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SOME PROBLEMS OF DIFFERENTIAL GEOMETRY OF ONE CLASS OF SPACES OF SUPPORTING ELEMENTS

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It is known that the space of supporting elements [1] represents a differentiable manifold which is a locally-topological product of this differentiable manifold (base) and a space of values of some differential–geometrical (supporting) object. It may be regarded in a well-known sense as an associated fiber bundle. This space belongs to the so called generalized spaces and each concrete definition of the supporting object gives a specific space with its own geometrical properties.

This paper is concerned with the class of spaces of vector densities u^i with an arbitrary weight p. The spaces represent a particular case of the space of supporting elements on one hand and generalize the space of linear elements on the other hand; in the metric case they generalize the Finsler spaces. At the beginning of this paper we present some fundamental notions of the theory of these spaces, e.g. the definition of the space, the construction of tangent spaces and their equipments, the covariant differential and the metric tensor. Then, using the Lie derivative, we shall examine some variational problems of these spaces.

1. FUNDAMENTAL NOTIONS

To each point of the differentiable manifold X_n, $n = \dim X_n$, which is the base space, there is associated the space of values of a contravariant vector density with an arbitrary weight p. The received manifold is called the space of supporting vector densities.

Transformations of the base space

$$x^{i'} = x^{i'}(x^1, \dots, x^n)$$

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induce transformations

$$u^{i'} = \Delta^{-p} u^i f_i^{i'} \qquad (i, j, \ldots; i', j', \ldots = 1, \ldots, n),$$

where

$$f_i^{i'} = \frac{\partial x^{i'}}{\partial x^i}$$

and

$$\Delta = \det (f_i^{i'}).$$

Here, (x^i, u^i) is called a supporting element and the point x is called its centre.

Constructing the tangent space of supporting elements we will follow the approach of Bliznikas [2]. For a supporting element $(x, u)_0$ consider the set of all differentiable functions $f(x, u)$, defined in a neighbourhood of $(x, u)_0$. Then its differentials $(df)_0$ form a vector space $\overset{*}{T}(x, u)_0$, whose basis is called a natural coframe. The dual basis of the dual space $T(x, u)_0$ of $\overset{*}{T}(x, u)_0$, is called a natural frame.

It is easy to find that the matrices of the corresponding groups of the transformations of spaces T and $\overset{*}{T}$ have the forms

$$\begin{pmatrix} f_{i'}^i & 0 \\ A_{i'}^j & B_{i'}^i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} f^{i'} & \tilde{A}_k^{i'} \\ 0 & \tilde{B}_i^{i'} \end{pmatrix} \quad \text{respectively,}$$

where

$$A_{i'}^j = -\Delta^p u^r \partial_s (\Delta^{-p} f_r^{j'}) f_{j'}^j, \qquad B_{i'}^i = \Delta^p f_{i'}^i,$$

and

$$\tilde{A}_r^{i'} = u^r \partial_k (\Delta^{-p} f_r^{i'}), \qquad \tilde{B}_i^{i'} B_{j'}^i = \delta_{j'}^{i'}.$$

As to the equipment of these spaces there holds the following Theorem.

Theorem. The tangent and cotangent spaces T and $\overset{*}{T}$ will be equipped invariantly if and only if there is given a field of objects Γ_i^j on the base X_n, satisfying the following transformation formulas:

$$\Gamma_{i'}^{j'} = \Delta^{-p} \Gamma_i^j f_{i'}^i f_j^{j'} - \tilde{A}_s^{j'} f_{i'}^s.$$

The objects of an affine connection on spaces of vector densities, introduced by Davies [3], will be denoted by $L_{jk}^i(x, u)$ and $C_{jk}^i(x, u)$. Recall that the functions $L_{jk}^i(x, u)$ are homogeneous, of degree zero in u, and $C_{jk}^i(x, u)$ define a tensor density of a weight $-p$, and are homogeneous of degree minus one in u. We note that given a field of objects L_{jk}^i we also obtain an equipment of the spaces T and $\overset{*}{T}$.

The covariant differential of a tensor density $T_{(j)}^{(i)} = T_{j_1 \ldots j_s}^{i_1 \ldots i_r}$ having the weight q is defined by

$$\delta T_{(j)}^{(i)} = \mathrm{d} T_{(j)}^{(i)} + \overbrace{L_{*k}^{*} T_{(j)}^{(i)}}^{q} \mathrm{d}x^k + \overbrace{C_{*k}^{*} T_{(j)}^{(i)}}^{q} \delta u^k$$

or,

$$\delta T^{(i)}_{(j)} = T^{(i)}_{(j),k}\mathrm{d}x^k + T^{(i)}_{(j):k}\delta u^k$$

where

$$T^{(i)}_{(j),k} = \partial_k T^{(i)}_{(j)} - \overbrace{T^{(i)}_{(j).l} L^{*}_{*k} u^l}^{p} + \overbrace{L^{*}_{*k} T^{(i)}_{(j)}}^{q},$$

$$T^{(i)}_{(j):k} = T^{(i)}_{(j).k} - \overbrace{T^{(i)}_{(j).l} C^{*}_{*k} u^l}^{p} + \overbrace{C^{*}_{*k} T^{(i)}_{(j)}}^{q}.$$

The tensor density $T^{(i)}_{(j),k}$ is called the covariant derivative of the first type, and $T^{(i)}_{(j):k}$ is called the covariant derivatives of the second type. In these formulas the symbol of B. L. Laptev [4] defined by the form

$$\overbrace{L^{*}_{*k} T^{(i)}_{(j)}}^{q} = \sum_{m=1}^{r} L^{i_m}_{lk} T^{i_1 \ldots l \ldots i_r}_{(j)} - \sum_{m=1}^{s} L^{l}_{j_m k} T^{(i)}_{j_1 \ldots l \ldots j_s} - q L^{l}_{lk} T^{(i)}_{(j)},$$

and the notations $\partial_i = \dfrac{\partial}{\partial x^i}$, $.i = \dfrac{\partial}{\partial u^i}$ are used.

A metric tensor of the considered spaces is defined in [3] by means of a scalar function $L(x, u)$, positively homogeneous, of degree 1 in u, by the formula

$$g_{ij}(x, u) = g^p \left(\frac{1}{2} L^2\right)_{ij}$$

where

$$g = \det(g_{ij}).$$

2. THE LIE DERIVATIVE AND ITS APPLICATIONS

In the considered space the Lie derivative of a tensor $T^{(i)}_{(j)}$ with respect to the vector field $v^i(x)$ can be written as follows [5]:

$$\mathcal{L}_v T^{(i)}_{(j)} = v^r T^{(i)}_{(j),r} + \overbrace{T^{(i)}_{(j),k} v^{*}{}_{,*} u^k}^{p} - \overbrace{v^{*}{}_{,*} T^{(i)}_{(j)}}^{q} +$$
$$+ 2v^r \overbrace{\Omega^{*}_{r*} T^{(i)}_{(j)}}^{q} - 2v^s \overbrace{T^{(i)}_{(j).r} \Omega^{*}_{s*} u^r}^{p},$$

where

$$\Omega^k_{rs} = L^k_{rs} - L^k_{sr}$$

is the tensor of torsion.

Solving variational problems in the considered space we suppose that the supporting object u^i has the same direction as the tangent vector to the curve $x^i = x^i(t)$, i.e., $u^i = \sqrt{g^p}\,\mathrm{d}x^i$, where $\sqrt{g^p}$ is a scalar density of weight p.

Herewith, we take into account that the variation of the supporting object arises not only due to the displacement of the point. Then the Lie derivative of the supporting object will not be, in general, equal to zero; in connection with this fact we use the so called complete Lie derivative [6], which, for a tensor $T^{(I)}_{(J)}$, can be written as

$$\overline{\mathscr{L}}_v T^{(I)}_{(J)} = \mathscr{L}_v T^{(I)}_{(J)} + T^{(I)}_{(J),k} \overline{\mathscr{L}}_v u^k,$$

where

$$\overline{\mathscr{L}}_v u^i = [(\ln \sqrt{g^p})_{,r} v^r{}_{,s} u^s + p v^r{}_{,r}]\, u^i$$

is a Lie derivative of the supporting object.

If we examine the first variation with respect to a vector field $v^i = v^i(x)$ of an integral

$$s = \int_{M_1}^{M_2} L(x, u)\, \mathrm{d}t, \tag{1}$$

which defines the arc length in the considered space, then

$$\delta_v s = \int_{M_1}^{M_2} \overline{\mathscr{L}}_v L(x, u)\, \mathrm{d}t.$$

Thus, we get the equation of extremals of the form [5]

$$\delta(l^i + pA^i) + A^i_{jk}(l^j + pA^j)\, \delta l^k = 0, \tag{2}$$

where

$$A^i_{jk} = L \sqrt{g^p} C^i_{jk}, \qquad A^i = g^{ki} A^m_{mk},$$

and

$$l^i = \frac{u^i}{L\sqrt{g^p}}$$

represent the unit vector of the supporting object.

In this way we can also obtain the invariant form of the second variation of the integral (1) of the arc length. If the extremals coincide with the autoparallels it can be expressed as follows:

$$\delta^2_v s = \int_{M_1}^{M_2} (g_{ij} v^i{}_{,0} v^j{}_{,0} - v^i v^j R_{i00j} - \sqrt{g^p} L_{0i0j} v^i v^j{}_{,0})\, \mathrm{d}s, \tag{3}$$

where "0" denotes the index k which is contracted with the unit vector l^k, and R, L are tensors of curvature [4].

Let there be given a hypersurface

$$x^i = x^i(t^1, \ldots, t^{n-1}), \qquad u^i = u^i(t^1, \ldots, t^{n-1}).$$

The mean curvature of this hypersurface is defined by

$$H = \frac{1}{n-1} g^{\alpha\beta} \alpha_{\alpha\beta} \qquad (\alpha, \beta \ldots = 1, \ldots, n-1),$$

where

$$g^{\alpha\beta}(t^1, \ldots, t^{n-1}) = g^{ij} X_i^\alpha X_j^\beta,$$

and

$$a_{\alpha\beta}(t', \ldots, t^{n-1}) = l_i(X^i_{\alpha\beta} + L^i_{\alpha\beta})$$

are the first and the second fundamental tensors, respectively.

Here we used the notation

$$X^i_\alpha = \frac{\partial x^i}{\partial t^\alpha}, \qquad X^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial t^\alpha \partial t^\beta}, \qquad X^\alpha_i = g_{ik} g^{\alpha\beta} X^k_\beta.$$

Now, we consider the variation of the mean curvature of the hypersurface, assuming that the supporting object has the orientation in the direction of the normal of the hypersurface.

Then

$$\delta_v H = \overline{\mathscr{L}}_v H,$$

but in view of $H_{.i} \overline{\mathscr{L}} u^i = 0$, we have

$$\delta_v H = \mathscr{L}_v H$$

and thus we obtain the equation of the extremal displacement for minimal hypersurface (i.e. $H = 0$)

(4) $$g^{\alpha\beta} X^k_\alpha X^s_\beta (v^0_{,k,s} + v^0 R_{oks0}) + 2A^{\alpha\beta\gamma} a_{\alpha\beta} v_{\gamma,0} + 2v^0 a^\alpha_\beta a^\beta_\alpha + P^k_{00k} v^0_{,0} = 0.$$

Here P is also a tensor of curvature on the considered space [4].

The obtained expressions (2), (3) and (4) generalize the analogous equations in the Finsler space.

Finally, we note that the results (2), (3) and (4) can also be obtained in the case when the supporting object is a covector density of weight $-p$.

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