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CONDITIONS FOR TRIVIAL PRINCIPAL TOLERANCES

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Definitions. By a *tolerance* on an algebra $\mathfrak{A}$ is meant a compatible reflexive and symmetric relation on $\mathfrak{A}$, i.e. a subalgebra of $\mathfrak{A} \times \mathfrak{A}$ with a reflexive and symmetric relation on $|\mathfrak{A}|$ as its support.

By a *principal tolerance* $T(a, b)$ on an algebra $\mathfrak{A}$ is meant the least tolerance on $\mathfrak{A}$ containing $[a, b] \in |\mathfrak{A}| \times |\mathfrak{A}|$.

An algebra $\mathfrak{A}$ is said *to have trivial principal tolerances* if every principal tolerance on $\mathfrak{A}$ is a congruence.

A class of algebras $\mathscr{V}$ is said *to have trivial principal tolerances* if every algebra from $\mathscr{V}$ has trivial principal tolerances.

Lemma 1. *Let $\mathfrak{A}$ be an algebra, $a, b, x, y \in |\mathfrak{A}|$. There holds $[x, y] \in T(a, b)$ iff there exist a natural number n, an $(n + 2)$-ary polynomial f on $\mathfrak{A}$ and elements $c_1, \dots, c_n \in |\mathfrak{A}|$ such that*

$$x = f(a, b, c_1, \dots, c_n)$$
$$y = f(b, a, c_1, \dots, c_n).$$

Proof will be omitted, cf. [1].

Lemma 2. *Let φ be a homomorphism of an algebra $\mathfrak{A}$ onto an algebra $\mathfrak{B}$. Then $T(\varphi a, \varphi b) = (\varphi \times \varphi)\, T(a, b)$.*

Proof. Let $[x, y] \in (\varphi \times \varphi)\, T(a, b)$. Then there exist elements $v, w \in |\mathfrak{A}|$ such that $[v, w] \in T(a, b)$ and $x = \varphi v$, $y = \varphi w$. By the lemma 1, there exist an $(n + 2)$-ary polynomial f and elements $c_1, \dots, c_n \in |\mathfrak{A}|$ such that $v = f(a, b, c_1, \dots, c_n)$, $w = f(b, a, c_1, \dots, c_n)$. Then

$$x = \varphi v = \varphi f(a, b, c_1, \dots, c_n) = f(\varphi a, \varphi b, \varphi c_1, \dots, \varphi c_n),$$
$$y = \varphi w = \varphi f(b, a, c_1, \dots, c_n) = f(\varphi b, \varphi a, \varphi c_1, \dots, \varphi c_n),$$

so that it holds $[x, y] \in T(\varphi a, \varphi b)$. We have obtained $(\varphi \times \varphi)\, T(a, b) \subseteq T(\varphi a, \varphi b)$.

Clearly $[\varphi a, \varphi b] \in (\varphi \times \varphi)\, T(a, b)$. Since $(\varphi \times \varphi)\, T(a, b)$ is reflexive, as φ is onto, and symmetric, and as a homomorphic image of a subalgebra of $\mathfrak{A} \times \mathfrak{A}$ a subalgebra of $\mathfrak{B} \times \mathfrak{B}$, so a tolerance on $\mathfrak{B}$, it follows $T(\varphi a, \varphi b) \subseteq (\varphi \times \varphi)\, T(a, b)$. Consequently $T(\varphi a, \varphi b) = (\varphi \times \varphi)\, T(a, b)$. Q.E.D.

Proposition. *An algebra $\mathfrak{A}$ satisfies (i) and (ii) iff it satisfies (iii).*

(i) $\mathfrak{A}$ has trivial principal tolerances

(ii) every principal congruences S, T on $\mathfrak{A}$ satisfy $STS = TST$

(iii) every principal tolerances S, T on $\mathfrak{A}$ satisfy $STS = TST$

Proof. Obviously (*i*) and (*ii*) implies (*iii*). Let (*iii*) hold, let T be a principal tolerance on $\mathfrak{A}$. Since Δ is a principal tolerance on $\mathfrak{A}$, it follows $T = \Delta T \Delta = T \Delta T = TT$. Thus T is a congruence. There holds (*i*). But (*i*) and (*iii*) implies (*i*) immediately. Q.E.D.

This proposition describes completely relations among the conditions (*i*), (*i*) and (*iii*). It will be illustrated in the following examples.

Example 1. Let $\mathscr{V}$ be the variety of all monounary algebras that satisfy identity $ffx = x$. Every $\mathscr{V}$-free algebra satisfies (*i*), but if it has at least two generators it does not satisfy (*ii*) and (*iii*):

Obviously $T(a, b) = \{[a, b], [b, a], [fa, fb], [fb, fa]\} \cup \Delta$.

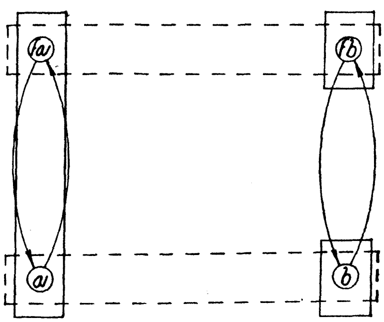
1. $fa = b$

In this case $T(a, b) = \{[a, b], [b, a]\} \cup \Delta$. It is a congruence.

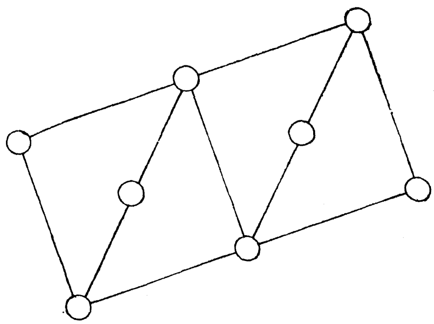
2. $fa \neq b$

In this case $fb \neq a$, because $fb = a$ would imply $fa = ffb = b$, and obviously $fa \neq a, fb \neq b$. So $T(a, b)$ is a congruence.

Let a, b be two distinct free generators of a $\mathscr{V}$-free algebra. We have $[b, fb] \in T(a, b)\, T(a, fa)\, T(a, b)$, but $[b, fb] \notin T(a, fa)\, T(a, b)\, T(a, fa)$.



Example 2. A simple algebra which is not tolerance simple satisfies (*ii*) but it does not satisfy (*i*) and (*iii*). An example of such an algebra is the following modular lattice.



Theorem 1. *Let $\mathscr{V}$ be a class of algebras and let $\mathscr{F}$ be a subclass of $\mathscr{V}$ such that $\mathscr{V} = \mathbf{H}\mathscr{F}$. The following conditions are equivalent:*

(*A*) *$\mathscr{V}$ has trivial principal tolerances*
(*B*) *every algebra $\mathfrak{A}$ from $\mathscr{F}$ satisfies (i) and (ii)*
(*C*) *every algebra $\mathfrak{A}$ from $\mathscr{F}$ satisfies (iii)*
(*D*) *every algebra $\mathfrak{A}$ from $\mathscr{V}$ satisfies (iii)*

Proof. $A \Rightarrow D$: Let $\mathscr{V}$ have trivial principal tolerances. Let $S = \Theta(a, b)$, $T = \Theta(c, d)$ be arbitrary principal tolerances i.e. principal congruences on $\mathfrak{A}$. We have $\mathbf{H}\mathscr{V} = \mathbf{HH}\mathscr{F} = \mathbf{H}\mathscr{F} = \mathscr{V}$, thus $\mathfrak{A}|_T$ is an algebra from $\mathscr{V}$. Denote by φ the quotient homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}|_T$. By the lemma 2, $T(\varphi a, \varphi b) = (\varphi \times \varphi)\, T(a, b)$. Let $[x, y] \in STS$, so there exists elements v, w such that $[x, v] \in S$, $[v, w] \in T$ and $[w, y] \in S$. Then $\varphi v = \varphi w$ and it holds $[\varphi x, \varphi v] \in T(\varphi a, \varphi b)$, $[\varphi v, \varphi y] \in T(\varphi a, \varphi b)$. Since $\mathfrak{A}|_T$ is an algebra from $\mathscr{V}$ and so it has trivial principal tolerances, we have $[\varphi x, \varphi y] \in T(\varphi a, \varphi b)$. It means that there exist elements $x_1, y_1 \in |\mathfrak{A}|$ such that $[x, x_1] \in T$, $[y, y_1] \in T$ and $[x_1, y_1] \in S$, and so $[x, y] \in TST$. We have obtained $STS \subseteq TST$ for arbitrary principal tolerances S, T on $\mathfrak{A}$. Thus $STS \subseteq TST \subseteq STS$ holds for arbitrary principal tolerances on $\mathfrak{A}$.

$D \Rightarrow C$ is obvious.

$C \Rightarrow B$: follows from the Proposition.

$B \Rightarrow A$: Let every algebra $\mathfrak{A}$ from $\mathscr{F}$ satisfy (*i*) and (*ii*). Let $\mathfrak{B}$ be an arbitrary algebra from $\mathscr{V}$. Since $\mathscr{V} = \mathbf{H}\mathscr{F}$, there exists an algebra $\mathfrak{A}$ from $\mathscr{F}$ and a homomorphism φ of $\mathfrak{A}$ onto $\mathfrak{B}$. Let $a, b, x, y, z \in |\mathfrak{B}|$ be such that $[x, y] \in T(a, b)$ and $[y, z] \in T(a, b)$. Choose $g_a \in \varphi^{-1}a$ and $g_b \in \varphi^{-1}b$. Then, by the lemma 2, $T(a, b) = (\varphi \times \varphi)\, T(g_a, g_b)$. Thus there exist elements $g_x, g_y, h_y, h_z \in |\mathfrak{A}|$ such that $[g_x, g_y] \in T(g_a, g_b)$, $[h_y, h_z] \in T(g_a, g_b)$ and $\varphi g_x = x$, $\varphi g_y = \varphi h_y = y$, $\varphi h_z = z$. By the assumption, $T(g_a, g_b)$ is a congruence. So if $g_y = h_y$, then $[g_x, h_z] \in T(g_a, g_b)$ and consequently $[x, z] \in T(a, b)$. If $g_y \neq h_y$, denote $S = T(g_y, h_y)$ and $T = T(g_a, g_b)$. We have $[g_x, h_z] \in TST$. By the assumption, there exist elements $g_1, g_2 \in |\mathfrak{A}|$ such that $[g_x, g_1] \in S$, $[g_1, g_2] \in T$ and $[g_2, h_z] \in S$. Since $S \subseteq \operatorname{Ker} \varphi$, it holds $\varphi g_x = \varphi g_1$ and $\varphi g_2 = \varphi h_z$. Hence $[x, z] = [\varphi g_x, \varphi h_z] =$

$= [\varphi g_1, \varphi g_2] \in (\varphi \times \varphi)\, T = T(a, b)$. We have obtained that $T(a, b)$ is a congruence. Q.E.D.

Theorem 2. *Let $\mathscr{V}$ be a variety of algebras. The following conditions are equivalent:*

*(**A**) $\mathscr{V}$ has trivial principal tolerances,*

*(**B**) every $\mathscr{V}$-free algebra $\mathfrak{A}$ satisfies (i) and (ii),*

*(**C**) every $\mathscr{V}$-free algebra $\mathfrak{A}$ satisfies (iii),*

*(**D**) every algebra $\mathfrak{A}$ from $\mathscr{V}$ satisfies (iii),*

*(**E**) for every natural number n, every $(n + 2)$-ary polynomials f_1, g, f_2 and every n-ary polynomials s, t, u, v such that*

$$f_1(s(x_1, \dots, x_n), t(x_1, \dots, x_n), x_1, \dots, x_n) = g(u(x_1, \dots, x_n), v(x_1, \dots, x_n), x_1, \dots, x_n),$$
$$f_2(t(x_1, \dots, x_n), s(x_1, \dots, x_n), x_1, \dots, x_n) = g(v(x_1, \dots, x_n), u(x_1, \dots, x_n), x_1, \dots, x_n),$$

holds in $\mathscr{V}$ there exist $(n + 2)$-ary polynomials g_1, f, g_2 such that

$$f_1(t(x_1, \dots, x_n), s(x_1, \dots, x_n), x_1, \dots, x_n) = g_1(u(x_1, \dots, x_n), v(x_1, \dots, x_n), x_1, \dots, x_n),$$
$$f(s(x_1, \dots, x_n), t(x_1, \dots, x_n), x_1, \dots, x_n) = g_1(v(x_1, \dots, x_n), u(x_1, \dots, x_n), x_1, \dots, x_n),$$
$$f(t(x_1, \dots, x_n), s(x_1, \dots, x_n), x_1, \dots, x_n) = g_2(u(x_1, \dots, x_n), v(x_1, \dots, x_n), x_1, \dots, x_n),$$
$$f_2(s(x_1, \dots, x_n), t(x_1, \dots, x_n), x_1, \dots, x_n) = g_2(v(x_1, \dots, x_n), u(x_1, \dots, x_n), x_1, \dots, x_n)$$

holds in $\mathscr{V}$.

Proof. Since every algebra from $\mathscr{V}$ is a homomorphic image of a $\mathscr{V}$-free algebra, we have $\boldsymbol{A} \Rightarrow \boldsymbol{D} \Rightarrow \boldsymbol{C} \Rightarrow \boldsymbol{B} \Rightarrow \boldsymbol{A}$ by theorem 1.

$\boldsymbol{C} \Rightarrow \boldsymbol{E}$: Suppose $\boldsymbol{C}$. Let f_1, g, f_2, s, t, u, v be polynomials satisfying the first two identities. Then $[f_1(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}), f_2(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x})] \in T(s(\boldsymbol{x}), t(\boldsymbol{x}))\, T(u(\boldsymbol{x}), v(\boldsymbol{x}))\, T(s(\boldsymbol{x}), t(\boldsymbol{x}))$, where $\boldsymbol{x}$ denotes $x_1, \dots, x_n$. It is true also for the $\mathscr{V}$-free algebra over n free generators, so applying $\boldsymbol{C}$ and lemma 1 we obtain polynomials g_1, f, g_2 in request.

$\boldsymbol{E} \Rightarrow \boldsymbol{C}$: Suppose $\boldsymbol{E}$. Let $\mathfrak{A}$ be a $\mathscr{V}$-free algebra, $a, b, c, d, x, y \in |\mathfrak{A}|$, $[x, y] \in T(a, b)\, T(c, d)\, T(a, b)$. By the lemma 1, there exist natural numbers m_p, m_q, m_r and an m_p-ary polynomial h_p, an m_q-ary polynomial h_q, an m_r-ary polynomial h_r, elements $p_1, \dots, p_{m_p}, q_1, \dots, q_{m_q}, r_1, \dots, r_{m_r}$ such that

$$h_p(a, b, p_1, \dots, p_{m_p}) = x;$$
$$h_p(b, a, p_1, \dots, p_{m_p}) = h_q(c, d, q_1, \dots, q_{m_q}),$$
$$h_r(a, b, r_1, \dots, r_{m_r}) = h_q(d, c, q_1, \dots, q_{m_q}),$$
$$h_r(b, a, r_1, \dots, r_{m_r}) = y.$$

There exists a finite set of free generators of $\mathfrak{A}$, denote it $\{x_1, \dots, x_n\}$, such that $a, b, c, d, p_1, \dots, p_{m_p}, q_1, \dots, q_{m_q}, r_1, \dots, r_{m_r}$ are elements of the subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by $\{x_1, \dots, x_n\}$, which is itself a $\mathscr{V}$-free algebra with the set of free generators $\{x_1, \ldots, x_n\}$. Thus there exist n-ary polynomials s, t, u, v such that

$$a = t(x_1, \ldots, x_n),$$
$$b = s(x_1, \ldots, x_n),$$
$$c = u(x_1, \ldots, x_n),$$
$$d = v(x_1, \ldots, x_n)$$

and $(n + 2)$-ary polynomials f_1, g, f_2 such that

$$h_p(w, z, p_1, \ldots, p_{m_p}) = f_1(w, z, x_1, \ldots, x_n),$$
$$h_q(w, z, q_1, \ldots, q_{m_q}) = g(w, z, x_1, \ldots, x_n),$$
$$h_r(w, z, r_1, \ldots, r_{m_r}) = f_2(w, z, x_1, \ldots, x_n)$$

holds for abitrary elements $w, z \in |\mathfrak{B}|$. Now we substitute a, b, c, d for w, z and then $t(\boldsymbol{x}), s(\boldsymbol{x}), u(\boldsymbol{x}), v(\boldsymbol{x})$ for a, b, c, d. We obtain the first two expresions from $\boldsymbol{E}$ and

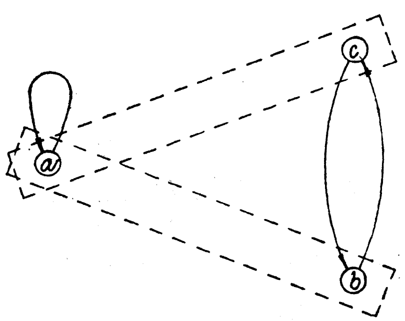
$$f_1(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}) = \boldsymbol{x},$$
$$f_2(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x}) = y.$$

Since $x_1, \ldots, x_n$ are free generators, the first two identities from $\boldsymbol{E}$ hold identically in the variety $\mathscr{V}$. Thus there exist $(n + 2)$-ary polynomials g_1, f, g_2 such that the last four identities from $\boldsymbol{E}$ hold in $\mathscr{V}$. But then $[x, y] \in T(c, d)\, T(a, b)\, T(c, d)$

Q.E.D.

The condition (*ii*) in $\boldsymbol{B}$ cannot be omitted. There exists a variety which has not trivial principal tolerances even though all free algebras of it have.

Example 3. The variety from example 1 has not trivial principal tolerances: Put $|\mathfrak{A}| = \{a, b, c\}, f = (a \mapsto a, b \mapsto c, c \mapsto b)$. Obviously $[a, c] \in T(a, b)$, but $[b, c] \notin T(a, b)$.



Example 4. The variety of distributive lattices has trivial principal tolerances (cf. [2]). We confirm that fact by proving the assertion $(\boldsymbol{E})$.

Let f_1, f_2, g, s, t, u, v be arbitrary lattice polynomials satisfying the conditions required. Denote by h_1, h_2 the following $(n + 2)$-ary lattice polynomials:

$$h_1(y, z, \boldsymbol{x}) \equiv f_1(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x}) \vee f_1(z, y, \boldsymbol{x}) \vee f_2(y, z, \boldsymbol{x}) \vee f_2(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}),$$
$$h_2(y, z, \boldsymbol{x}) \equiv f_1(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x}) \wedge f_1(z, y, \boldsymbol{x}) \wedge f_2(y, z, \boldsymbol{x}) \wedge f_2(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}).$$

It is clear that

$$h_2(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x}) \leqq h_2(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}) \leqq h_1(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}) \leqq h_1(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x}),$$

and if we denote

$$j(y, z, \boldsymbol{x}) \equiv (g(u(\boldsymbol{x}), v(\boldsymbol{x}), \boldsymbol{x}) \wedge g(y, z, \boldsymbol{x})) \vee (g(z, y, \boldsymbol{x}) \wedge g(v(\boldsymbol{x}), u(\boldsymbol{x}), \boldsymbol{x}),$$

then we obtain

$$h_1(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}) = j(u(\boldsymbol{x}), v(\boldsymbol{x}), \boldsymbol{x})$$

and

$$h_2(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}) = j(v(\boldsymbol{x}), u(\boldsymbol{x}), \boldsymbol{x}).$$

From the $\vee\wedge$-representation of h_1 and h_2 and in view of the above we conclude that there exist n-ary lattice polynomials a_1, a_2, b_1, b_2 such that

$$\begin{aligned}
h_1(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x}) &= (a_1(\boldsymbol{x}) \wedge (s(\boldsymbol{x}) \vee t(\boldsymbol{x}))) \vee b_1(\boldsymbol{x}),\\
h_1(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}) &= (a_1(\boldsymbol{x}) \wedge (s(\boldsymbol{x}) \wedge t(\boldsymbol{x}))) \vee b_1(\boldsymbol{x}),\\
h_2(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}) &= (a_2(\boldsymbol{x}) \wedge (s(\boldsymbol{x}) \vee t(\boldsymbol{x}))) \vee b_2(\boldsymbol{x}),\\
h_2(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x}) &= (a_2(\boldsymbol{x}) \wedge (s(\boldsymbol{x}) \wedge t(\boldsymbol{x}))) \vee b_2(\boldsymbol{x}),
\end{aligned}$$

be means of which we can construct the desired polynomials. First we define auxiliary $(n + 2)$-ary lattice polynomials k_1, k_2, l by

$$\begin{aligned}
k_1(y, z, \boldsymbol{x}) &\equiv j(y, z, \boldsymbol{x}) \vee ((a_1(\boldsymbol{x}) \vee a_2(x)) \wedge (s(\boldsymbol{x}) \vee t(\boldsymbol{x}))),\\
k_2(y, z, \boldsymbol{x}) &\equiv j(y, z, \boldsymbol{x}) \wedge ((((a_1(\boldsymbol{x}) \vee a_2(x)) \wedge (s(\boldsymbol{x}) \wedge t(\boldsymbol{x}))) \vee b_2(\boldsymbol{x})),\\
l(y, z, \boldsymbol{x}) &\equiv (((s(\boldsymbol{x}) \wedge y) \vee (z \wedge t(\boldsymbol{x}))) \wedge (a_1(\boldsymbol{x}) \vee a_2(\boldsymbol{x}))) \vee b_2(\boldsymbol{x}).
\end{aligned}$$

We have

$$\begin{aligned}
k_1(u(\boldsymbol{x}), v(\boldsymbol{x}), \boldsymbol{x}) &= h_1(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}) \vee ((a_1(\boldsymbol{x}) \vee a_2(\boldsymbol{x})) \wedge (s(\boldsymbol{x}) \vee t(\boldsymbol{x}))) =\\
&= ((a_1(\boldsymbol{x}) \wedge (s(\boldsymbol{x}) \wedge t(\boldsymbol{x}))) \vee b_1(\boldsymbol{x})) \vee\\
&\quad \vee ((a_1(\boldsymbol{x}) \vee a_2(\boldsymbol{x})) \wedge (s(\boldsymbol{x}) \vee t(\boldsymbol{x})))) =\\
&= ((a_1(\boldsymbol{x}) \vee a_2(\boldsymbol{x})) \wedge (s(\boldsymbol{x}) \vee t(\boldsymbol{x}))) \vee b_1(\boldsymbol{x}) =\\
&= (a_1(\boldsymbol{x}) \wedge (s(\boldsymbol{x}) \vee t(\boldsymbol{x}))) \vee (a_2(\boldsymbol{x}) \wedge (s(\boldsymbol{x}) \vee t(\boldsymbol{x}))) \vee b_1(\boldsymbol{x}) =\\
&= h_1(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x}),
\end{aligned}$$

$$\begin{aligned}
k_1(v(\boldsymbol{x}), u(\boldsymbol{x}), \boldsymbol{x}) &= h_2(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}) \vee ((a_1(\boldsymbol{x}) \vee a_2(\boldsymbol{x})) \wedge (s(\boldsymbol{x}) \vee t(\boldsymbol{x}))) =\\
&= ((a_2(\boldsymbol{x}) \wedge (s(\boldsymbol{x}) \vee t(\boldsymbol{x}))) \vee b_2(\boldsymbol{x})) \vee\\
&\quad \vee ((a_1(\boldsymbol{x}) \vee a_2(\boldsymbol{x})) \wedge (s(\boldsymbol{x}) \vee t(\boldsymbol{x}))) =\\
&= ((a_1(\boldsymbol{x}) \vee a_2(\boldsymbol{x})) \wedge (s(\boldsymbol{x}) \vee t(\boldsymbol{x}))) \vee b_2(\boldsymbol{x}) = l(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x}),
\end{aligned}$$

$$
\begin{aligned}
k_2(u(\boldsymbol{x}), v(\boldsymbol{x}), \boldsymbol{x}) &= h_1(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}) \wedge \\
&\quad \wedge (((a_1(\boldsymbol{x}) \vee a_2(\boldsymbol{x})) \wedge (s(\boldsymbol{x}) \wedge t(\boldsymbol{x}))) \vee b_2(\boldsymbol{x})) = \\
&= (((a_1(\boldsymbol{x}) \wedge (s(\boldsymbol{x}) \wedge t(\boldsymbol{x}))) \vee b_1(\boldsymbol{x})) \wedge \\
&\quad \wedge (((a_1(\boldsymbol{x}) \vee a_2(\boldsymbol{x})) \wedge (s(\boldsymbol{x}) \wedge t(\boldsymbol{x}))) \vee b_2(\boldsymbol{x})) = \\
&= ((a_1(\boldsymbol{x}) \vee a_2(\boldsymbol{x})) \wedge (s(\boldsymbol{x}) \wedge t(\boldsymbol{x}))) \vee b_2(\boldsymbol{x}) = \\
&= l(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}),
\end{aligned}
$$

$$
\begin{aligned}
k_2(v(\boldsymbol{x}), u(\boldsymbol{x}), \boldsymbol{x}) &= h_2(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}) \wedge (((a_1(\boldsymbol{x}) \vee a_2(\boldsymbol{x})) \wedge (s(\boldsymbol{x}) \wedge t(\boldsymbol{x}))) \vee b_2(\boldsymbol{x})) = \\
&= ((a_2(\boldsymbol{x}) \wedge (s(\boldsymbol{x}) \vee t(\boldsymbol{x}))) \vee b_2(\boldsymbol{x})) \wedge \\
&\quad \wedge (((a_1(\boldsymbol{x}) \vee a_2(\boldsymbol{x})) \wedge (s(\boldsymbol{x}) \wedge t(\boldsymbol{x}))) \vee b_2(\boldsymbol{x})) = \\
&= (a_2(\boldsymbol{x}) \wedge (s(\boldsymbol{x}) \wedge t(\boldsymbol{x}))) \vee b_2(\boldsymbol{x}) = \\
&= h_2(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x}).
\end{aligned}
$$

Now, the polynomials g_1, g_2, f are as follows:

$$
\begin{aligned}
g_1(y, z, \boldsymbol{x}) &\equiv (f_1(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}) \wedge k_1(y, z, \boldsymbol{x})) \vee (f_2(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x}) \wedge k_2(z, y, \boldsymbol{x})), \\
g_2(y, z, \boldsymbol{x}) &\equiv (f_1(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}) \wedge k_2(y, z, \boldsymbol{x})) \vee (f_2(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x}) \wedge k_1(z, y, \boldsymbol{x})), \\
f(y, z, \boldsymbol{x}) &\equiv (f_1(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}) \wedge l(y, z, \boldsymbol{x})) \vee (f_2(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x}) \wedge l(z, y, \boldsymbol{x})).
\end{aligned}
$$

Indeed, this construction yields

$$
\begin{aligned}
g_1(u(\boldsymbol{x}), v(\boldsymbol{x}), \boldsymbol{x}) &= (f_1(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}) \wedge h_1(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x})) \vee \\
&\quad \vee (f_2(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x}) \wedge h_2(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x})) = \\
&= f_1(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}),
\end{aligned}
$$

$$
\begin{aligned}
g_1(v(\boldsymbol{x}), u(\boldsymbol{x}), \boldsymbol{x}) &= (f_1(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}) \wedge l(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x})) \vee \\
&\quad \vee (f_2(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x}) \wedge l(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x})) = \\
&= f(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x}),
\end{aligned}
$$

$$
\begin{aligned}
g_2(u(\boldsymbol{x}), v(\boldsymbol{x}), \boldsymbol{x}) &= (f_1(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}) \wedge l(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x})) \vee \\
&\quad \vee (f_2(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x}) \wedge l(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x})) = \\
&= f(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}),
\end{aligned}
$$

$$
\begin{aligned}
g_2(v(\boldsymbol{x}), u(\boldsymbol{x}), \boldsymbol{x}) &= (f_1(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}) \wedge h_2(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x})) \vee \\
&\quad \vee (f_2(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x}) \wedge h_1(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x})) = \\
&= f_2(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x}).
\end{aligned}
$$

Noted by the referee. For motivation and some other results see [3], which has appeared in the meantime.

REFERENCES

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