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GLOBAL TRANSFORMATIONS OF LINEAR DIFFERENTIAL EQUATIONS AND QUADRATIC FUNCTIONALS, I

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In the present paper the unified approach to study of extremal properties of quadratic functionals of the type

$$J(y) = \int_0^B [y'^2(t) + q(t)\,y^2(t)]\,\mathrm{d}t$$

is introduced, where $q(t) \in C^0_{(0,B]}$ and $y(t)$ are A-admissible functions on $[0, B]$:

i) $y(t) \in C^0_{[0,B]}$; $y(0) = y(B) = 0$,

ii) $y(t)$ is absolutely continuous and $y'^2(t)$ is integrable on each closed subinterval of the interval $(0, B]$.

The method is based on Borůvka's theory of global transformations of second order linear differential equations.

Throughout the paper only Lebesgue integral is used (briefly L-integral). At the end of the paper there is given a new explicit class of quadratic functionals which achieve absolute minimum equal to 0 on A-admissible functions i.e.

$$\liminf_{E \to 0+} J(y)\,|_E^B \geqq 0 \tag{I}$$

for each A-admissible function $y(t)$ on $[0, B]$. If the lowest limit of (I) exists, then it is zero since $y(t) \equiv 0$ for $t \in [0, B]$ is also A-admissible on $[0, B]$. Further on this lowest limit will be called the minimal nonnegative limit of the considered quadratic functional.

The problem was systematically studied mainly by Leighton. Since 1936 he has written several extensive papers, where he found sufficient and necessary conditions for the existence of the minimal nonnegative limit by means of classical methods depending on the type of admissible functions $y(t)$ and the form of functional [8], [9], [10], [11].

Recently the problem has been partially studied by W. A. Coppel [2], who has extended some of Leighton's results to functionals of higher orders. It was also

Krbila [4]–[7] who dealt with the quadratic functionals. Using Borůvka's theory of transformations he studied special properties of regular quadratic functionals. In his considerations he used mainly Riemann integral.

In the present paper a new approach to mentioned problems is introduced. This approach to the study of arbitrary quadratic functionals is based on the basic result of Borůvka's theory consisting in the fact that each linear differential equation of the second order on its whole definition interval can be globally transformed to the equation $y'' = -y$ on a suitable interval. First by classical means of direct evaluation there are derived properties of one special functional corresponding to Euler equation $y'' + y = 0$ and then by means of global transformation the results are extended to general functionals.

Functional of the type $(-\bar{1})$

Notation. Functional of the type

$$J(y) = \int_0^B [Y'^2(T) + \bar{Q}(T)\, Y^2(T)]\, dT,$$

where $Q(T) \in C^0_{(0,B]}$ (fixed); $Y(T)$ are A-adissible on $[0, B]$ and corresponding L-integral is taken on an open interval $(0, B)$, is denoted by $(\bar{Q})$. The associated Euler equation ((E-equation) to the functionals $(\bar{Q})$ is denoted by (Q).

Then (Q): $Y''(T) = Q(T)\, Y(T)$ is defined on the the open interval $(0, B)$. Specially the functional

$$J(y) = \int_0^b [y'^2(t) - y^2(t)]\, dt$$

is denoted by $(-\bar{1})$ and E-equation by (-1).

Thus the equation

$$(-1) \qquad y''(t) = -y(t)$$

is defined on the interval $(0, b)$.

Since Euler equations (Q) and (-1) are defined on open intervals $(0, B)$ and $(0, b)$, we use all notions from Borůvka's theory of transformations [1]. Let us mention some of the most important notions used in this paper. Each ordered couple of linearly independent integrals u, v of the differential equation

$$(\mathrm{q}) \qquad y''(t) = q(t)\, y(t), \qquad t \in j = (a, b)$$

is called the base of the equation. Let $t \in j$ and u denote an arbitrary integral of (q) with the property $u(t) = 0$. The number $x \in j$ is called conjugate to the number t if $u(x) = 0$. The first zero x to the right of t is called the first conjugate point to t. The function assigning the first conjugate point to t is called the basic central dispersion of the first kind and is denoted by φ.

Let R_1 (S_1) denote the set of all numbers $t \in j$ to which there exist conjugate numbers to the left (to the right). If $R_1 \cup S_1 = \emptyset$, the equation (q) is called disconjugate. All the integrals of (q) are of the same oscillatory character, i.e. they have either finite or infinite number of zeros in j. In the first case the equation (q) is of finite type and by the type of (q) is meant the integer m defined as follows: The equation (q) has an integral with m zeros in j but none of its integral has $m + 1$ zeros in j. In the second case the equation (q) is of infinite type and we assign ∞ as its type. More in detail (q) is called oscillatory from the left or from the right, or both side oscillatory if the zeros of its integrals accumulate to the left end a, or to the right end b, or to both ends a, b of the interval j, respectively.

For each equation (q) we can define its kind: general or special. The equation (q) is of general kind if there exists a base (u, v) such that u, v have exactly $m - 1$ zeros in j. If there is no such a base, the equation (q) is of special kind. The couple consisting of the type and the kind of an equation is called its character.

A phase of the base (u, v) is a function $\alpha(t)$ continuous on j satisfying

$$\operatorname{tq} \alpha(t) = \frac{u(t)}{v(t)}$$

everywhere on j with the exception of zeros of denominator. The phase α is called normal if it has a zero in j. Evidently each equation (q) admits a phase with an arbitrary given zero. Consider equation (q), $t \in j$ and equation (Q), $T \in J$. The phase α of (q) and the phase A of (Q) are called similar if $\alpha(j) = A(J)$. This situation occurs exactly if

$$\lim_{t \to a+} \alpha(t) = \lim_{T \to A+} A(T) \quad \text{and} \quad \lim_{t \to b-} \alpha(t) = \lim_{T \to B-} A(T).$$

If equations (q), (Q) are of the same character, then to each phase of the first equation there exists a similar phase of the second one.

Lemma 1. ([3] p. 426)
The function

$$1, \cos x, \cos 2x, \dots \ (*)$$

and

$$\sin x, \sin 2x, \dots \ (**)$$

form complete orthogonal system in the space $L_2[-\pi, \pi]$. Each system (*) and (**) is orthogonal and complete on the interval $[0, \pi]$.

Lemma 2. ([3] p. 448)
If a function f with period 2π is absolutely continuous and its derivation belongs to $L_2[-\pi, \pi]$, then Fourier series of the function f converges to the function f uniformly on the whole real axis.

Lemma 3. ([3] p. 335)
If $\psi_n(x) \geqq 0$ and

$$\sum_{n=1}^{\infty} \int_A \psi_n(x)\,d\mu < \infty,$$

then the series $\sum_{n=1}^{\infty} \psi_n(x)$ converges almost everywhere on A and it holds

$$\int_A \left(\sum_{n=1}^{\infty} \psi_n(x)\right) d\mu = \sum_{n=1}^{\infty} \int_A \psi_n(x)\,d\mu.$$

From these assertions follows immediately the following lemma.

Lemma 4.
Let $y(t)$ be an A-admissible function on $[0, b]$. Then there exists one and only one denumerable system of real numbers

$$K_n = \frac{2}{b} \int_0^b y(t) \sin\left(\frac{n\pi}{b} t\right) dt$$

such that Fourier series of the function $y(t)$, i.e.

$$\sum_{n=1}^{\infty} K_n \sin\left(\frac{n\pi}{b} t\right)$$

converges uniformly to the function $y(t)$, and Fourier series of $y'(t)$ is of form

$$\sum_{n=1}^{\infty} \frac{n\pi}{b} K_n \cos\left(\frac{n\pi}{b} t\right), \qquad t \in (0, b].$$

Now we give the main theorem of the first part of the paper.

Theorem 1. *Let $y(t)$ be A-admissible functions on $[0, b]$. Then*

$$\text{(II)} \qquad \liminf_{e \to 0+} \int_e^b [y'^2(t) - y^2(t)]\,dt \geqq 0$$

for each function $y(t)$ if and only if $0 < b \leqq \pi$. If $b < \pi$, then the equality in (II) occurs exactly for $y(t) \equiv 0$. If $b = \pi$, *then the equality in (II) occurs exactly for the system of functions $y(t) = k \,.\, \sin t$, where $k \in R$. If $b > \pi$, then the relation* (II) *is not satisfied for all A-admissible functions on $[0, b]$.*

Proof. From Lemmas 3, 4 it follows

$$\liminf_{e \to 0+} \int_e^b (y'^2(t) - y^2(t))\,dt =$$

$$= \liminf_{e \to 0+} \sum_{n=1}^{\infty} \int_e^b \left[\left(\frac{n\pi}{b} K_n \cos \frac{n\pi}{b} t\right)^2 - \left(K_n \sin \frac{n\pi}{b} t\right)^2\right] dt =$$

$$= \inf \frac{b}{2} \sum_{n=1}^{\infty} K_n^2 \,.\, \left[\left(\frac{n\pi}{b}\right)^2 - 1\right].$$

From this there follow immediately the assertions of Theorem 1, and the proof is complete.

Corollary 1. The relation (II) is satisfied if and only if the associate E-equation (-1): $y''(t) = -y(t)$ is of the type 1 (disconjugate) on the open interval $(0, b)$. If $b < \pi$, then (-1) is of the type 1 and general. If $b = \pi$, then (-1) is of the type 1 and special. If $b = \pi$, then the integral of the equation (-1) $\sin t$ extended continuously to the closed interval $[0, \pi]$ is A-admissible function on $[0, \pi]$, for which the functional $(-\bar{1})$ achieves its minimal limit equal to zero. From the properties of integrals of the equation (-1) it follows immediately that this solution up to its constant multiple is the only one of the above property.

Functional of the type $(\bar{Q})$

Assumption 1. Assume associate E-equations of functionals $(-\bar{1})$ and $(\bar{Q})$, i.e.

$$(-1) \qquad y''(t) = -y(t), \qquad t \in j = (0, b),$$

$$(Q) \qquad Y''(T) = Q(T)\, Y(T), \qquad T \in J = (0, B)$$

are of the same character, i.e. they are globally transformable on their whole intervals of definitions.

Remark 1. By [1] there exist functions $X(t)$, $M(t)$ for $t \in (0, b)$ transformating the equation (Q) to (-1) on the whole intervals of definitions, i.e. for each integral $Y(T)$, $T \in (0, B)$ of (Q), the function

$$y(t) = M(t)\, Y(X(t)), \qquad \text{where} \qquad M(t) = c \,.\, |\, X'(t)\,|^{-1/2}, \qquad c \in R$$

is an integral of the equation (-1) on $(0, b)$. The function $M(t)$ is called a multiplier of the function $M(t)$ and the function $X(t)$ is called a kernel of the global transformation $(Q, -1)$.

Similarly there exist functions $x(T)$, $m(T)$ for $T \in (0, B)$ such that for each integral $y(t)$, $t \in (0, b)$ of (-1) the function

$$Y(T) = m(T)\, y(x(T)), \qquad \text{where} \qquad m(T) = \frac{1}{c \,.\, |\, x'(T)\,|^{1/2}}$$

is an integral of (Q) on the whole interval $(0, B)$. The function $m(T)$ is a multiplicator and $x(T)$ a kernel of the transformation $(-1, Q)$.

O. Borůvka [1] has shown that as a kernel of the global transformation $(Q, -1)$ there can be taken a function

$$X(t) = A^{-1}\alpha(t) \qquad \text{for} \qquad t \in (0, b),$$

where $A(T)$ and $\alpha(t)$ are similar normal phases of the equations (Q) and (-1), respectively. The transformation $(-1, Q)$ given by kernel

$$x(T) = \alpha^{-1}A(T) \qquad \text{for} \qquad T \in (0, B)$$

is inverse to the transformation (Q, -1).

Each kernel $X(t)$ or $x(T)$ is of the following basic properties $X(j) = J$; $X(t) \in C_j^3$; $X'(t) \neq 0$ for all $t \in j$ or $x(T) = j$; $x(T) \in C_J^3$; $x'(T) \neq 0$ for all $T \in J$, respectively. If the equations (q), (Q) of the same character are of the finite type or oscillatory, there exist their global transformations with increasing and decreasing kernels.

Lemma 5. *Let Assumption 1 be fulfilled and $X(t)$ be an arbitrary increasing kernel of a global transformation* (Q, -1) *continuously extended to the closed interval* $[0, b]$. *Further let $Y(T)$ be an arbitrary A-admissible function on* $[0, B]$. *Then*

$$y(t) = \frac{YX(t)}{X'(t)^{1/2}}$$

is A-admissible function on the interval $[0, b]$.

Similarly if we consider a transformation $t = x(T)$, where $x(T)$ is the kernel of the inverse transformation $(-1, Q)$ *continuously extended to the closed interval* $[0, B]$, *then the function*

$$Y(T) = \frac{y(x(T))}{x'(T)^{1/2}}$$

is A-admissible function on the interval $[0, B]$ *for each A-admissible function $y(t)$ on* $[0, b]$. *In this way a certain one-to-one mapping of the set of all A-admissible functions $Y(T)$ on $[0, B]$ onto the set of all A-admissible functions $y(t)$ on $[0, b]$ is defined.*

Proof. From the properties of the function $X(t)$ and from A-admissibility of $Y(T)$ it follows:

a) If $Y(T) \in C^0_{[0,B]}$, $Y(0) = Y(B) = 0$, then $Y[X(t)] . [X'(t)]^{-1/2} \in C^0_{[0,B]}$; $YX(0) . [X'(0)]^{-1/2} = Y[X(b)] . [X'(b)]^{-1/2} = 0$,

b) If $Y(T)$ is absolutely continuous and $Y'^2(T)$ is L-integrable on each closed subinterval of the interval $(0, B]$, then $\dfrac{YX(t)}{\sqrt{X'(t)}}$ is absolutely continuous and $\left[\dfrac{YX(t)}{\sqrt{X'(t)}}\right]'^2$ is L-integrable on each closed subinterval of the interval $(0, b]$.

The assertion a) follows immediately. The assertion b) is valid due to [3] p. 414 and 378. The second half of the assertion can be proved similarly to the first half of the assertion of Lemma 5. The mapping is one-to-one due to properties of the functions $X(t)$, $x(T)$, the kernels of global transformations (Q, -1) and (-1, Q). The proof is complete.

Corollary 2. Due to [1], Corollary 1 and Lemma 5 it follows immediately: If $U(T)$ is an integral of the equation (Q) vanishing at 0, then

$$U[X(t)] \, . \, X'(t)^{-1/2} = k \, . \sin t, \qquad k \in R,$$

i.e. the integral of the equation (-1) vanishing at zero. Analogously the function

$$U(T) = \sin [x(T)] \, . \, [x'(T)]^{-1/2}$$

is an integral of the equation (Q) vanishing at zero. If the equation (Q) is disconjugated and special on $(0, B)$, then this integral $U(T)$ can be continuously extended to $[0, B]$ such that it is an A-admissible function on $[0, B]$.

Recall one of the results from [1].

Lemma 6. *Let (Q) and (-1) be of the same character. Then a function $X(t)$ is a kernel of the global transformation* (Q, -1) *if and only if $X(t)$ is a solution of Kummer equation*

$$-\{X, t\} + Q(X(t)) \, . \, X'^2(t) = -1,$$

where

$$-\{X, t\} = -\frac{1}{2} \frac{X'''(t)}{X'(t)} + \frac{3}{4} \frac{X''^2(t)}{X'^2(t)}.$$

Lemma 7. *Let Assumption 1 be satisfied. Then there exists a one-to-one mapping of the set of A-admissible functions $y(t)$ on $[0, b]$ onto the set of A-admissible functions $Y(T)$ on $[0, B]$ such that for each two corresponding A-admissible functions $y(t)$ and $Y(T)$ it holds*:

$$\text{(III)} \qquad \liminf_{e \to 0_+} \int_e^b [y'^2(t) - y^2(t)] \, dt = \liminf_{E \to 0_+} \int_E^B (Y'^2(T) + Q(T) \, Y^2(T)] \, dT.$$

Proof. Consider the one-to-one mapping described in Lemma 5. Then from Lemma 5 and Lemma 6 we get:

$$\int_e^b [y'^2(t) - y^2(t)] \, dt =$$

$$= \int_e^b [Y'^2(X(t)) + Q(X(t)) \, Y^2(X(t))] \, X'(t) \, dt - \frac{Y^2(X(t)) \, X''(t)}{2X'^2(t)} \Bigg|_e^b,$$

where $X(t)$ is an increasing kernel of the transformation $(Q, -1)$. From the properties of the functions $Y(X(t))$, $X(t)$ it follows immediately

$$\lim_{e \to 0_+} \left(-\frac{Y(X(t)) \, X''(t)}{2X'^2(t)} \right) = 0.$$

Substituting $T = X(t)$, we get

a) $dT = X'(t) \, dt$,

b" $X(b) = B$; $X(e) = E$ and if $e \to 0_+$, then $E \to 0_+$.

c) $X(t)$ satisfies all requirements of the substitution method for L-integral: $X(t)$ is finite, continuous, increasing, mapping the interval $[e, b]$ on the interval

$[E, B]$ and it has a finite nonvanishing continuous derivative $x'(t) > 0$. After substituting we get immediately the relation (III).

Remark 2. If one side in (III) is greater than or equal to 0, then the same is true also for the other side. Equality (III) can be obtained also by substituting $t = x(T)$ to the right side of (III).

Definition. Let the equation (q): $y''(t) = q(t)\, y(t)$ be defined in the interval $(0, d)$ and choose b: $0 < b < d$. Let a point $a_1 \in (0, b)$ be such that there exists $\varphi(a_1)$, i.e. the first conjugate point to a_1. Then the first conjugate point to 0 is defined as $\lim_{a \to 0_+} \varphi(a)$. If $\varphi(a)$ does not exist for any $a \in (0, b)$, we say that the point 0 has not the first conjugate point on the interval $[0, d)$.

Remark 3. If the first conjugate point to 0 coincides with the point 0, then the function q cannot be continuously extended to the point 0.

Introduce without proof two theorems from [11] (Theorems 2.2 and 2.3).

Lemma 8. *If the point* 0 *does not coincide with its first conjugate point, then there exists a solution W of E-equation (q) of the functional (I) such that $W(T) \not\equiv 0$ and for each solution $Z(T)$ independent of $W(T)$ it holds*

$$\lim_{T \to 0_+} \frac{W(T)}{Z(T)} = 0.$$

Any solution with this property is linearly dependent on the solution $W(T)$. The first conjugate point to 0, *if it exists, is the first positive zero of the solution $W(T)$, and the first positive zero of the solution $W(T)$. if it exists, is the first conjugate point to* 0.

Lemma 9. *If there exists the minimal nonnegative limit of the functional (I), then the interval $[0, B)$ does not contain the first conjugate point to* 0.

Now we can give the necessary and sufficient condition for the existence of the minimal nonnegative limit for functionals of the type $(\bar{Q})$.

Theorem 2. *Let $Y(T)$ be A-admissible functions on $[0, B]$ and $Q(T) \in C^0_{(0,B]}$ be a fixed chosen function. Then*

$$\text{(IV)} \qquad \liminf_{E \to 0_+} \int_E^B [Y'^2(T) + Q(T)\, Y^2(T)]\, dt \geqq 0$$

for the functions $Y(T)$ if and only if the associated E-equation

$$\text{(Q)} \qquad Y''(T) = Q(T)\, Y(T)$$

is disconjugate on the open interval $(0, B)$, i.e. it is of type 1. If it is of type 1 and general, then the equality in (IV) occurs exactly for the integral $Y(T) \equiv 0$.

If it is of type 1 and special, then the equality in (IV) occurs exactly for the integral $Y(T)$ of (Q) continuously prolonged on the whole interval $[0, B]$ with $Y(0) = Y(B) = 0$ and for integrals depending on $Y(T)$. If E-equation (Q) is not disconjugate on $(0, B)$, then the relation (IV) is not satisfied for all A-admissible functions $Y(T)$ on $[0, B]$.

Proof. $\Rightarrow$ follows immediately from Lemmas 8, 9. $\Leftarrow$ immediately from Corollaries 1, 2, Lemma 7, Remark 2.

Corollary 3. By [1] Theorem 2 is equivalent to the assertion: The minimal non-negative limit of the functional ($\bar{Q}$) exists if and only if there exists an increasing phase $A(T)$ of (Q) such that

$$\lim_{T \to 0+} A(T) = 0; \qquad \lim_{T \to B-} A(T) \leqq \pi.$$

If $\lim\limits_{T \to B-} A(T) < \pi$, then the equality in (IV) occurs just for $Y(T) \equiv 0$.

If $\lim\limits_{T \to B+} A(T) = \pi$, then the equality in (IV) occurs exactly for the functions

$U(T) = K \dfrac{\sin (A(T))}{\sqrt{A'(T)}}$, where $K \in R$.

Lemma 10. ([1] p. 140 or [12])

Let $y''(t) = q(t)\, y(t)$ be defined for $t \in j = (-\infty, \infty)$ with the basic central dispersion of the first kind φ. Let α be a fixed first phase of (q) and f an arbitrary π-periodic function of the class $C^2_{(-\infty, \infty)}$ with the following property

$$f(0) = f'(0) = 0; \qquad \int_0^\pi \frac{\exp(-2f(\sigma)) - 1}{\sin^2 \sigma} \, d\sigma = 0.$$

Then all the differential equations with the same basic central dispersion of the first kind ψ are exactly the equations

$$(q_1) \qquad y'' = \{q + [f''\alpha + f'^2\alpha + 2f'\alpha \operatorname{cotg} \alpha]\, \alpha'^2\}\, y.$$

Corollary 4. The solution of (q_1) determined by $y(0) = 0$, $y'(0) = 1$ can be expressed in the form

$$y_1(t) = \frac{1}{|\alpha'(0)|^{1/2}} \frac{e^{f\alpha(t)} \sin \alpha(t)}{|\alpha'(t)|^{1/2}},$$

where the function f satisfies the properties of Lemma 10 and $\alpha(t)$ is a fixed phase of (q) with zero at 0, i.e. $\alpha(0) = 0$.

Proof. From the assumption that $\alpha(t)$ is a phase of (q) it follows immediately

$$q(t) = -\frac{1}{2} \frac{\alpha'''(t)}{\alpha'(t)} + \frac{3}{4} \frac{\alpha''^2(t)}{\alpha'^2(t)} - \alpha'^2(t).$$

Choose $y(t) = \dfrac{e^{f\alpha(t)} \sin \alpha(t)}{|\alpha'(t)|^{1/2}}$.

By direct evaluation we get

$$y'(t) = \alpha'(t)^{1/2}\, y(t)\, f'\alpha(t) + \alpha'(t)^{1/2}\, e^{f\alpha(t)} \cos \alpha(t) - \frac{y(t)\,\alpha''(t)}{2\alpha'(t)^{3/2}},$$

$$y''(t) = \left[-\frac{1\alpha'''(t)}{2\alpha'(t)} + \frac{3}{4}\frac{\alpha''^2(t)}{\alpha'^2(t)} - \alpha'^2(t) \right] y(t) +$$

$$+ [f'\alpha(t) + f'^2\alpha(t) + 2f'\alpha(t) \operatorname{cotg} \alpha(t)\, \alpha'^2(t)]\, y(t).$$

Assuming $\alpha(0) = 0$, we get immediately

$$y_1(0) = 0; \qquad y_1'(0) = 1.$$

Lemma 11. ([19] Theorem 3.6)
Each solution of a differential equation (q) *is periodic or halfperiodic with period* c $(c > 0)$ *having exactly* n *zeros in* $[t_0, t_0 + c)$ *if and only if*

$$q = -\frac{1}{2}\left[\frac{P''}{P' + \varepsilon\pi n/c}\right]' + \frac{1}{4}\left[\frac{P''}{P' + \varepsilon\pi n/c}\right]^2 - (P' + \varepsilon\pi n/c)^2,$$

where $P(t + c) = P(t)$, $P \in C^3$, $P'(t) + \varepsilon\pi n/c \neq 0$, $\varepsilon = 1$ *or* $\varepsilon = -1$.

Corollary 5. All differential equations (q_1) with the basic central dispersion of the first kind $\varphi(t + b) = \varphi(t)$ are given by the relation

$$(\mathrm{q}_1)\quad y''(t) = \left\{ -\frac{\pi^2}{b^2} + \left[f''\left(\frac{\pi}{b}t\right) + f'^2\left(\frac{\pi}{b}t\right) + 2f'\left(\frac{\pi}{b}t\right) \operatorname{cotg}\left(\frac{\pi}{b}t\right) \right] \frac{\pi^2}{b^2} \right\} y(t).$$

Then

$$\liminf_{e \to 0+} \int_e^b [y'^2(t) + q_1(t)\, y^2(t)] \geqq 0$$

for each A-admissible function $y(t)$ on $[0, b]$. This functional achieves its absolute minimum equal to zero exactly on the integrals

$$y(t) = K e^{f\left(\frac{\pi}{b}t\right)} \sin \frac{\pi}{b} t,$$

where $K \in R$ and the functions f satisfy the conditions given in Lemma 10.

Remark 5. Using Corollary 5 and Lemma 11, we can also determine the class of functionals $(\bar{Q})$ having the nonnegative minimal limit. However, Lemma 11 cannot be used for direct evaluation of A-admissible functions $y(t)$ on which these functionals $(\bar{Q})$ achieve their absolute minimum equal to 0.

Remark 6. The base of the described method is the global transformation (Q, —1) or its inverse $(-1, \mathrm{Q})$ which transforms globally not only the corresponding Euler equations (Q) and (-1) but also the corresponding functionals $(\bar{Q})$ and $(-\bar{1})$

together with the classes of A-admissible functions. The method given in the paper may be extended to more general one, e.g. of the type

$$(\mathrm{p}, \mathrm{q}) \qquad \int_0^B [p(t)\, y'^2(t) + q(t)\, y^2(t)]\, \mathrm{d}t,$$

where $(p(t) > 0,\ q(t)) \in C^0_{(0,B]}$ are fixed and $y(t)$ are A-admissible (even more general) function on $[0, B]$.

F. Neuman [14] extended Borůvka's global transformation of linear differential equations of the second order to equations of the n-th order, $n \geqq 2$. Using it, it would be possible to extend this unified method to investigating extremal properties of more general functionals whose associated Euler equations is a linear differential equation of the 2n-th order.

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