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TWO-POINT BOUNDARY VALUE PROBLEMS FOR SECOND ORDER SYSTEMS

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In this paper we shall consider

(1) $$x'' = f(t, x, x')$$

together with

(2) $$x(a) = A, \qquad x(b) = B$$

or

(3) $$x(a) = A, \qquad x'(b) = B$$

or

(4) $$x'(a) = A, \qquad x(b) = B$$

where $f \in C([a, b] \times R^n \times R^n, R^n)$, and prove the following

Theorem. *Let for all (t, u_1, v_1), $(t, u_2, v_2) \in [a, b] \times R^n \times R^n$, the function f satisfy Lipschitz condition*

(5) $$| f(t, u_1, v_1) - f(t, u_2, v_2) | \leqq L_0 | u_1 - u_2 | + L_1 | v_1 - v_2 |$$

(component — wise) where L_0 and L_1 are $n \times n$ nonnegative matrices. Then, there exists a unique solution (α) of (1), (2) provided

(6) $$\varrho\left(\frac{1}{\pi^2} L_0(b - a)^2 + \frac{4}{\pi^2} L_1(b - a)\right) < 1$$

or

(7) $$\varrho\left(\frac{5}{48} L_0(b - a)^2 + \frac{2}{5} L_1(b - a)\right) < 1$$

or

(8) $$\varrho\left(\frac{\sqrt{3} - 1}{4\sqrt{3}} L_0(b - a)^2 + \frac{1}{3} L_1(b - a)\right) < 1$$

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[1]) This paper was done while the authors were visiting at the Istituto Matematico "Ulisse Dini", Universita degli Studi, Firenze.

(β) *of* (1), (3) (*or* (4)) *provided*

(9) $$\varrho\left(\frac{4}{\pi^2}L_0(b-a)^2+\frac{2}{\pi}L_1(b-a)\right)<1$$

or

(10) $$\varrho\left(\frac{1}{2}L_0(b-a)^2+\frac{1}{2}L_1(b-a)\right)<1$$

In (6) − (10), $\varrho(P)$ denotes the spectral radius of the nonnegative matrix P.

Several known results are deduced or compared as following remarks. An undecided case is mentioned in the last.

The proof needs the following particular case of more general (Kantorovich [6], Schröder [8]) Contraction mapping:

Lemma 1. *Let F be a generalized Banach space* ($\|\cdot\|_G \to R^n_+$, see [4]) *and let* $T: E \to E$ *be such that for all* $x, y \in E$ *and for some positive integer k*

$$\| T^k x - T^k y \|_G \leqq K \| x - y \|_G,$$

where K is $n \times n$ *nonnegative matrix with* $\varrho(K) < 1$. *Then, T has a unique fixed point* x^*.

The homogenous boundary value problem

$$x'' = 0; \qquad x(a) = 0, \qquad x(b) = 0$$

has $G(t, s)$ as the Green's function, where

$$G(t,s) = -\begin{cases} \dfrac{(b-t)(s-a)}{(b-a)}, & a \leqq s \leqq t \leqq b, \\ \dfrac{(b-s)(t-a)}{(b-a)}, & a \leqq t \leqq s \leqq b. \end{cases}$$

We shall need some estimates related to $G(t, s)$ which are collected in

Lemma 2. *The following hold;*

(i) $\int_a^b | G(t,s) | \, ds = \dfrac{1}{2}(t-a)(b-t) = \varphi_1(t) \leqq \dfrac{(b-a)^2}{8}\psi_1(t), \psi_1(t) =$ $= \sin\dfrac{\pi(t-a)}{(b-a)}$,

(ii) $\int_a^b | G_t(t,s) | \, ds = \dfrac{(t-a)^2+(b-t)^2}{2(b-a)} = \varphi_2(t) \leqq \dfrac{(b-a)^2}{2\pi}\psi_2(t)$,

$$\psi_2(t) = \left[\frac{2}{(b-a)}\sin\frac{\pi(t-a)}{(b-a)} + \frac{\pi(b-2t+a)}{(b-a)^2}\cos\frac{\pi(t-a)}{(b-a)}\right],$$

(iii) $\varphi_2(t) \leqq \frac{5}{4} \psi_3(t)$,

$$\psi_3(t) = \left[\frac{2}{5}(b-a) - \frac{12}{5} \frac{(t-a)^2 (b-t)^2}{(b-a)^3}\right],$$

(iv) $\int_a^b | G(t,s) | \varphi_1(s) \, ds \leqq \frac{5}{48} (b-a)^2 \varphi_1(t)$,

(v) $\int_a^b | G(t,s) | \psi_1(s) \, ds = \frac{(b-a)^2}{\pi^2} \psi_1(t)$,

(vi) $\int_a^b | G(t,s) | \varphi_2(s) \, ds \leqq \frac{8}{25} (b-a) \varphi_1(t)$,

(vii) $\int_a^b | G(t,s) | \psi_2(s) \, ds \leqq \frac{4}{\pi^2} (b-a) \psi_1(t)$,

(viii) $\int_a^b | G_t(t,s) | \varphi_1(s) \, ds \leqq \frac{5}{48} (b-a)^2 \psi_3(t)$,

(ix) $\int_a^b | G_t(t,s) | \varphi_1(s) \, ds \leqq \frac{\sqrt{3}-1}{4\sqrt{3}} (b-a)^2 \varphi_2(t)$,

(x) $\int_a^b | G_t(t,s) | \psi_1(s) \, ds = \frac{(b-a)^2}{\pi^2} \psi_2(t)$,

(xi) $\int_a^b | G_t(t,s) | \varphi_2(s) \, ds \leqq \frac{1}{3} (b-a) \varphi_2(t)$,

(xii) $\int_a^b | G_t(t,s) | \psi_2(s) \, ds \leqq \frac{4}{\pi^2} (b-a) \psi_2(t)$,

(xiii) $\int_a^b | G_t(t,s) | \psi_3(s) \, ds \leqq \frac{2}{5} (b-a) \psi_3(t)$.

Proof. The proof involves some elementary computations.

The homogenous boundary value problem

$$x'' = 0; \qquad x(a) = 0, \qquad x'(b) = 0$$

has $g(t,s)$ as the Green's function, where

$$g(t,s) = \begin{cases} (s-a), & a \leqq s \leqq t \leqq b, \\ (t-a), & a \leqq t \leqq s \leqq b. \end{cases}$$

Lemma 3. *The following hold:*

(i) $\int_a^b | g(t,s) | \, ds = \frac{1}{2}(t-a)(2b-t-a) = p_1(t) \leqq \frac{2}{\pi}(b-a)$

$$q_1(t) = \sin \frac{\pi(t-a)}{2(b-a)},$$

(ii) $\int_a^b |g_t(t, s)| ds = (b - t) = p_2(t) \leqq \frac{2(b-a)^2}{\pi} q_2(t)$,

$$q_2(t) = \frac{\pi}{2(b-a)} \cos \frac{\pi(t-a)}{2(b-a)},$$

(iii) $\int_a^b |g(t, s)| p_1(s) ds \leqq \frac{5}{12}(b-a)^2 p_1(t)$,

(iv) $\int_a^b |g(t, s)| q_1(s) ds = \frac{4(b-a)^2}{\pi^2} q_1(t)$,

(v) $\int_a^b |g(t, s)| p_2(s) ds \leqq \frac{1}{2}(b-a) p_1(t)$,

(vi) $\int_a^b |g(t, s)| q_2(s) ds \leqq \frac{2(b-a)}{\pi} q_1(t)$,

(vii) $\int_a^b |g_t(t, s)| p_1(s) ds \leqq \frac{1}{2}(b-a)^2 p_2(t)$,

(viii) $\int_a^b |g_t(t, s)| q_1(s) ds = \frac{4(b-a)^2}{\pi^2} q_2(t)$,

(ix) $\int_a^b |g_t(t, s)| p_2(s) ds \leqq \frac{1}{2}(b-a) p_2(t)$,

(x) $\int_a^b |g_t(t, s)| q_2(s) ds \leqq \frac{2(b-a)}{\pi} q_2(t)$.

The homogenous boundary value problem

$$x'' = 0; \quad x'(a) = 0, \quad x(b) = 0$$

has $h(t, s)$ as the Green's function, where

$$h(t, s) = -\begin{cases} (b-t), & a \leqq s \leqq t \leqq b, \\ (b-s), & a \leqq t \leqq s \leqq b. \end{cases}$$

Lemma 4. *The following hold:*

(i) $\int_a^b |h(t, s)| ds = \frac{1}{2}(b-t)(b+t-2a) = c_1(t) \leqq$

$\leqq \frac{2}{\pi}(b-a)^2 d_1(t), d_1(t) = \sin \frac{\pi(b-t)}{2(b-a)}$,

(ii) $\int_a^b |h_t(t, s) ds = (t-a) = c_2(t) \leqq \frac{2(b-a)^2}{\pi} d_2(t)$,

$$d_2(t) = \frac{\pi}{2(b-a)} \cos \frac{\pi(b-t)}{2(b-a)},$$

(iii) *all* (iii) – (×) *of lemma* 3, *with replacing* g *to* h, p_1 *to* c_1, q_1 *to* d_1, p_2 *to* c_2 *and* q_2 *to* d_2.

Proof of the theorem. The problem (1), (2) is equivalent to

$$x(t) = A + (B - A)\frac{(t-a)}{(b-a)} + \int_a^b G(t, s)\, f(s, x(s), x'(s))\, ds.$$

We define an operator T on $\mathscr{S} = (C^{(1)}[a, b], R^n)$, by

$$Tx(t) = A + (B - A)\frac{(t-a)}{(b-a)} + \int_a^b G(t, s)\, f(s, x(s), x'(s))\, ds. \tag{11}$$

If $x \in \mathscr{S}$, the generalized norm is defined by

$$\| x \|_G = \max\left(\max_{a \leq t \leq b} | x(t) |, \max_{a \leq t \leq b} | x'(t) |\right),$$

where $| x(t) | = (| x_1(t) |, \ldots, | x_n(t) |)^T$.

For all $x(t), y(t) \in \mathscr{S}$, we have from (11) and lemma 2

$$| Tx(t) - Ty(t) | \leqq \int_a^b | G(t, s) | [L_0 | x(s) - y(s) | + L_1 | x'(s) - y'(s) |]\, ds \leqq$$

$$\leqq (L_0 + L_1)\frac{(b-a)^2}{8} \| x - y \|_G \psi_1(t)$$

and

$$| (Tx)'(t) - (Ty)'(t) | \leqq (L_0 + L_1)\frac{(b-a)^2}{2\pi} \| x - y \|_m \psi_G(t).$$

Thus, for the operator T^2, we find from lemma 2

$$| T^2x(t) - T^2y(t) | \leqq$$

$$\leqq \int_a^b | G(t, s) | \left[L_0(L_0 + L_1)\frac{(b-a)^2}{8}\psi_1(s) + L_1(L_0 + L_1)\frac{(b-a)^2}{2\pi}\psi_2(s)\right] \times$$

$$\times \| x - y \|_G\, ds \leqq \left(\frac{1}{\pi^2} L_0(b-a)^2 + \frac{4}{\pi^2} L_1(b-a)\right)(L_0 + L_1)\frac{(b-a)^2}{2\pi} \times$$

$$\times \| x - y \|_G \psi_1(t).$$

Similarly

$$| (T_x^2 x)'(t) - (T_y^2)'(t) | \leqq \left(\frac{1}{\pi^2} L_0(b-a)^2 + \frac{4}{\pi^2} L_1(b-a)\right)(L_0 + L_1)\frac{(b-a)^2}{2\pi} \times$$

$$\times \| x - y \|_G \psi_2(t).$$

Inductively, for a positive integer m we have

$$| T^m x(t) - T^m y(t) | \leqq \left(\frac{1}{\pi^2} L_0(b-a)^2 + \frac{4}{\pi^2} L_1(b-a)\right)^{m-1} (L_0 + L_1)\frac{(b-a)^2}{a\pi} \times$$

$$\times \| x - y \|_G \psi_1(t),$$

$$|(T^m x)'(t) - (T^m y)'(t)| \leqq \left(\frac{1}{\pi^2} L_0(b-a)^2 + \frac{4}{\pi^2} L_1(b-a)\right)^{m-1} \times$$

$$\times (L_0 + L_1) \frac{(b - a^2)}{2\pi} \| x - y \|_G \psi_2(t).$$

Hence for the operator T^m

$$\| T^m x - T^m y \|_G \leqq \left(\frac{1}{\pi^2} L_0(b-a)^2 + \frac{4}{\pi^2} L_1(b-a)\right)^{m-1} (L_0 + L_1) \frac{(b-a)^2}{2\pi} \times$$

$$\times \max\left(\max_{a \leqq t \leqq b} \psi_1(t), \max_{a \leqq t \leqq b} \psi_2(t)\right) \| x - y \|_G .$$

From (6), $\left(\frac{1}{\pi^2} L_0(b-a)^2 + \frac{4}{\pi^2} L_1(b-a)\right)^k$ tends to 0 as k tends to infinity and hence there exists a number m such that

$$\varrho\left(\left[\frac{1}{\pi^2} L_0(b-a)^2 + \frac{4}{\pi^2} L_1(b-a)\right]^{m-1} (L_0 + L_1) \frac{(b-a)^2}{2\pi} \times \right.$$

$$\left. \times \max\left(\max_{a \leqq t \leqq b} \psi_1(t), \max_{a \leqq t \leqq b} \psi_2(t)\right)\right) < 1$$

and the conclusion follows from lemma 1. Other parts are proved analogously using the estimates obtained in the lemmas.

Remark 1. The scalar boundary value problem

$$\begin{aligned} &x^{(2n)} = g(t, x) \\ &x^{(2i)}(a) = A_i, \qquad x^{(2i)}(b) = B_i; \qquad 0 \leqq i \leqq n-1 \end{aligned} \tag{12}$$

where $g(t, x)$ is continuous and for all (t, x), $(t, y) \in [a, b] \times R$

$$| g(t, x) - g(t, y) | \leqq k | x - y |$$

has a unique solution provided

$$\frac{k}{\pi^{2n}} (b-a)^{2n} < 1. \tag{13}$$

In fact the problem (12) is equivalent to the system

$$\begin{aligned} &x_i'' = x_{i+1}, \qquad 1 \leqq i \leqq n-1 \\ &x_n'' = g(t, x_1) \\ &x(a) = A, \qquad x(b) = B \end{aligned}$$

and condition (6) reduces to $\varrho(\Delta) < 1$, where

$$\Delta = \frac{(b-a)^2}{\pi^2}\begin{bmatrix} 0 & 1 & 0 & . & \dots & . & 0 \\ 0 & 0 & 1 & . & \dots & . & 0 \\ & & & & \dots & & \\ 0 & 0 & 0 & . & \dots & . & 1 \\ k & 0 & 0 & . & \dots & . & 0 \end{bmatrix},$$

which is the same as (13).

This result is also proved in ([2], theorem 3.6) using different methods and cover a particular case of Usmani [9].

Remark 2. If f is independent of x', then

1. Conditions (6) and (9) are best possible since the uncoupled system

$$x_i'' + kx_i = 0, \qquad 1 \leqq i \leqq n$$
$$x(a) = x(b) = 0$$

where $\frac{k}{\pi^2}(b-a)^2 = 1$ has infinite number of solutions:

$$x_i(t) = c_1 \sin \sqrt{k}(t-a),$$

where c_1 is arbitrary constant.

Also the system

$$x_i'' + x_{i+1} = 0, \qquad 1 \leqq i \leqq n-1$$
$$x_n'' + kx_1 = 0$$
$$x(a) = x(b) = 0$$

where $\frac{k}{\pi^{2n}}(b-a)^{2n} = 1$ has infinite number of solutions:

$$x_i(t) = c_2 k^{(i-1)/n} \sin k^{1/2n}(t-a),$$

where c_2 is arbitrary constant.

2. Theorem 3.2 obtained in [1] can be improved to: If condition (6) is satisfied then, there exist a solution (unique solution) of (1) satisfying the periodic boudary conditions

$$x(a) = x(b)$$
$$x'(a) = x'(b)$$

if and only if there exists some p (unique p) for which

$$x'(b, p) - x'(a, p) = 0$$

where $x(t, p)$ is the solution of (1) satisfying $x(a) = x(b) = p$.

Remark 3. Condition (6) is the natural generalization of the result obtained by Lettenmeyer [7] for scalar problems and non-comparable with (7) or (8). Finding best possible results here similar to obtained in [3] remains undecided.

Remark 4. It will be desirable to find similar results for infinite systems considered in [5], [10] and references therein.

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