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LAPLACIAN IN GENERAL RIEMANNIAN STRUCTURES*

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0. This paper contains some introductory remarks on the concepts of divergence and laplacian in general Riemannian structures. First, on the algebraic level, we study such a structure in the R-module of all vector fields, where R is a K-algebra. Next, we apply these algebraic concepts to extend the notion of the laplacian to the category of Riemannian differential spaces. In particular, we give the relationship between the laplacian and the first Beltrami invariant, which appears, for example, in the formula for laplacian of the composition of smooth functions.

1. Algebraic level. Let K be a field and R be a K-algebra i.e., R has a structure of commutative ring with the unit, and a structure of vector space over K, satisfying $(a\alpha)\,\beta = a(\alpha\beta)$ for any a of K and any α and β of R. We will assume that K is a subring of R. Then R-linearity of all considered mappings will imply their K-linearity. We define (see [1] and [13]) the R-module of all R-vector fields as follows. A K-linear endomorphism $X\colon R \to R$ such that $X(\alpha\beta) = \alpha X(\beta) + \beta X(\alpha)$ for α and β of R is called an R-vector field. For any R-vector fields X and Y and any λ of R we set

$$(X + Y)(\alpha) = X(\alpha) + Y(\alpha), \qquad (\lambda X)(\alpha) = \lambda X(\alpha), \qquad \alpha \in R.$$

Then we have defined the structure of an R-module in the set of all vector fields. This R-module is called the R-module of all R-vector fields and will be denoted by $V(R)$.

An R-linear symmetric mapping $G\colon V(R) \times V(R) \to R$ such that for any R-linear mapping $L\colon V(R) \to R$ there exists exactly one Y in $V(R)$ such that $G(X, Y) = L(X)$ for any X in $V(R)$ is called (see [4]) a metric structure on R. Such an R-linear metric structure also defines, for any R-linear mapping $U\colon V(R) \to R$, exactly one covariant derivative, ∇, (see [4]) such that

$$\nabla_X G = U(X)\, G \qquad \text{and} \qquad \nabla_X Y - \nabla_Y X - [X, Y] = 0 \qquad \text{for } X \text{ and } X \text{ in } R.$$

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In the special case $U = 0$ we have the Levi–Civita covariant derivative of G.

Let End $V(R)$ be the R-module of all R-linear mappings

$$f\colon V(R) \to V(R). \tag{1}$$

Consider an R-linear mapping

$$t\colon \text{End } V(R) \to R \tag{2}$$

such that

$$t(d\alpha \,.\, Y) = Y(\alpha) \qquad \text{for any } \alpha \text{ in } R \text{ and any } Y \text{ in } V(R), \tag{3}$$

where $d\alpha\colon V(R) \to R$ denotes the R-linear mapping defined by the formula $d\alpha(X) = X(\alpha)$ for X in $V(R)$. This mapping is called the differential of α. Here $d\alpha \,.\, Y$ denotes the endomorphism (1) such that $f(X) = d\alpha(X)\, Y$ for any X in $V(R)$. A mapping (2) satisfying (3) will be called a quasi-trace in R.

Let us assume that we have got the structure (G, ∇, t), where G is a metric structure on R, ∇ is an abstract covariant derivative on $V(R)$ (see [3]) and t is a quasi-trace in R. First, we define the (∇, t)-divergence, div, setting for any Y in $V(R)$

$$\text{div } Y = t(X \mapsto \nabla_X Y). \tag{4}$$

Now, we can define the (G, ∇, t)-laplacian, $\Delta\alpha$, of an element α of R by the formula

$$\Delta\alpha = \text{div grad } \alpha. \tag{5}$$

If ∇ is the Levi–Civita covariant derivative of G, the (G, ∇, t)-laplacian will be called the (G, t)-laplacian. We will write $\langle X, Y\rangle$ instead of $G(X, Y)$.

Proposition 1. For any α and β or R, any Y in $V(R)$ and any a of K we have

$$\text{div}\,(\alpha Y) = d\alpha(Y) + \alpha \text{ div } Y, \tag{6}$$

$$\Delta(\alpha + \beta) = \Delta\alpha + \Delta\beta, \qquad \Delta(a\alpha) = a\Delta\alpha \tag{7}$$

and

$$\Delta(\alpha\beta) = \alpha\Delta\beta + \beta\Delta\alpha + 2\langle \text{grad } \alpha, \text{grad } \beta\rangle. \tag{8}$$

For any polynomial ω in one variable over K and any α of R we have

$$\Delta(\omega(\alpha)) = \omega'(\alpha)\,\Delta\alpha + \omega''(\alpha)\,\Delta_1\alpha, \tag{9}$$

where $\Delta_1\alpha$ is, so called, first Beltrami differential invariant of α defined by the formula

$$\Delta_1\alpha = \langle \text{grad } \alpha, \text{grad } \alpha\rangle \tag{10}$$

and ω' denotes the derivative of the polynomial ω.

Proof: The formula (6) is an immediate consequence of (3). K-linearity of grad and div yield (7). Let us take α and β of R. For any X in $V(R)$ we have

$$\langle \text{grad}\,(\alpha\beta), X\rangle = X(\alpha\beta) = \alpha X(\beta) + \beta X(\alpha) =$$
$$= \alpha\langle \text{grad}\,\beta, X\rangle + \beta\langle \text{grad}\,\alpha, X\rangle = \langle \alpha\,\text{grad}\,\beta + \beta\,\text{grad}\,\alpha, X\rangle.$$

Hence it follows that

(11) $$\text{grad}\,(\alpha\beta) = \alpha\,\text{grad}\,\beta + \beta\,\text{grad}\,\alpha.$$

We remark also that

(12) $$d\alpha(\text{grad}\,\beta) = \langle \text{grad}\,\alpha, \text{grad}\,\beta\rangle.$$

From (5), (11), (6) and (12) we get

$$\Delta(\alpha\beta) = \text{div}\,(\alpha\,\text{grad}\,\beta) + \text{div}\,(\beta\,\text{grad}\,\alpha) = d\alpha(\text{grad}\,\beta) + \alpha\Delta\beta +$$
$$+ d\beta(\text{grad}\,\alpha) + \beta\Delta\alpha = \alpha\Delta\beta + \beta\Delta\alpha + 2\langle \text{grad}\,\alpha, \text{grad}\,\beta\rangle,$$

To prove the second part of Proposition it sufficies to apply the formula (8) and K-linearity of Δ.

2. Riemannian differential space. Let (M, C) be a differential space (see [3]). Then C is in a natural way an R-algebra, and we have the R-module $V(C)$ of all C-vector fields (see [2] and [4]). The differential space (M, C) together with metric structure G on the R-algebra C is said to be a Riemannian differential space. If t is a quasi-trace in C, then we have the (G, t)-laplacian Δ will be called the t-laplacian in the Riemannain differential space $((M, C), G)$.

Proposition 2. The t-laplacian Δ is an R-linear mapping such that for any α, β in C and for any real function ω of class C^∞ on R the formula (8) and

(13) $$\Delta(\omega \circ \alpha) = \omega' \circ \alpha\Delta\alpha + \omega'' \circ \alpha\Delta_1\alpha$$

hold.

Proof: For any X in $V(C)$ any α in C and any real function ω of class C^∞ on R we have (see [2] and [4])

$$X(\omega \circ \alpha) = \omega' \circ \alpha\,.\,X(\alpha).$$

Hence it follows that

$$\langle \text{grad}\,(\omega \circ \alpha), X\rangle = X(\omega \circ \alpha) = \omega' \circ \alpha X(\alpha) = \omega' \circ \alpha\langle \text{grad}\,\alpha, X\rangle =$$
$$= \langle \omega' \circ \alpha\,\text{grad}\,\alpha, X\rangle.$$

Thus

(14) $$\text{grad}\,(\omega \circ \alpha) = \omega' \circ \alpha\,.\,\text{grad}\,\alpha.$$

Applying (6), we get

(15) $$\Delta(\omega \circ \alpha) = d(\omega' \circ \alpha)\,(\text{grad}\,\alpha) + \omega' \circ \alpha\Delta\alpha.$$

From (12) it follows that

$$d(\omega' \circ \alpha)(\text{grad}\,\alpha) = \langle \text{grad}\,(\omega' \circ \alpha), \text{grad}\,\alpha \rangle.$$

Hence by (14) we have

$$d(\omega' \circ \alpha)(\text{grad}\,\alpha) = \langle \omega'' \circ \alpha\, \text{grad}\,\alpha, \text{grad}\,\alpha \rangle = \omega'' \circ \alpha \Delta_1 \alpha.$$

This formula and (14) yield (13).

The important case is when the differential space is of finite dimension i.e., when each point of M has a neighbourhood with a local base of vector fields (see [4]). In such a space every C-linear mapping

$$f: V(C) \to V(C) \tag{16}$$

induces the mapping $\bar{f}$: $T(M, C) \to T(M, C)$ of the tangent bundle (see [2]) of (M, C), linear on each tangent space $T_p(M, C)$, in such a way that $\bar{f}(\bar{X}(p)) = \overline{f(X)}(p)$ for any point p of M, where, for each p, $\bar{X}(p)$ is the vector of $T_p(M, C)$ such that $\bar{X}(p)(\alpha) = X(\alpha)(p)$ for any α of C. We have got then for each p of M the endomorphism $\bar{f} \mid T_p(M, C)$ of the finitely dimensional space $T_p(M, C)$. Setting

$$t(f)(p) = tr(\bar{f} \mid T_p(M, C)) \qquad \text{for } p \text{ of } M, \tag{17}$$

we get the quasi-trace t in C defined in the canonical way. The quasi-trace t defined for every endomorphism (16) by the formula (17) is called the trace on (M, C). Therefore, in a Riemannian differential space of finite dimension we have well defined concepts: trace, gradient, Levi–Civita covariant derivative. Thus, we have got the concepts of divergence and laplacian for real smooth functions on such spaces. The considered concepts coincide with the suitable ones for differentiable manifolds. In particular, for any Riemannian differential space of finite dimension we are given a well defined concept of harmonic function.

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