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TRANSITIVELY ACTING MONOIDS OF LOCAL AUTOMORPHISMS OF LOCALLY FINITE TREES

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INTRODUCTION

Charles Wells characterizes in paper [10] among others locally finite trees with the transitive action of monoids of their local automorphisms (i.e. isotone transformations restrictions of which onto bounded inervals are order isomorphisms) and automorphism groups (Theorem 1) on the supports of these trees. By a nontrivial locally finite tree (a forest in more convenient terminology), is meant at least a two-element partially ordered set whose all dual principal ideals are well ordered having the ordinal number at most ω (i.e. the ordinal of the set N of all positive integers). This paper, using results of papers [5], [6], [10] and [11], is concerned with certain algebraic and topological characterizations of locally finite and lower forests with transitively acting local automorphism monoids on their carrier sets. The significance of main theorems also consists in the fact that they show that under assumption of the transitive action of local automorphism monoids on locally finite forests these can be endowed by topologies of Alexandroff [2], [3] (i.e. quasi-discrete in the sense of [4] chapt. V.) such that the local automorphism monoid of the forest coincides with the monoid of all local homeomorphisms or with the monoid of all closed continuous selfmaps of the space in question. Moreover for any pair of different points there are topologies with the above mentioned properties semiseparating these points.

1. PRELIMINARIES

The terminology concerning trees used in literature (from the various points of view) is not unique—cf. [7], [10]. In accordance with [10] we say that an ordered set $(T, \leqq)$ is an *upper locally finite forest* if every its dual principal ideal is well ordered with the ordinal at most ω. An ordered set $(T, \leqq)$ is said to be a *lower locally finite forest* if $(T, \leqq^{-1})$ is an upper locally finite forest. A connected upper (lower) locally finite forest is called an *upper* (*lower*) *locally finite tree*. Thus an upper locally finite tree is an ordered set which is an upper semilattice whose all bounded intervals are finite chains. The root of an upper (lower) tree is the greatest (least) element of this tree. We say that a forest is *antirooted* of none of its maximal trees has a root. By Max $(T, \leqq)$ (Min $(T, \leqq)$) we denote the set of all maximal (minimal) elements of $(T, \leqq)$. The principal (dual) ideal of $(T, \leqq)$ generated by an element $x \in T$ is denoted by $(x]_{\leqq}$ or $(x]$ ($[x)_{\leqq}$ or $[x)$). Interval with the initial element s and the terminal element t, i.e. the set $\{x \mid s \leqq x \leqq t\}$ is denoted by $[s, t]$; $s \prec t$ means $[s, t] \neq \{s, t\}$.

An isotone selfmap f of a forest (tree) $(T, \leqq)$ is said to be a *local automorphism* of $(T, \leqq)$ if for any pair of elements $s, t \in T$, $s < t$ the restriction $f \mid [s, t]$ is an isomorphism of the chain $[s, t]$ onto the chain $[f(s), f(t)]$. The monoid of all local automorphisms of $(T, \leqq)$ will be denoted by LA$(T, \leqq)$. As usually a monoid S (with the unity e) is said to be acting on a set X if there is given a mapping, $\pi : S \times X \to X$ such that $\pi(e, x) = x$, $\pi(s_1 s_2, x) = \pi(s_1, \pi(s_2, x))$ for all $x \in X$, $s_1, s_2 \in S$. For a submonoid F of the symmetric monoid $(X^X, .)$ of X we put $\pi(f, x) = f(x)$ for $f \in F$, $x \in X$. The monoid F *acts transitively* on X if for any pair of elements $x_1, x_2 \in X$ there is $f \in F$ such that $f(x_1) = x_2$.

Let f be a transformation—i.e. a selfmap—of a set T. The monoid $C_T(f) = \{g \in T^T \mid fg = gf\}$ is called a centralizer of the transformation f (in the symmetric monoid $(T^T, .)$). In the agreement with [6] we define a binary operation $\circ$ on a locally finite tree $(T, \leqq)$ as follows: Suppose $(T, \leqq)$ is an upper locally finite tree. For $s, t \in T$ we put $\delta(s, t) = \text{card}\,[t, \sup\{s, t\}] - \text{card}\,[s, \sup\{s, t\}]$ (both cardinals on the right hand are finite). Then we put $s \circ t = t^+$ (where t^+ is the successor of t) if $\delta(s, t) \geqq 0$ and $t \notin \max(T, \leqq)$. If t is the greatest element of $(T, \leqq)$ then $s = t$ and we put $s \circ t = t$. Further, $s \circ t = s^+$ whenever $\delta(s, t) < 0$. If $(T, \leqq)$ is a lower locally finite tree we consider the tree $(T, \leqq^{-1})$ and the multiplication of its elements is defined with respect to the ordering $\leqq^{-1}$ which creats the structure of an upper tree on T. The endomorphism monoid of the groupoid $(T_t, \circ)$ will be denoted by End $(T_t, \circ)$.

The notion of (finite) tree algebra was introduced by Ladislav Nebeský in [8]. The generalization of this notion for the case of infinite supports is investigated by Bohdan Zelinka in [11] where among others he characterizes tree algebras realized by (unoriented) tree graphs. We recall first some necessary terms from [11]. A *tree algebra* $\mathscr{A} = (T, P)$ is an algebra with the support T and with one ternary operation P which satisfies the following conditions for arbitrary elements t, u, v, w of T:

1. $P(t, t, u) = t$,
2. $P(t, u, v) = P(u, t, v) = P(t, v, u)$,

3. $P(P(t, u, v), u, w) = P(t, u, P(v, u, w))$,

4. $P(t, u, w) \neq P(u, v, w) \neq P(t, v, w) \Rightarrow P(t, u, w) = P(t, v, w)$.

Let $\mathscr{A} = (T, P)$ be a tree algebra, $t, u \in T$. The bounded segment of $\mathscr{A}$ determined by t and u is the set $S(t, u) = \{x \in T \mid P(t, u, x) = x\}$. Let $x_0, x_1, x_2, \ldots$ be an infinite sequence of pairwise different elements of T with these properties: $S(x_0, x_i) \subset S(x_0, x_{i+1})$ for each positive integer i and for each $y \in T$ there exists a positive integer i such that $x_i \notin S(x_0, y)$. Then the set $D_{x_0} = \bigcup_{i=1}^{\infty} S(x_0, x_i)$ is called an unbounded segment of $\mathscr{A}$ with the initial element x_0. A tree algebra $\mathscr{A}$ is called discrete if the segment $S(u, v)$ in $\mathscr{A}$ for any two elements t, u of its support is finite. Theorem 12 from [11] says that there is one-to-one correspondence between exactly discrete tree algebras and tree graphs $\mathrm{G} = (T, H)$ given by the rule: $P(t, u, v)$ is the (single) common vertex of the path in G connecting t and u, the path in G connecting t and v and the path in G connecting u and v.

Let $(T, \leqq)$ be a locally finite forest in the sense of the above definition. Denote by $(T, \varrho_{\leqq})$ the reflexion of an orientated graph (in the category of orientated graphs and homomorphisms) determinated by the symmetrization, i.e. $t\varrho_{\leqq}s$ iff $t \leqq s$ or $s \leqq t$. If $\{(T_\iota, \leqq) \mid \iota \in I\}$ is the family of all maximal trees of $(T, \leqq)$ then by $\mathscr{A}_\iota = (T_\iota, P_\iota)$, for $\iota \in I$, we denote the discrete tree algebra corresponding to the (unorientated) tree graph $(T_\iota, \varrho_{\leqq})$. Suppose $\mathscr{A} = (T, P)$ contains at least one unbounted segment D_t and denote by $E(D_t)$ the end of $\mathscr{A}$ determined by D_t, i.e. the set of all unbounded segments of $\mathscr{A}$ such that the intersection of each of them with D_t is also an unbounded segment. Define on $E(D_t)$ a ternary operation P^* in this way: $P^*(D_u, D_v, D_w) = D_u \cap D_v \cap D_w$ for any triad D_u, D_v, D_w of elements from $E(D_t)$. It is easy to verify that $(E(D_t), P^*)$ is a tree algebra—called an end tree algebra determined by the unbounded segment D_t of the algebra $\mathscr{A}$—isomorphic to the algebra $\mathscr{A}$; the corresponding isomorphism φ: $(T, P) \rightarrow (E(D_t), P^*)$ is defined by $\varphi(x) = D_x \in E(D_t)$ for any $x \in T$.

Concerning the retract theory for general ordered sets see [9] and the other papers quoted in the bibliography of [9]. A subset Q of an ordered set $(S, \leqq)$ is a retract of $(S, \leqq)$ if there is an order preserving map $g : (S, \leqq) \rightarrow (Q, \leqq)$ (called a retraction) which is the identity map on Q. In the agreement with this notion an ordered subset $(A, \leqq)$ of a locally finite forest $(T, \leqq)$ is said to be an LA-*retract* of $(T, \leqq)$ if there is a local automorphism $g : (T, \leqq) \rightarrow (A, \leqq)$—called an LA-*retraction*—such that $g \mid A = \mathrm{id}_A$. It is to be noted that some results from the retract theory of ordered sets are transferable onto the considered case, e.g. every maximal chain of a locally finite tree is its LA-retract (cf. [9], p. 104). The minimality of an LA-retract means the minimality with respect to the natural ordering by set inclusion.

By an Alexandroff topological space (called also quasi-discrete), we mean in the agreement with [2], [3] a pair (X, τ) where X is a set and τ is a completely additive topological closure operation on X. Quasi-discrete T_0-spaces were investigated more in detail probably for the first time in [1]; from here it follows the utility of the notion of continuous closed mappings (simplicial mappings) of such spaces. The monoid of all closed deformations (i.e. closed continuous selfmaps) of a topological space (X, τ) with the usual composition of mappings as the multiplication is denoted by $\mathrm{S}(X, \tau)$. A mapping f of (X, τ) into (Y, σ) is called a local homeomorphism if for any point $x \in X$ there exists a τ-neighbourhood O_x of x such that the restriction $f \mid O_x$ is a homeomorphism of O_x onto $f(O_x)$. The monoid of all local homeomorphisms of (X, τ) into itself will be denoted by $\mathrm{LH}(X, \tau)$. Points x, y of a topological space are called semiseparated ([4]) if there are neighbourhoods O_x, O_y of x, y respectively such that $y \notin O_x$ and $x \notin O_y$. If moreover $O_x \cap O_y = \emptyset$ then points x, y are called separated. A topological space is said to be perfect if it does not contain isolated points.

Ends of proofs are denoted by the symbol $\square$.

2. CHARACTERIZATIONS OF THE TRANSITIVE ACTION

Main results of this paper are formulated in this paragraph.

2.1. Theorem. *Let $(T, \leqq)$ be a locally finite upper or lower forest, $\{(T_\iota, \leqq) \mid \iota \in I\}$ be the family of all maximal trees of $(T, \leqq)$. The following conditions are equivalent:*

1° *The monoid* $\mathrm{LA}(T, \leqq)$ *acts transitively on the set T.*

2° *The ordered set $(T, \leqq)$ has no maximal or minimal elements.*

3° *For any $\iota \in I$ and each element t of the discrete tree algebra $\mathscr{A}_\iota = (T_\iota, P_\iota)$ there exist $\leqq$-comparable elements $x, y \in T_\iota$, $x \neq t \neq y$ such that $P_\iota(x, y, t) = t$.*

4° *For any $\iota \in I$ and any element $t \in T_\iota$ there exist at least two different end tree algebras $(E(D_t), P^*)$, $(E(D_t'), P^*)$ such that $D_t' \cup D_t$ is the minimal* LA*-retract of the tree $(T_\iota, \leqq)$.*

5° *For any $\iota \in I$, $(T_\iota, \circ)$ is a simple groupoid with the property* $\mathrm{End}(T_\iota, \circ) = \mathrm{LA}(T_\iota, \leqq)$.

2.2. Theorem. *Let $\mathscr{A} = (M, P)$ be a discrete tree algebra, $\leqq$ be an ordering on the set M such that $(M, \leqq)$ is an upper or lower semilattice. The automorphism group* $\mathrm{Aut}(M, \leqq)$ *acts transitively on M iff the following two conditions are satisfied:*

(i) *For any element $t \in M$ there exist $\leqq$-comparable elements $x, y \in M$, $x \neq t \neq y$ such that $P(x, y, t) = t$.*

(ii) *For any pair of elements $t, u \in M$ we have* $\mathrm{card}\{x \mid x \prec t\} = \mathrm{card}\,\{x \mid x \prec u\}$ *if $(M, \leqq)$ is an upper semilattice, or* $\mathrm{card}\,\{x \mid t \prec x\} = \mathrm{card}\,\{x \mid u \prec x\}$ *if $(M, \leqq)$ is a lower semilattice.*

2.3. Theorem. *Let $(T, \leqq)$ be a locally finite upper or lower antirooted forest, $\{(T_\iota, \leqq) \mid \iota \in I\}$ be the family of all maximal trees of $(T, \leqq)$. The following conditions are equivalent:*

1° *The monoid* $\mathrm{LA}(T, \leqq)$ *acts transitively on the set* T.

2° *There exists a perfect Alexandroff topology* τ *on the set* T *such that* $\mathrm{LA}(T, \leqq) = \mathrm{S}(T, \tau)$.

3° *For any pair of different elements* $a, b \in T$ *there exists a perfect Alexandroff topology* $\tau_{a,b}$ *on the set* T *such that* $\mathrm{LA}(T, \leqq) = \mathrm{S}(T, \tau_{a,b})$ *and points* a, b *are semiseparated in the space* $(T, \tau_{a,b})$.

4° *There exists an Alexandroff topology* σ *on the set* T *such, that* $\mathrm{LA}(T, \leqq) = \mathrm{LH}(T, \sigma)$ *and the cardinality of the* σ*-closure of any nonempty subset of* T *is infinite.*

5° *For any pair of different elements* $a, b \in T$ *there exists an Alexandroff topology* $\sigma_{a,b}$ *on* T *having the properties from the conditions* 4° *and such that points* a, b *are semiseparated in the space* $(T, \sigma_{a,b})$.

2.4. Remark. The monoid of local homeomorphisms $\mathrm{LH}(T, \sigma)$ from conditions 4°, 5° of Theorem 2.3 can be replaced by the monoid of open deformations (i.e. open continuous selfmaps of the considered space) and assertions remain valid. We show now that Theorem 2.3 is justified, i.e. used notions-as closed deformation, open deformation, local homeomorphism do not coincide in the case of Alexandroff – i.e. quasi-discrete spaces.

Consider the set of all nonnegative integers $\mathbf{N}$ and the left order topology τ^- on $\mathbf{N}$; thus closures of singletons $\{n\} \subset \mathbf{N}$ are sets $\tau^-\{n\} = \{n, n+1, n+2, \ldots\}$. Define a mapping $f: \mathbf{N} \to \mathbf{N}$ by putting $f(0) = 1$, $f(n) = n$ for $n > 1$. Evidently $f \in \mathrm{S}(\mathbf{N}, \tau^-)$, $f \notin \mathrm{LH}(\mathbf{N}, \tau^-)$ and f is not an open deformation of the space $(\mathbf{N}, \tau^-)$ as well. But on the other hand f is an open deformation of the dual space $(\mathbf{N}, \tau^+)$, $f \notin \mathrm{S}(\mathbf{N}, \tau^+)$ and simultaneulosly $f \notin \mathrm{LH}(\mathbf{N}, \tau^+)$. Further, put $g(n) = n + 1$ for each $n \in \mathbf{N}$. The mapping g is a topological embedding of the space $(\mathbf{N}, \tau^-)$ into itself, hence $g \in \mathrm{LH}(\mathbf{N}, \tau^-) \cap \mathrm{LH}(\mathbf{N}, \tau^+)$. However g is not an open deformation of the space $(\mathbf{N}, \tau^-)$ and $g \notin \mathrm{S}(\mathbf{N}, \tau^+)$ as well.

3. PROOFS OF CHARACTERIZATION THEOREMS

Proofs of theorems introduced in the previous paragraph will be divided into the sequence of proofs of auxiliary assertions.

3.1. Lemma. *Let* $(T, \leqq)$ *be an upper or lower locally finite forest. The monoid* $\mathrm{LA}(T, \leqq)$ *acts transitively on the set* T *iff* $\mathrm{Max}\,(T, \leqq) \cup \mathrm{Min}\,(T, \leqq) = \emptyset$.

Proof. The assertion follows immediately from [10] – assertion (a) of Theorem 1. □

3.2. Lemma. *Let* $\{(T_\iota, \leqq) \mid \iota \in I\}$ *be the family of all maximal trees of a locally finite forest* $(T, \leqq)$. *The following conditions are equivalent:*

(i) $\mathrm{Max}\,(T, \leqq) \cup \mathrm{Min}\,(T, \leqq) = \emptyset$.

(ii) *For any $\iota \in I$ and each element t of the discrete tree algebra $\mathscr{A}_\iota = (T_\iota, P_\iota)$ there exist $\leqq$-comparable elements $x, y \in T_\iota$, $x \neq t \neq y$ such that $P_\iota(x, y, t) = t$.*

(iii) *For any $\iota \in I$ and any element $t \in T_\iota$ there exist at least two different end tree algebras $(E(D_t), P^*)$, $(E(D_t'), P^*)$ such that $D_t' \cup D_t$ is the minimal* LA*-retract of the tree $(T_\iota, \leqq)$.*

Proof. Implication (i) $\Rightarrow$ (ii) follows immediately from the definition of the discrete tree algebra $\mathscr{A}_\iota$ realized by the tree $(T_\iota, \leqq)$; condition (ii) says in other words that any element of the tree algebra $\mathscr{A}_\iota$ is an internal element of a suitable bounded segment which is a chain with respect to the ordering $\leqq$.

(ii) $\Rightarrow$ (iii): Let $\iota \in I$ be an arbitrary index, $t \in T_\iota$ be an arbitrary element. Condition (ii) implies that the degree of the vertex t of the tree graph (T_ι, ϱ_ι) is at least 2 and moreover $x \prec t \prec y$ for a suitable notation. Using the mathematical induction we construct chains D_t, D_t' of the type ω, ω^* respectively such that $D_t' \cup D_t$ is a chain (with respect to $\leqq$) of the type $\omega^* + \omega$, hence tree algebras $(E(D_t), P^*)$, $(E(D_t'), P^*)$ are different. For any element $x \in T_\iota$ denote by $f(x)$ the unique element of $D_t' \cup D_t$ such that $\delta(x, f(x)) = 0$. Then $f: T_\iota \to D_t' \cup D_t$ is an LA-retraction and $D_t' \cup D_t$ is evidently a minimal LA-retract of the tree $(T_\iota, \leqq)$.

(iii) $\Rightarrow$ (i): We prove this implication for an upper forest. The proof for the case of a lower forest is similar. Admit first $\operatorname{Max}(T, \leqq) \neq \emptyset$. For $a \in \operatorname{Max}(T, \leqq)$ denote by $(T_i, \leqq)$ the tree the greatest element of which is a. According to (iii) the tree $(T_\iota, \leqq)$ contains at least two different infinite chains D_a, D_a' (of the type ω^*) each of which is an LA-retract of $(T_\iota, \leqq)$ (the corresponding retractions are defined similarly as above). Hence $D_a' \cup D_a$ can not be a minimal retract.

Now admit $\operatorname{Max}(T, \leqq) = 0$ and $\operatorname{Min}(T, \leqq) \neq 0$. Let $b \in \operatorname{Min}(T, \leqq)$ be an arbitrary element and $\iota \in I$ an index such that $b \in T_i$. By (iii) there exist two different unbounded segments D_b, D_b' of the tree algebra $\mathscr{A}_\iota = (T_\iota, P_\iota)$ such that $D_b' \cup D_b$ is a minimal LA-retract of $(T_\iota, \leqq)$. Then $D_b' \cup D_b$ has no upper bound. Since in the opposite case, if c denotes an arbitrary upper bound of $D_b' \cup D_b$ then any local automorphism g assigning to an element $x > a$ an element $g(x) \in D_b' \cup D_b$, is not an LA-retraction. Since $D_b' \cup D_b$ is unbounded (as a subset of $(T_\iota, \leqq)$), exactly one of segments D_b, D_b' – say D_b is a chain of the type ω. Since $D_b \neq D_b'$, the set $D_b \cap D_b'$ is finite. Denote by d the greatest element of $D_b \cap D_b'$ and put $R = (D_b \div D_b') \cup \{d\}$ (where $\div$ means the symmetric difference of sets). Since $(R, \leqq)$ is a chain of the type $\omega^* + \omega$, it is an LA-retract of $(T_\iota, \leqq)$. But $R \subsetneqq D_b' \cup D_b$, which is a contradiction. Therefore $\min(T, \leqq) \cup \max(T, \leqq) = \emptyset$. $\square$

3.3. Lemma. *Let $(T, \leqq)$ be a locally finite forest. Then* $\max(T, \leqq) \cup \min(T, \leqq) = 0$ *iff for every maximal tree $(T_\iota, \leqq)$ of $(T, \leqq)$ the groupoid $(T_\iota, \circ)$ is simple and* $\operatorname{End}(T_\iota, \circ) = \operatorname{LA}(T_\iota, \leqq)$.

Proof. (We consider the case of an upper forest only; for the case of lower forests the proof is quite similar).

Suppose $\text{Max}(T, \leqq) \cup \text{Min}(T, \leqq) = \emptyset$. Let $(T_\iota, \leqq)$ be an arbitrary maximal tree of $(T, \leqq)$, $A \neq \emptyset$ be a both-side ideal of the groupoid $(T_\iota, \circ)$. Admit $T_\iota \setminus A \neq \emptyset$ and $t_0 \in T_\iota \setminus A$. Since t_0 is not a minimal element of $(T_\iota, \leqq)$, there is $t_1 \in T_\iota$, such that $t_1^+ = t_0$ and $t_1 \notin A$ (for $t_1 \circ t_1 = t_0$). Since $t \circ t = t^+$ for each element $t \in T$, we have A is a dual ideal of the ordered set $(T_\iota, \leqq)$, thus $t_0 < a$ for some $a \in A$. Then $a \circ t_1 = t_1 \circ a = t_0 \notin A$, a contradiction. Hence $A = T_\iota$.

Now we prove the equality $\text{LA}(T_\iota, \leqq) = \text{End}(T_\iota, \circ)$. Suppose $f \in \text{LA}(T_\iota, \leqq)$, $a, b \in T_\iota$. If $\delta(a, b) \leqq 0$ we have $a \circ b = a^+, f(a \circ b) = f(a^+)$. Put $c = a \vee b$. Then $\text{card}[f(a), f(c)] = \text{card}[a, c] \geqq \text{card}[b, c] = \text{card}[f(b), f(c)]$, thus $\delta(f(a), f(b)) \leqq$ $\leqq 0$ and from here $f(a) \circ f(b) = (f(a))^+$. Since f maps the two-element chain $[a, a^+]$ isomorphically onto the two-element chain $[f(a), f(a^+)]$, we have $(f(a))^+ =$ $= f(a^+)$, thus $f(a \circ b) = f(a) \circ f(b)$, i.e. $\text{LA}(T_\iota, \leqq) \subset \text{End}(T_\iota, \circ)$. Suppose now $f \in \text{End}(T_\iota, \circ)$, $a, b \in T_\iota$, $a < b$. Let $t_0 < t_1 < \ldots < t_n$ be a chain such that $t_i \in T_\iota$, $t_0 = a$, $t_n = b$, $t_{i+1} = t_i^+$ for $i = 0, 1, \ldots, n-1$. Then $t_0 \circ t_0 = t_0^+ = t_1$, consequently $f(t_1) = f(t_0 \circ t_0) = f(t_0) \circ f(t_0) = (f(t_0))^+$. From here we get that the interval $[a, b]$ of the tree $(T_\iota, \leqq)$ is isomorphic with the interval $[f(a), f(b)]$, hence $f \in \text{LA}(T_i, \leqq)$, i.e. $\text{End}(T_\iota, \circ) \subset \text{LA}(T_\iota, \leqq)$. Therefore $\text{LA}(T_\iota, \leqq) =$ $= \text{End}(T_\iota, \circ)$.

Now admit $\text{Min}(T_\iota, \leqq) \neq \emptyset$ for some maximal subtree of the forest $(T, \leqq)$. Suppose $t_0 \in \text{Min}(T_\iota, \leqq)$ and put $A = T_\iota \setminus \{t_0\}$. Then for every pair of elements $t \in T$, $a \in A$ we have $t \circ a \in A$, $a \circ t \in A$, which means that A is a proper ideal of the groupoid $(T_\iota, \circ)$, which contradicts the assumption. Hence $\text{Min}(T_\iota, \leqq) = \emptyset$. Admit the tree $(T_\iota, \leqq)$ has the greatest element s. By the assumption $\text{LA}(T_\iota, \leqq) =$ $= \text{End}(T_\iota, \circ)$ and $s \circ s = s$, consequently $f(s) = f(s \circ s) = f(s) \circ f(s)$, i.e. $f(s) = s$ for every local automorphism f of the tree $(T_\iota, \leqq)$. Since $\text{Min}(T_\iota, \leqq) = \emptyset$, the tree $(T_\iota, \leqq)$ contains a decreasing chain of the type ω^* with the greatest element s, $s = t_0 > t_1 > \ldots$ Put $g(t_i) = t_{i+1}$ for $i = 1, 2, \ldots$, and $g(t) = g(t_i)$ for any element $t \in T_i$ with the property $\delta(t_i, t) = 0$. Then $g \in \text{LA}(T_\iota, \leqq)$, but $g(s) = t_1 \neq s$, a contradiction. Hence $\text{Max}(T, \leqq) \cup \text{Min}(T, \leqq) = \emptyset$. $\square$

3.4. Lemma. *Let* $(T, \leqq)$ *be a locally finite upper (lower) forest. If* $\text{Max}(T, \leqq) = \emptyset$ $(\text{Min}(T, \leqq) = \emptyset)$ *then there exists a transformation* $f \in T^T$ *with the property* $C_T(f) = \text{LA}(T, \leqq)$.

Proof. A locally finite upper forest $(T, \leqq)$ without maximal elements is a functional graph of some transformation f of the set T (without fixed points). For any element $t \in T$ we have $f(t) = t^+$. Suppose $g \in C_T(f)$, $t \in T$. Then $(g(t))^+ =$ $= f(g(t)) = g(f(t)) = g(t^+)$. Using the mathematical induction we get that for every pair $t_1, t_2 \in T$, $t_1 < t_2$ the restriction $g \mid [t_1, t_2]$ is an order isomorphism of the interval $[t_1, t_2]$ onto the interval $[g(t_1), g(t_2)]$. If $g \in \text{LA}(T, \leqq)$, $t \in T$, then $f(g(t)) = (g(t))^+ = g(t^+) = g(f(t))$, thus $g \in C_T(f)$. Therefore we have $C_T(f) =$ $= \text{LA}(T, \leqq)$ in the case of an upper forest. If $(T, \leqq)$ is a lower forest then each of its elements has at most one predecessor. For $t \in T \setminus \mathrm{Min}\,(T, \leqq)$ we put $f(t) = s$, where $s^+ = t$; if $t \in \mathrm{Min}\,(T, \leqq)$ we put $f(t) = t$. The other part of the proof is similar to the above one. □

3.5. Remark. With respect to Lemma 3.4 we get easily the following assertion: If $(T, \leqq)$ is an upper locally finite forest, $f : T \to T$ is a mapping such that $f(t) = t^+$ for every $t \in T \setminus \mathrm{Max}\,(T, \leqq)$ and $f(t) = t$ for $t \in \mathrm{Max}\,(T, \leqq)$ then $\mathrm{C}_T(f) = \mathrm{LA}(T, \leqq)$ iff $\mathrm{Max}\,(T, \leqq) = \emptyset$.

3.6. Lemma. *Let $(T, \leqq)$ be a locally finite tree without maximal and minimal elements. For every pair of different elements $a, b \in T$ there exists a perfect Alexandroff topology $\tau_{a,b}$ on T such that* $\mathrm{LA}(T, \leqq) = \mathrm{S}(T, \tau_{a,b})$ *and points a, b are semiseparated in the space $(T, \tau_{a,b})$.*

Proof. If $(T, \leqq)$ is an upper locally finite tree, we put $f_{\leqq}(x) = x^+$ for any $x \in T$; if $(T, \leqq)$ is a lower tree, we put $f_{\leqq}(x) = x^{+*}$ where x^{+*} is a successor of x with respect to the ordering $\leqq^{-1}$ (inverse to $\leqq$). Thus we can suppose without lose of generality that $(T, \leqq)$ is an upper locally finite tree. In what follows we write only f instead of $f_{\leqq}$.

Let a, b be arbitrary elements. Assume $a \parallel b$ in $(T, \leqq)$. Put $\tau_{a,b} = \tau_f$ ([5]) p. 84) i.e. $\tau_{a,b}X = \tau_f X = X \cup \bigcup_{2 \leqq k < \omega} f^k(X)$ for any subset X of T. Since $\mathrm{Min}\,(T, \leqq) = \{x \mid f^{-1}(x) = \emptyset\} = \emptyset$, the space $(T, \tau_{a,b})$ does not contain isolated points and points a, b are separated (e.g. their $\tau_{a,b}$-neighbourhoods formed by principal ideals $(a]_{\leqq}, (b]_{\leqq}$ are disjoint) and thus semiseparated. By Lemma 3.4 we have $\mathrm{C}_T(f) = \mathrm{LA}(T, \leqq)$ and by [5] Theorem 3.3 the equality $\mathrm{C}_T(f) = \mathrm{S}(T, \tau_{a,b})$ holds. Hence $\mathrm{LA}\,(T, \leqq) = \mathrm{S}(T, \tau_{a,b})$. Now suppose a, b are comparable. Assume $a < b$ (the case $b < a$ is similar). If $a^+ = b$, we put $\tau_{a,b} = \tau_f$ again. Since the least $\tau_{a,b}$-neighbourhood $O_b = (b] \setminus \{x \mid x \prec b\}$ of the point b does not contain the point a, these points are semiseparated. Assume $n \geqq 2$, $a \prec a_1 \prec a_2 \prec \dots \prec a_{n-1} \prec b$. We put $\tau_{a,b} = \tau_f^{(n)}$ ([5] p. 84), i.e. $\tau_f^{(n)}X = X \cup \bigcup_{n+1 \leqq k < \omega} f^k(X)$ for any subset X of T. Since the least neighbourhood of an arbitrary point x of the space $(T, \tau_f^{(n)})$ is the set $O_x = \{t \mid f^k(t) = x, k \in \mathrm{N} \setminus \{1, 2, \dots, n\}\}$, we have $a \notin O_b$, hence points a, b are semiseparated in the space $(T, \tau_f^{(n)})$. Further, $\mathrm{Min}\,(T, \leqq) = \emptyset$, consequently the space $(T, \tau_f^{(n)})$ is perfect.

Now we prove the equality $\mathrm{C}_T(f) = \mathrm{S}(T, \tau_f^{(n)})$. Suppose $g \in \mathrm{C}_T(f)$, $t \in T$. We have $g\tau_f^{(n)}\{t\} = \{gf^k(t) \mid k = 0, n+1, n+2, \dots\} = \{f^k g(t) \mid k = 0, n+1, n+2, \dots\} = \tau_f^{(n)}\{g(t)\}$ and thus $g\tau_f^{(n)}X = g(\tau_f^{(n)} \bigcup_{t \in X} \{t\}) = g(\bigcup_{t \in X} \tau_f^{(n)}\{t\}) = \bigcup_{t \in X} \tau_f^{(n)}\{g(t)\} = \tau_f^{(n)} \bigcup_{t \in X} \{g(t)\} = \tau_f^{(n)} g(X)$. Evidently $g\tau_f^{(n)}\emptyset = \tau_f^{(n)} g(\emptyset)$, thus $g \in \mathrm{S}(T, \tau_f^{(n)})$. Hence $\mathrm{C}_T(f) \subset \mathrm{S}(T, \tau_f^{(n)})$. Now assume $g \in \mathrm{S}(T, \tau_f^{(n)})$, $t \in T$. Admit $f^{n+2}g(t) \lneqq gf^n(t)$. Since $g\tau_f^{(n)}\{t\} = \tau_f^{(n)}\{g(t)\}$ and $g\tau_f^{(n)}\{f^n(t)\} = \tau_f^{(n)}\{gf^n(t)\}$, we have $g^{-1}([f^{n+1}g(t))_{\leqq} \setminus ([f^{n+1}gf^n(t))_{\leqq} \cup \{gf^n(t)\})) \subset [f^{n+1}(t))_{\leqq} \setminus [f^{n+1}f^n(t))_{\leqq} = [f^{n+1}(t))_{\leqq} \setminus [f^{2n+1}(t))_{\leqq}$.

If we admit the existence of an element $s \in [f^{2n+1}(t))_{\leq}$ such that $g(s) \in [f^{n+1}g(t))_{\leq} \setminus$ $\setminus ([f^{n+1}gf^n(t))_{\leq} \cup \{gf^n(t)\})$, we get $g\tau_f^{(n)}\{f^n(t)\} \neq \tau_f^{(n)}\{gf^n(t)\}$. But card $[[f^{n+1}(t))_{\leq} \setminus$ $\setminus [f^{2n+1}(t))_{\leq}] = n$, card $[[f^{n+1}g(t))_{\leq} \setminus ([f^{n+1}gf^n(t))_{\leq} \cup \{gf^n(t)\}) = n +$ $+ \text{card}\,\{x \in T \mid f^{n+1}g(t) \leqq x < gf^n(t)\} \geqq n + 1$, a contradiction. Thus for any element $t \in T$ the quality

$$(1) \qquad gf^n(t) \leqq f^n g(t)$$

holds. Suppose $gf^n(t) = f^{n+1}g(t)$. Since $g(x) \in \{f^{n+1}g(t)\} \cup [f^{2n+2}g(t))_{\leq}$ for each $x \in [f^{2n+1}(t))_{\leq}$, we have $g[f^{n+1}(t), f^{2n}(t)] = [f^{n+2}g(t), f^{2n+1}g(t)]$. Consider the element $gf^{n+1}(t)$. Since $f^{2n+2}g(t) \notin \tau_f^{(n)}\{gf^{n+1}(t)\}$, simultaneously $\{f^{n+1}g(t)\} \cup$ $\cup [f^{2n+2}g(t))_{\leq} \subset g[f^{2n+1}(t))_{\leq}, f^{2n+2}g(t) < g(x)$ for any $x \in [f^{2n+2}(t))_{\leq}$, we have $gf^{2n+1}(t) = f^{2n+2}g(t)$. Therefore we get the inclusion

$$(2) \qquad g[f^{n+1}(t))_{\leq} \subset [f^{n+2}g(t))_{\leq}.$$

Further, $f^{n+1}g(t) \in \tau_f^{(n)}\{g(t)\}$, $g \in S(T, \tau_f^{(n)})$ thus for some element $x \in \tau_f^{(n)}\{t\}$ it holds $g(x) = f^{n+1}g(t)$, which contradicts the inclusion (2) for $\tau_f^{(n)}\{t\} = \{t\} \cup$ $\cup [f^{n+1}(t))_{\leq}$.

Suppose $gf^n(t) \in [fg(t), f^{n-1}g(t)]$. Let $u \in T$ be an element such that $f(u) = t$. Admit $g(u) < g(t)$. Since $f^n g(t) \in \tau_f^{(n)}\{g(u)\}$, we have $g^{-1}(f^n g(t)) \cap [f^{n+1}(t))_{\leq} \neq \emptyset$, further $x \in T, f^{n+1}(t) \leqq x$ implies $g(x) \notin [fg(t), f^n g(t)]$, hence $g(t) \leqq g(u)$, a contradiction. The assumption $gf^n(t) < g(t)$ leads to a contradiction again. Indeed: Admit $gf^n(t) < g(t)$. Then $f^n g(t) \in \tau_f^{(n)}\{gf^n(t)\}$, thus there exists $x_0 \in \tau^{(n)}\{f^n(t)\}$ such that $g(x_0) = f^n g(t)$. But $x_0 \in \tau_f^{(n)}\{t\}$ for $f^{2n+1}(t) \leqq x_0$ and simultaneously $g(x_0) = f^n g(t) \notin \tau_f^{(n)}\{g(t)\}$, a contradiction.

We have got up to now that for every element $x \in T$ exactly one of the following cases occurs:

$$1^\circ \quad gf^n(x) = g(x), \qquad\qquad 2^\circ \quad gf^n(x) = f^n g(x).$$

It is to be noted for the sake of completeness that the case $gf^n(n) \parallel f^n g(x)$, i.e. $gf^n(x) \parallel g(x)$ leads to a contradiction, since $\text{Min}\,(T, \leqq) = \emptyset$ thus $f^{n+1}(y) = x$ for some $y \in (x]_{\leq}$ hence $f^n(x) \in \tau_f^{(n)}\{y\}$, $x \in \tau_f^{(n)}\{y\}$. From here $g(x) \in \tau_f^{(n)}\{g(y)\}$, i.e. $g(x) \nparallel g(y)$ and at the same time $gf^n(x) \notin \tau_f^{(n)}\{g(y)\}$, which contradicts the fact that g is a closed deformation of the space $(T, \tau_f^{(n)})$.

Let $t \in T$ be an arbitrary element. Suppose 1° holds, i.e. $gf^n(t) = g(t)$. Consider the element $f^{2n}(t) = f^n(f^n(t))$. Since the equality $gf^n f^n(t) = f^n gf^n(t)$ contradicts the assumption $g \in S(T, \tau_f^{(n)})$ for $f^{2n}(t) \in \tau_f^{(n)}\{t\}$ and $f^n gf^n(t) = f^n g(t) \notin \tau_f^{(n)}\{g(t)\}$, we have $gf^{2n}(t) = g(t)$. In the same way we get $gf^{3n}(t) = g(t)$ and $gf^{kn}(t) = g(t)$ for any $k \geqq 1$. On the other hand $f^{n+1}g(t) \in \tau_f^{(n)}\{g(t)\}$ thus there exists $x_0 \in \tau_f^{(n)}\{t\}$ (i.e. $x_0 \in [f^{n+1}(t))_{\leq}$) with the property $g(x_0) = f^{n+1}g(t)$. Let k be a positive integer such that $f^{kn}(t) < x_0 < f^{(k+1)n}(t)$. Then $f^{n+1}(x_0) < f^{n+1}f^{(k+1)n}(t) = f^{(k+2)n+1}(t)$, thus $f^{n+1}(x_0) \leqq f^{(k+2)n}(t)$, hence $f^{(k+2)n}(t) \in \tau_f^{(n)}\{x_0\}$. From here $g(t) = gf^{(k+2)n}(t) \in$

$\in \tau_f^{(n)}\{g(x_0)\} = \tau_f^{(n)}\{f^{n+1}g(t)\}$, a contradiction. Therefore $gf^n(t) = f^n g(t)$ for every element $t \in T$.

Let $g \in \mathrm{S}(T, \tau_f^{(n)})$ be an arbitrary closed deformation, $t \in T$. We have $\tau_f^{(n-1)}\{t\} = \tau_f^{(n)}\{t\} \cup \{f^n(t)\}$, thus $g(\tau_f^{(n-1)}\{t\}) = g(\tau_f^{(n)}\{t\}) \cup \{gf^n(t)\} = \tau_f^{(n)}\{g(t)\} \cup \{f^n g(t)\} = \tau_f^{(n-1)}\{g(t)\}$, hence $g(\tau_f^{(n-1)}X) = \tau_f^{(n-1)}g(X)$ for any subset $X \subset T$. We get that monoids of closed deformations form a chain with respect to the set inclusion:

$$\mathrm{S}(T, \tau_f^{(n)}) \subset \mathrm{S}(T, \tau_f^{(n-1)}) \subset \ldots \subset \mathrm{S}(T, \tau_f).$$

But $\mathrm{S}(T, \tau_f) = \mathrm{C}_T(f)$ by the above mentioned Theorem 3.3 from [5] which implies $\mathrm{S}(T, \tau_f^{(n)}) \subset \mathrm{C}_T(f)$. With respect to the above proved opposite inclusion we get the equality. The proof is complete. □

3.7. Remark. The generalization of Lemma 3.6 for the case of locally finite forest is evident, since it follows from the proof of the mentioned lemma that it is inessential whether a selfmap of a locally finite tree or a mapping of a tree into another one is considered. (All trees are supposed to be without maximal and minimal elements). Thus the notion of a tree in Lemma 3.6 can be replaced by the notion of a forest and the assertion remains true.

3.8. Lemma. *Let $(T, \leqq)$ be a locally finite upper (lower) antirooted forest, τ be a perfect Alexandroff topology on T such that $\mathrm{S}(T, \tau) = \mathrm{LA}(T, \leqq)$. Then* $\mathrm{Min}\,(T, \leqq) = \emptyset$ $(\mathrm{Max}\,(T, \leqq) = \emptyset)$.

Proof. Suppose $(T, \leqq)$ is an upper forest. Since $\mathrm{Max}\,(T, \leqq) = \emptyset$, putting $f(t) = t^+$ for any $t \in T$, we have $\mathrm{C}_T(f) = \mathrm{LA}(T, \leqq)$ by Remark 3.5. Let $\{(T_\iota, \leqq) \mid \iota \in I\}$ be the collection of all maximal subtrees of the forest $(T, \leqq)$. Since $\mathrm{C}_T(f) = \mathrm{S}(T, \tau)$, by [5] Theorem 4.1 exactly one of the following cases occurs:

1° $\mathrm{Min}\,(T_\iota, \leqq) = \emptyset$ for any $\iota \in I$.

2° $\iota \in I$ implies that either $(T_\iota, \leqq)$ is a chain of the type $\omega^* + \omega$ or $T_\iota = K_\iota \cup \mathrm{Min}\,(T_\iota, \leqq)$, where $(K_\iota, \leqq)$ is a chain of the type $\omega^* + \omega$ and $\mathrm{Min}\,(T_\iota, \leqq) \neq \emptyset$.

3° $\iota \in I$ implies that either $(T_\iota, \leqq)$ is a chain of the type $\omega^* + \omega$ or $T_\iota = \mathrm{Min}\,(T_\iota, \leqq) \cup T'_\iota$, where $\mathrm{Min}\,(T_\iota, \leqq) \neq \emptyset$ $(\mathrm{card}\, T'_\iota = \aleph_0)$ and for every pair of elements $s \in \mathrm{Min}\,(T_\iota, \leqq)$, $t \in T'_\iota$ it holds $s < t$.

With respect to the above conditions 1°, 2°, 3° and the equality $\mathrm{C}_T(f) = \mathrm{S}(T, \tau)$ we get easily that all components T_ι, $\iota \in I$ are closed subsets of (T, τ). Admit $\mathrm{Min}\,(T_\iota, \leqq) \neq \emptyset$ for some $\iota \in I$. Let $t \in \mathrm{Min}\,(T_\iota, \leqq)$ be an arbitrary element and put $X = T_\iota \setminus \{t\}$. If $t^+ = x^+$ for some $x \in X$, we define a mapping $g : T \to T$ in such a way: $g(t) = x$, $g(s) = s$ for $s \in T$, $s \neq t$. Clearly, $g \in \mathrm{C}_T(f) = \mathrm{S}(T_\iota, \tau)$ thus $\tau g(X) = g(\tau X) = X$, hence $X = \tau g(X) = \tau\tau g(X) = \tau X$. If the element $x \neq t$ having the property $x^+ = t^+$ does not exist (i.e. $(T_\iota, \leqq)$ is a chain of the type ω) we define $g : T \to T$ as follows: $g(s) = s^+$ for $s \in T_\iota$, $g(s) = s$ for $s \in T \setminus T_\iota$. The assumption $t \in \tau X$ implies (with respect to the fact $g \in S(T, \tau)$ and the τ-closedness of T_t) that $\tau g(X) = g(\tau X) = X$, i.e. $t \in \tau X = X$ — a contradiction. Hence $t \notin \tau X$, i.e. the set X is closed. Then $\{t\}$ is an open subset the existence of which contradicts the assumption of the perfectness of the space (T, τ). Consequently the set $\operatorname{Min}(T_t, \leqq)$ is empty.

It $(T, \leqq)$ is a lower forest, we define the transformation f using predecessors of elements of the forest $(T, \leqq)$ and the proof of the equality $\operatorname{Max}(T, \leqq) = \emptyset$ coincides with the just previous one. □

3.9. Lemma. *If $(T, \leqq)$ is a locally finite antirooted forest then* $S(T, \tau) = LA(T, \leqq)$ *for some perfect Alexandroff topology τ on the set T iff there exists an Alexandroff topology σ on T such that* $LH(T, \sigma) = LA(T, \leqq)$ *and the σ-closure of any nonempty subset of the set T is infinite.*

Proof. Let τ be a perfect topology of Alexandroff on T such that $S(T, \tau) = LA(T, \leqq)$. Denote by σ the Alexandroff topology dual to τ. The least σ-neighbourhood of any point $x \in T$ is the τ-closure $\tau\{x\}$ thus $S(T, \tau) = LH(T, \sigma)$. Since $(T, \leqq)$ does not contain any maximal or minimal element and every local automorphism $g \in LA(T, \leqq)$ such that the g-image of T is a chain of the type $\omega^* + \omega$ is a closed deformation of (T, τ), we have $\tau\{t\}$ is a chain for each $t \in T$. Further, $f^n \in S(T, \tau)$ for each $n \in N$ (where $f(t) = t^+$ for $t \in T$) thus $\tau\{t\}$ is a cofinal subset of $(T, \leqq)$ (in the case of an upper forest) and coinitial subset of $(T, \leqq)$ in the case of lower forest. Consequently, σ-closures of singletons are infinite. Now σ is supposed to be an Alexandroff topology on T satisfying conditions from the lemma. Then σ^* (the dual topology to σ) has this property: $S(T, \sigma^*) = LA(T, \leqq^{-1}) = LA(T, \leqq)$ and the space (T, σ^*) is perfect. □

3.10. Lemma. *Let $(T, \leqq)$ be a locally finite forest. For any pair of different elements $a, b \in T$ there exists a perfect Alexandroff topology $\tau_{a,b}$ such that* $S(T, \tau_{a,b}) = LA(T, \leqq)$ *and points a, b are in the space $(T, \tau_{a,b})$ semiseparated iff for any pair of different elements $c, d \in T$ there exists an Alexandroff topology $\sigma_{c,d}$ on T possessing the following properties*:

(i) $LH(T, \sigma_{c,d}) = LA(T, \leqq)$,

(ii) card $\sigma_{c,d}X \geqq \aleph_0$ *for every nonempty subset $X \subset T$,*

(iii) *points c, d are semiseparated in the space $(T, \sigma_{c,d})$.*

Proof. Similarly as in the proof of Lemma 3.9 using the dual topology we get the equivalence of the above stated conditions. □

Proof of Theorem 2.1: The scheme of the proof is the following one ($1° - 5°$ are corresponding conditions):

$$\begin{tikzcd} 1^\circ \arrow[r, Leftrightarrow] & 2^\circ \arrow[r, Rightarrow] & 3^\circ \arrow[dl, Rightarrow] \\ 5^\circ \arrow[ur, Rightarrow] & 4^\circ \arrow[u, Rightarrow] & \end{tikzcd}$$

The equivalence $1° \Leftrightarrow 2°$ is given by Lemma 3.1, implications $2° \Rightarrow 3° \Rightarrow 4° \Rightarrow 2°$ by Lemma 3.2 and the equivalence $2° \Leftrightarrow 5°$ is given by Lemma 3.3. □

Proof of Theorem 2.2 follows from Theorem 2.1 with respect to [10] Theorem 1, part (b). □

Proof of Theorem 2.3: The scheme of implications is as follows:

$$\begin{tikzcd}
& 2^\circ \arrow[dl, Rightarrow] & & 4^\circ \arrow[dl, Leftrightarrow] \\
1^\circ \arrow[rr, Rightarrow] & & 3^\circ \arrow[ul, Rightarrow] \arrow[rr, Leftrightarrow] & & 5^\circ
\end{tikzcd}$$

The implication $1° \Rightarrow 3°$ follows from Lemma 3.6 with respect to Remark 3.7 and Lemma 3.1. The implication $3° \Rightarrow 2°$ is trivial, $2° \Rightarrow 1°$ is given by Lemma 3.8 with respect to Lemma 3.1. The equivalence $3° \Leftrightarrow 4°$ is established in Lemma 3.9 and the equivalence $3° \Leftrightarrow 5°$ in Lemma 3.10. □

3.11. Remark. Characterizations contained in Theorem 2.3 have been obtained under the essential assumption of the antirootedness of considered trees. Characterizations without this assumption or ocassionally with the assumption of existence of branches in all trees seem to be an open problem. Another special problem solving of which does not follow from the above considerations is the question of the realizability of local automorphism monoids of finite trees by closed deformations of a topological space.

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