neighbourhood of zero in $G(A_I)$ since $U(R_{w_i}) = w_i^{-1}(0)$ is open in $(\boldsymbol{Q}, \boldsymbol{T}_I)$ for $\boldsymbol{T}_I = \sup \{\boldsymbol{T}_{wi} : i \in I\}$. On the other hand

$$\varphi(U_{w_i,a}/U(A_I)) = \{\alpha \in \boldsymbol{Z}^{(I)} : \alpha_i > a\} \qquad (= B)$$

as follows using the approximation theorem for valuations in $\boldsymbol{Q}$. Since for every $\alpha \in B$ we have $\alpha + H_i \subset B$, B is open in $\boldsymbol{F}$ and, therefore, φ is a homeomorphism.

Now, let $(\hat{\boldsymbol{Q}}, \hat{\boldsymbol{T}}_I)$ be the completion of $(\boldsymbol{Q}, \boldsymbol{T}_I)$ and let $\hat{A}_I$ be the closure of A_I in $\hat{\boldsymbol{Q}}_I$. It is well known that $\hat{\boldsymbol{Q}}_I$ has zero divisors, so that $(\hat{\boldsymbol{Q}}_I, \hat{\boldsymbol{T}}_I, \hat{A}_I)$ cannot be a representation of any tl-group. On the other hand, it may be a general representation and, in fact, we shall prove the following main result for $G = \boldsymbol{Z}^{(I)}$:

Theorem 2. $(\hat{G}, \hat{\boldsymbol{F}}) \sim (\hat{\boldsymbol{Q}}_I, \hat{\boldsymbol{T}}_I, \hat{A}_I)$.

The proof of this theorem will be a consequence of several independent propositions which describe structures of $\hat{\boldsymbol{Q}}_I$ and $\hat{G}$, respectively. As we have mentioned above, for an investigation of algebraic properties of $\hat{\boldsymbol{Q}}_I$ and $\hat{G}$ we use a method which is based on a notion of an enlargement from the tools of nonstandard analysis. Included solely for the convenience of the reader, we introduced the basic facts about enlargements.

For any set $A = A_0$ of individuals, the *superstructure* on A is the set $\mathscr{A} = \bigcup_{n \in N} A_n$, where A_{n+1} is the set of all subsets of $A_0 \cup A_n$. The first ordèr language $\mathscr{L}$ we need is a simple modification of a clasical one, namely, we assume that all constants of $\mathscr{L}$ are in $1-1$ correspondence with elements of $\mathscr{A}$ and identify the constants with the corresponding elements. Well-formed formulae (WFF) and sentences (WFS) are defined as usual with the restriction that all quantifiers must have form $(\forall x \in C)$ or $(\exists x \in C)$ with C a constant (i.e. $C \in \mathscr{A}$). Now, let A, B be two sets with superstructures $\mathscr{A}$, $\mathscr{B}$, respectively, and let

$$* : \mathscr{A} \to \mathscr{B}$$

be a map of $\mathscr{A}$ into $\mathscr{B}$. We write $^*C$ for $*(C)$. Let $^*\mathscr{A} = \bigcup_{n \in N} {}^*A_n$ (since $A_n \in \mathscr{A}$). Given a WFF α, we denote by $^*\alpha$ the formulae obtained from α by replaceing in it each constant $C \in \mathscr{A}$ by $^*C$. Elements of the form $^*C$ $(C \in \mathscr{A})$ are called *standard*, their elements are called *internal*. A $1-1$ map $* : \mathscr{A} \to \mathscr{B}$ is then called a *strict monomorphism* if

(1) $^*\emptyset = \emptyset$,

(2) for every $y \in {}^*\mathscr{A}$, $y \subseteq {}^*\mathscr{A}$ holds,

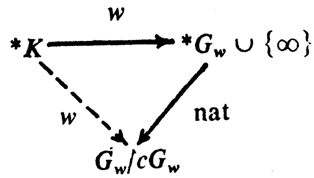
(3) for every WFS α, $\mathscr{A} \models \alpha$ iff $\mathscr{B} \models {}^*\alpha$. A binary relation R in $\mathscr{A}$ is said to be *concurrent* if, for any finite number of elements $a_1, \ldots, a_m \in D_1(R) = \{x : (\exists y)\,(x, y) \in R\}$, there exists b such that $(a_k, b) \in R$ for $k = 1, \ldots, m$. Then a strict monomorphism $* : \mathscr{A} \to \mathscr{B}$ is called *enlarging* and $^*\mathscr{A}$ an *enlargement* of $\mathscr{A}$, if, for each concurrent relation R in $\mathscr{A}$ there is some $b \in {}^*\mathscr{A}$ such that $({}^*a, b) \in {}^*R$ for all $a \in D_1(R)$, simultaneously.

If $^*\mathscr{A}$ is an enlargement of $\mathscr{A}$, where $\mathscr{A}$ is a superstructure on A, we say frequently that $^*A$ $(\in {}^*\mathscr{A})$ is an *enlargement* of A $(\in \mathscr{A})$. For any $X \subseteq A$ we may consider X as a subset of $^*X$ and, furthermore, for any binary relation $R \subseteq X \times Y$, $X, Y \subseteq A$, we have $R \subseteq {}^*R$.

Now, let K be a field with a topology $T = \sup\,(T_w : w \in \Omega)$ and let $\mathscr{K}$ be the superstructure on $K_0 = K \cup \Omega \cup \bigcup_{w \in \Omega} G_w$, $\mathscr{K} = \bigcup_{n \in N} K_n$, and let $^*\mathscr{K}$ be an enlargement of $\mathscr{K}$. Using the property (3), it can be proved that $^*K$ $(\in {}^*\mathscr{K})$ is a field, $K \subset {}^*K$ is a subfield and $^*w$ $(w \in \Omega \in \mathscr{K})$ is a valuation on $^*K$ with a value group $^*G_w$ such that the diagram

$$\begin{array}{ccc} {}^*K & \xrightarrow{\;{}^*w\;} & {}^*G_w \cup \{\infty\} \\ \uparrow & & \uparrow \\ K & \xrightarrow{\;w\;} & G_w \cup \{\infty\} \end{array}$$

comutates. Let cG_w be the convex closure of G_w in $^*G_w$ and let w be a valuation on $^*K$ completing the diagram



Let M_w be the maximal ideal of R_w and let

$$M = \bigcap_{w \in \Omega} M_w.$$

Then M is a subgroup of $(^*K, +)$ and on the factor group $^*K/M$ we may define a topology such that a subbase B of neighbourhoods of 0 consists of the sets

$$U^*_{w,\alpha}/M = {}^*(U_{w,\alpha})/M = \{x + M \in {}^*K/M : {}^*w(x) > \alpha\},$$

where $w \in \Omega$, $\alpha \in G_w^+$. Clearly, $^*K/M$ is then a topological group.

Now, under the injection $x \mapsto x + M$, $x \in K$, we may identify K with a subgroup in $^*K/M$. Let $\bar{K}$ be the closure of K in $^*K/M$. In [3] we have proved that $\bar{K}$ is homeomorphic with the completion $\hat{K}$ of K.

Let $\mathscr{K}$ be now the superstructure on the field $K = \boldsymbol{Q}$. Let P_I (Ω_I) be the set of all i-th prime numbers p_i (p_i – adic valuations on $\boldsymbol{Q}$) for $i \in I$ and let μ be a WFS such that

$$\mu = (\forall p \in N)(\forall x \in N)(\forall y \in N)(p \in P_I \Rightarrow (p \neq 1 \wedge (p = x \,.\, y \Rightarrow x = 1 \vee y = 1)).$$

Then $\mathscr{K} \models \mu$ states that P_I is a set of prime numbers in N. Since $^*\mathscr{K} \models {}^*\mu$, the set $^*P_I$ is a set of „prime numbers„ in $^*N$. Analogously, let γ be a WFS which states that for every $p \in P_I$ there exists a p-adic valuation w_p in $\boldsymbol{Q}$ with a value group $\boldsymbol{Z}$. Since $\mathscr{K} \models \gamma$, we have $^*\mathscr{K} \models {}^*\gamma$ and it follows that for every $p \in {}^*P_I$ there exists a "p-adic" valuation w_p in a field $^*\boldsymbol{Q}$ with a value group $^*\boldsymbol{Z}$. Clearly, for $p \in P_I \subset {}^*P_I$ we have $w_p = {}^*w_p$.

The following proposition describes fully the set $U(\hat{\boldsymbol{Q}}_I)$. The elements of $\hat{\boldsymbol{Q}}_I$ we denote by $\boldsymbol{x}$ $(= x + M)$, where $x \in {}^*\boldsymbol{Q}$.

Proposition 3. *Let $x \in {}^*\boldsymbol{Q}$. Then $\boldsymbol{x} \in U(\hat{\boldsymbol{Q}}_I)$ if and only if $^*w_i(x) \in \boldsymbol{Z}$ for each $i \in I$.*

Proof. Let $\boldsymbol{x} \in U(\hat{\boldsymbol{Q}}_I)$. If there exists $i \in I$ such that $^*w_i(x) = \omega$ for some $\omega \in {}^*N - N$, then for $\boldsymbol{y} \in \hat{\boldsymbol{Q}}_I$ such that $\boldsymbol{x} \,.\, \boldsymbol{y} = 1$ we have $^*w_i(x \,.\, y - 1) \in {}^*N - N$ and it follows $^*w_i(y) = -\omega$. Then for any $z \in \boldsymbol{Q}$ we have

$$-\omega = {}^*w_i(y - z) < n, \qquad \forall n \in N,$$

and $\boldsymbol{y} \notin \hat{\boldsymbol{Q}}_I$, a contradiction.

Conversely, without loss of generality we may suppose that $^*w_i(x) = -a_i$ for each $i \in I$ and $a_i \in N$. Then according to [3], Prop. 3.5, $\tilde{w}_i(\boldsymbol{z}) = {}^*w_i(z) \geqq 0$,

$i \in I$, where $z = x^{-1}$ and $\tilde{w}_i$ is the continuous extension of a valuation w_i onto a (Manis) valuation in a ring $\hat{Q}_I$. By [1], Prop. 6 and Lemma 11, we obtain $\hat{A}_I = \bigcap_{i \in I} R_{\tilde{w}_i}$ and it follows $z \in \hat{A}_I \subseteq \hat{Q}_I$. Hence, for every pair $(i, a) \in I \times N$ there exists $y_{i,a} \in A_I$ such that

$$*w_i(z - y_{i,a}) > a + 2a_i.$$

Since $*w_i(z) = a_i < a + 2a_i$, we have $y_{i,a} \neq 0$ and $y_{i,a}^{-1} \in Q$,

$$a_i = *w_i(z) = w_i(y_{i,a}).$$

Then we obtain

$$*w_i(x - y_{i,a}^{-1}) = *w_i(x(y_{i,a} - z)\, y_{i,a}^{-1}) > -a_i + a + 2a_i - a_i = a.$$

Therefore, we have proved

$$\forall (i, a) \in I \times N \, \exists z_{i,a} \in Q \text{ such that } *w_i(x - z_{i,a}) > a.$$

Now, let $i_1, \dots, i_m \in I$, $a_1, \dots, a_m \in N$. Using the approximation theorem for valuations in Q we may find an element $y \in Q$ such that

$$w_{i_t}(y - z_{i_t, a_t}) > a_t, \qquad t = 1, \dots, m.$$

Hence,

$$*w_{i_t}(x - z) = *w_{i_t}(x - z_{i_t,a_t} + z_{i_t,a_t} - y) > a_t, \qquad 1 \leqq t \leqq m,$$

and it follows $x \in \hat{Q}_I$. Clearly, $x \,.\, z = 1$ in $\hat{Q}_I$ and $x \in U(\hat{Q}_I)$.

To show that $(\hat{Q}_I, \hat{T}_I, \hat{A}_I)$ is a general representation we have to prove that $U(\hat{Q}_I)$ with induced topology is a topological group (and not only a topological semigroup).

Proposition 4. *$(U(\hat{Q}_I), \cdot, \hat{T}_I \mid U(\hat{Q}_I))$ is a topological group.*

Proof. We show that a map $x \mapsto x^{-1}$ is continuous. In fact, let $U = (1 + \bigcap_{t=1}^{n} U_{\tilde{w}_{i_t,a_t}}) \cap U(\hat{Q}_I)$ be an arbitrary neighbourhood of 1. Since (Q, T_I) is a topological field, there exists a neighbourhood $V = (1 + \bigcap_{s=1}^{m} U_{w_{j_s,b_s}}) \cap Q^*$ of 1 in Q such that

$$V^{-1} \subseteq 1 + \bigcap_{t=1}^{n} U_{w_{i_t,a_t}} = U.$$

Let $z \in V = (1 + \bigcap_{s=1}^{m} U_{\tilde{w}_{j_s,b_s}}) \cap U(\hat{Q}_I)$. Then by Prop. 3, $*w_i(z) \in Z$ for every $i \in I$ and $*w_{j_s}(z - 1) > b_s$, $s = 1, \dots, m$. Without loss of generality we may assume that $\{w_{j_1}, \dots, w_{j_m}\} \cap \{w_{i_1}, \dots, w_{i_n}\} = \emptyset$. Since $z \in \hat{Q}_I$, there exists $x \in Q$ such that

$$*w_{j_s}(z - x) > b_s, \qquad s = 1, \dots, m,$$
$$*w_{i_t}(z - x) > \max(a_t + 2w_{i_t}(z), w_{i_t}(z)), \qquad t = 1, \dots, n.$$

Since $w_{j_s}(x - 1) = {}^*w_{j_s}(x - z + z - 1) > b_s$, we have $x \in V$ and $x^{-1} \in U$. Then it is easy to see that

$$^*w_{i_t}(z^{-1} - 1) = {}^*w_{i_t}(z^{-1} - x^{-1} + x^{-1} - 1) > a_t, \qquad 1 \leqq t \leqq n,$$

and $V^{-1} \subseteq U$.

Now, the same method of enlargement we may use for investigation of properties of the completion $\hat{G}$ of $(Z^{(I)}, F)$. As in a case of topological fields we may do it in a more general way.

To do it, let G be a tl-group with a subbase $\mathscr{H}$ of zero consisting of prime l-ideals, $\mathscr{H} = \{H_i : i \in J\}$. Let $\mathscr{G}$ be a superstructure on the set $G_0 = G \cup J$ and let $^*\mathscr{G}$ be an enlargement of $\mathscr{G}$. Let

$$H = \bigcap_{j \in J} {}^*H_j .$$

Then H is an o-ideal of $^*G$ and in a group $^*G$ we may define a topology in such a way that $\{^*H_j : j \in J\}$ is a subbase of neighbourhoods of zero. Clearly, $^*G$ is a tl-group. Since H is closed l-ideal of $^*G$, we may consider a factor tl-group $^*G/H$. Then the canonical map $G \to {}^*G/H$ is an injection as follows from the fact $^*H_j \cap G = H_j$, j $\in J$. Then the following proposition holds.

Proposition 5. *The closure cG of G in *G/H is tl-isomorphic with the completion $\hat{G}$ of G.*

Proof. At first, $\hat{G}$ may be considered to be the factor set of the set of all Cauchy filters in G. Elements of this factor set will be denoted by $\bar{\alpha}$, their elements (i.e. Cauchy filters) by $\underline{\alpha}$, $\underline{\beta}$, etc. Then $\underline{\alpha}, \underline{\beta} \in \bar{\gamma}$ iff $\underline{\alpha} \cap \underline{\beta}$ is a Cauchy filter in G. The base of neighbourhoods of 0 in $\hat{G}$ consists of the sets

$$[\bigcap_{i=1}^{n} H_i] = \{\bar{\alpha} ; \bigcap_{i=1}^{n} H_i \in \underline{\alpha}\}, \qquad H_i \in \mathscr{H}.$$

The operations in $\hat{G}$ are defined as follows:

$$\bar{\alpha} + \bar{\beta} = \bar{\gamma} \quad \text{iff } \bar{\gamma} \text{ is a filter with a base } \underline{\alpha} + \underline{\beta},$$
$$\bar{\alpha} \wedge \bar{\beta} = \bar{\gamma} \quad \text{iff } \bar{\gamma} \text{ is a filter with a base } \alpha \wedge \underline{\beta}.$$

Let $\bar{\alpha} \in \hat{G}$. We define a binary relation R in $\mathscr{G}$ as follows:

$$(X, Y) \in R \qquad \text{iff } X, Y \in \underline{\alpha}, X \subseteq Y.$$

Then R is a concurrent relation and there exists $X \in {}^*\underline{\alpha}$ such that $X \subseteq {}^*Y$ for all $Y \in$. An element X with this property will be called an *infinitesimal element* of $^*\underline{\alpha}$. Let $\alpha \in X$. Then we define a map $\varrho : \hat{G} \to cG$,

$$\varrho(\bar{\alpha}) = \alpha + H.$$

This definition is correct. In fact, let $\beta \in X$ and let $i \in J$. Since $\underline{\alpha}$ is a Cauchy filter, there exists $Y \in \alpha$ such that $Y - Y \subseteq H_i$. Then $X \subset {}^*Y$ and $\beta - \alpha \in X - X \subseteq$ $\subseteq {}^*Y - {}^*Y \subseteq {}^*H_i$. Thus, $\alpha + H = \beta + H$. Let Z be any other infinitesimal element of ${}^*\underline{\alpha}$ and let $\gamma \in Z$. Then $Z \cap X \in {}^*\underline{\alpha}$ is infinitesimal and for $\omega \in Z \cap X$ we have $\omega - \alpha$, $\omega - \gamma \in H$, hence, $\alpha - \gamma \in H$. Finally, let $\underline{\beta} \in \bar{\alpha}$ and let T be infinitesimal in ${}^*\underline{\beta}$, $\beta \in T$. Since $\underline{\alpha} \cap \underline{\beta}$ is a Cauchy filter, for any $i \in J$ there exists $Y \in \underline{\alpha} \cap \underline{\beta}$ such that $Y - Y \subseteq H_i$. and $\alpha - \beta \in X - T \subseteq {}^*Y - {}^*Y \subseteq {}^*H_i$. Thus, $\alpha + H = \beta + H$.

Further, $\varrho(\bar{\alpha}) \in cG$. In fact, let $\varrho(\bar{\alpha}) = \alpha + H$ where α is an element of an infinitesimal element X of ${}^*\underline{\alpha}$. Then $\{\{\beta + H : \beta \in Y\} : Y \in \underline{\alpha}\}$ is a base (in G) of a filter F in ${}^*G/H$ and it is easy to see that $\lim F = \alpha + H$. It follows $\alpha + H \in cG$.

ϱ is injective. Indeed, let $\alpha + H = \varrho(\bar{\alpha}) = p(\bar{\beta}) = \beta + H$, where $\alpha(\beta)$ is an element of an infinitesimal element $X(Y)$ of ${}^*\underline{\alpha}$ (${}^*\underline{\beta}$). Then there exist $A \in \underline{\alpha}$ and $B \in \underline{\beta}$ such that $A - A \subseteq \bigcap H_i$, $B - B \subseteq \bigcap H_i$, $A \cup B \in \underline{\alpha} \cup \underline{\beta}$. Then it is easy to see that $A \cup B - A \cup B \subseteq \bigcap H_i$ and it follows $\bar{\alpha} = \bar{\beta}$.

Analogously it may be proved that ϱ is surjective and if we consider cG to be a subgroup of a factor group ${}^*G/H$, then from the fact

$$(\alpha - \beta) + H = \varrho(\varrho^{-1}(\alpha + H) - \varrho^{-1}(\beta + H)) \in cG$$

for $\alpha + H$, $\beta + H \in cG$ it follows that ϱ is a group isomorphism. Similarly it may be done that ϱ is an o-isomorphism and homeomorphism.

Proposition 6. *For every $i \in J$, $\hat{H}_i = {}^*H_i/H \cap \hat{G}$ is the closure of H_i in $\hat{G}$. The set $\hat{\mathscr{H}} = \{\hat{H}_i : i \in J\}$ is a realizator of G and $\hat{\mathscr{H}}$ is a subbase of neighbourhoods of* 0 *in $\hat{G}$.*

Proof. The first part of the proposition follows immediatelly from the fact that H_i is a dense subset in ${}^*H_i/H \cap G$. It may be easily seen that $\hat{H}_i$ is a prime l-ideal in $\hat{G}$ and

$$\bigcap_{i \in J} \hat{H}_i = \bigcap_{i \in J} ({}^*H_i/H \cap \hat{G}) = \bigcap_{i \in J} ({}^*H_i/H) \cap \hat{G} = \{0\}.$$

Moreover, the topology in $\hat{G}$ is induced from the one in ${}^*G/H$, i.e. the subbase of neighbourhoods of zero consists of the sets ${}^*H_i/H \cap \hat{G} = \hat{H}_i$.

Now, we are able to prove the theorem 2. At first, using the nonstandard construction of $\hat{G}$ we may fully describe elements of $\hat{G}$. So, let $G = Z^{(I)}$ for $I \subseteq N$, card $I = \aleph_0$, and let ${}^*G$ be an enlargement of G, $\alpha \in {}^*G$. Then $\alpha + H \in \hat{G}$ if and only if $\alpha_i \in Z$ for every $i \in I$. This follows immediately from Prop. 6, where ${}^*H_i = \{\beta \in {}^*G : \beta_i = 0\}$, $i \in I$. Let $\hat{w}$ be a semi-valuation associated with a ring $\hat{A}_I$, i.e.

$$\hat{w} : U(\hat{Q}_I) \to U(\hat{Q}_I)/U(\hat{A}_I)$$

is a canonical map and let $x \in U(\hat{Q}_I)$. According to Proposition 3, ${}^*w_i(x) \in Z$ for every $i \in I$. Moreover, interpreting a suitable WFS in ${}^*Q$ we may find an element $i_0 \in {}^*I$ such that $w_i(x) = 0$ for any $i \in {}^*I$, $i > i_0$. Since

${}^*G = \{\alpha \in {}^*Z^{*I} : \alpha$ is internal and there exists $i_0 \in {}^*I$ such that $\alpha_i = 0, \forall i > i_0\}$, we may find an element $\alpha \in {}^*G$ such that

$$\alpha_i = {}^*w_i(x) \in Z, \qquad i \in I.$$

We define a map ϱ in the following way:

$$\varrho : U(\hat{Q}_I)/U\hat{A}_I) \to \hat{G},$$
$$\varrho(\hat{w}(x)) = \alpha + H.$$

If x and y are elements in Q such that ${}^*w_i(x) = {}^*w_i(y) \in Z$ for every $i \in I$, then we have $\tilde{w}_i(x) = \tilde{w}_i(y)$ by [3] and since $x, y \in U(\hat{Q}_I)$, we have $x \,.\, y^{-1}$, $y \,.\, x^{-1} \in \bigcap R_{\tilde{w}_i} = \hat{A}_I$ by [1]. It follows $\hat{w}(x) = \hat{w}(y)$ and the definition of ϱ is correct. It is clear that ϱ is an o-isomorphism. Since $\varrho^{-1}(\hat{H}_i) = U(R_{\tilde{w}_i})$, $i \in I$, ϱ is open and continuous, hence, ϱ is a homeomorphism. Since $U(\hat{Q}_I)/U(\hat{A}_I)$ is a topological group, by [2], Lemma 1, it is a tl-group which is tl-isomorphic to $\hat{G}$. Hence, the theorem is proved.

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