Wiesław Sasin; Zbigniew Żekanovski

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ON SOME SHEAVES OVER A DIFFERENTIAL SPACE*

W. SASIN, Z. ŻEKANOWSKI

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INTRODUCTION

Let C be a non empty set of real functions defined on a set M. The set M will be interpreted as a topological space with weakest topology τ_C in which all functions from C are continuous.

It is known ([7]) that the set C is called the differential structure on M iff the set C is closed with respect to the lokalization ($C = C_M$) and C is closed with respect to the superpositions with the smooth functions on R^n.

It is easy to show that if C is the set of real functions on M closed with respect to the superposition with the smooth functions on R^n then C is a linear ring over R containing all constant functions and that topological space (M, τ_C) is a C-regular ([7)]).

The pair (M, C), where C is a differential structure on M is called the differential space.

Similarly as in theory of differential manifolds we define a tangent vector to the differential space (M, C) at the point $p \in M$ as well as the smooth tangent vector field on (M, C) ([7]).

The set M_p of all tangent vectors to differential space (M, C) at the point $p \in M$ has a natural structure of linear space over R and the set $\mathfrak{X}(M)$ of all smooth tangent vector fields on (M, C) has a natural structure of C-module.

In this paper by $\mathfrak{C}$ we shall denote the sheaf of all smooth real functions on (M, C) and by $\mathfrak{X}$ we shall denote the sheaf of all smooth tangent vector fields on (M, C).

A sheaf $\mathfrak{N}$ over differential space (M, C) is called the sheaf of $\mathfrak{C}$-modules ([2]) if

(i) $\mathfrak{N}(U)$ is $\mathfrak{C}(U)$-module for every open $U \in \tau_C$,

(ii) $\varrho^U_V(\alpha \cdot \xi) = \alpha \mid V \cdot \varrho^U_V(\xi)$ for $\alpha \in \mathfrak{C}(U)$ and $\xi \in \mathfrak{N}(U)$,

where $V \subset U$ and $\varrho^U_V : \mathfrak{N}(U) \to \mathfrak{N}(V)$ is restricting homomorphism in the sheaf $\mathfrak{N}$.

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2. THE SHEAVES OF $\mathfrak{C}$-MODULES OVER A DIFFERENTIAL SPACE

Let $\mathfrak{N}$ be an arbitrary sheaf of $\mathfrak{C}$-modules over a differential space (M, C). It is not difficult to prove.

Lemma 1. *If $U \in \tau_C$ and $\eta \in \mathfrak{N}(U)$ then for any point $p \in U$ there exists an open neighbourhood $B \in p$ and $\bar{\eta} \in \mathfrak{N}(M)$ such*

$$\varrho_B^U(\eta) = \varrho_B^M(\eta),$$

or equivalently, as we will write usually

$$\eta \mid B = \bar{\eta} \mid B.$$

Now let $\mathfrak{N}_1, \ldots, \mathfrak{N}_k, \mathfrak{N}_{k+1}, k \in N$ be any sheaves of $\mathfrak{C}$-modules over a differential space (M, C).
We introduce the following definition

Definition 1. Any map

$$f : \mathfrak{N}_1(U) \times \ldots \times \mathfrak{N}_k(U) \to \mathfrak{N}_{k+1}(U)$$

satisfying the condition:

(LF) $\quad$ *if* $\eta_i \mid V = \eta_i' \mid V, \quad \eta_i, \eta_i' \in \mathfrak{N}(U), \quad i = 1, 2, \ldots, k, \; V \subset U$

and $V \in \tau_C$ then

$$f(\eta_1, \ldots, \eta_k) \mid V = f(\eta_1', \ldots, \eta_k') \mid V,$$

will be called the *LF*-mapping of $\mathfrak{C}(U)$-modules $\mathfrak{N}_1(U), \mathfrak{N}_2(U), \ldots, \mathfrak{N}_k(U)$ into $\mathfrak{C}(U)$-module $\mathfrak{N}_{k+1}(U)$.

The set of all LF-mappings of $\mathfrak{C}(U)$-modules $\mathfrak{N}_1(U), \ldots, \mathfrak{N}_k(U)$ into $\mathfrak{C}(U)$-module $\mathfrak{N}_{k+1}(U)$ will be denoted by $\mathrm{LF}(\mathfrak{N}_1(U), \ldots, \mathfrak{N}_k(U); \mathfrak{N}_{k+1}(U))$.

Evidently this set can be equipped with the structure of $\mathfrak{C}(U)$-module.

Now we shall give some examples of LF-mappings important in the theory of differential space.

1. A smooth tangent vector field on differential space defined as a map $X : C \to C$ satisfying well known condition is of course LF-mapping.
2. For any smooth tangent vector fields $X, Y, X \circ Y$ is an LF-mapping, too.
3. One can easy show that the operator of exterior derivative is also an LF-mapping.
4. Likely a linear connection D in a module $\mathfrak{N}$, treated as a map $D : \mathfrak{N}(U) \to A^1(\mathfrak{X}(U), \mathfrak{N}(U))$ satisfying the condition

$$D(\alpha\xi) = \mathrm{d}\alpha \,.\, \xi + \alpha D\xi,$$

for any $\alpha \in \mathfrak{C}(U)$ and $\xi \in \mathfrak{N}(U)$, is an LF-mapping, too.

We shall prove.

Lemma 2. *If* $f_i : \mathfrak{N}_1(U) \times \ldots \times \mathfrak{N}_k(U) \to \mathfrak{N}_{k+1}(U)$, $i = 1, 2$ *are the LF-mappings satisfyinŋ condition*

(1) $$f_1(\eta_1 \mid U, \ldots, \eta_k \mid U) = f_2(\eta_1 \mid U, \ldots, \eta_k \mid U),$$

for all $(\eta_1, \ldots, \eta_k) \in \mathfrak{N}_1(V) \times \ldots \times \mathfrak{N}_k(V)$, where $U \subset V$, $U, V \in \tau_C$ then $f_1 = f_2$.

Proof. Let $(\eta_1, \ldots, \eta_k) \in \mathfrak{N}_1(U) \times \ldots \times \mathfrak{N}_k(U)$ and let there be an open covering of U such that for any $B \in \mathfrak{B}$ there exists $(\xi_1^B, \ldots, \xi_k^B) \in \mathfrak{N}_1(V) \times \ldots \times \mathfrak{N}_k(V)$ such that

$$\eta_i \mid B = \xi_i^B \mid B,$$

for any $i = 1, 2, \ldots, k$. Hence if f_i, $i = 1, 2$ are LF-mappings then

(2) $$f_1(\eta_1, \ldots, \eta_k) \mid B = f_2(\xi_1^B \mid U, \ldots, \xi_k^B \mid U) \mid B,$$

and

(3) $$f_2(\eta_1, \ldots, \eta_k) \mid B = f_2(\xi_1^B \mid U, \ldots, \xi_k^B \mid U) \mid B.$$

From (1), (2) and (3) we get

(4) $$f_1(\eta_1, \ldots, \eta_k) \mid B = f_2(\eta_1, \ldots, \eta_k) \mid B,$$

for all $B \in \mathfrak{B}$. From (4) and definition of sheaf we obtain

$$f_1(\eta_1, \ldots, \eta_k) = f_2(\eta_1, \ldots, \eta_k),$$

for any $(\eta_1, \ldots, \eta_k) \in \mathfrak{N}_1(U) \times \ldots \times \mathfrak{N}_k(U)$ or equivalently

$$f_1 = f_2.$$

Lemma 3. *For any LF-mapping* $f : \mathfrak{N}_1(U) \times \ldots \times \mathfrak{N}_k(U) \to \mathfrak{N}_{k+1}(U)$ *and for any open set* $V \subset U$ *there exists one and only one LF-mapping*

$$f_V : \mathfrak{N}_1(V) \times \ldots \times \mathfrak{N}_k(V) \to \mathfrak{N}_{k+1}(V),$$

such that

$$f_V(\eta_1 \mid V, \ldots, \eta_k \mid V) = f(\eta_1, \ldots, \eta_k) \mid V,$$

for all $(\eta_1, \ldots, \eta_k) \in \mathfrak{N}_1(U) \times \ldots \times \mathfrak{N}_k(U)$.

Proof: Let $(\xi_1, \ldots, \xi_k) \in \mathfrak{N}_1(V) \times \ldots \times \mathfrak{N}_k(V)$ and $\mathfrak{B}$ be an open covering of V such that for any $B \in \mathfrak{B}$ there exists $(\bar{\eta}_1^B, \ldots, \bar{\eta}_k^B) \in \mathfrak{N}_1(U) \times \ldots \times \mathfrak{N}_k(U)$ such that

$$\xi_i \mid B = \bar{\eta}_i^B \mid B,$$

for $i = 1, 2, \ldots, k$.

Now, let $f : \mathfrak{N}_1(U) \times \ldots \times \mathfrak{N}_k(U) \to \mathfrak{N}_{k+1}(U)$ be an LF-mapping. Let us consider a family

(5) $$(\varrho_B^U(f(\bar{\eta}_1^B, \ldots, \bar{\eta}_k^B)))_{B \in \mathfrak{B}},$$

of an elements of $\mathfrak{C}(B)$-modules $\mathfrak{N}_{k+1}(B)$ for $B \in \mathfrak{B}$.

Of course the elements of family (5) depend upon the choice of $(\xi_1, \ldots, \xi_k) \in \in \mathfrak{N}_1(V) \times \ldots \times \mathfrak{N}_k(V)$.

Now we shall show that the elements of the family (5), are agreeable on the intersections of sets of the covering. Indeed, let $B, B' \in \mathfrak{B}$ and $B \cap B' \neq \emptyset$. Then evidently

$$\xi_i \mid B \cap B' = \bar{\eta}_i^B \mid B \cap B' = \bar{\eta}_i^{B'} \mid B \cap B',$$

for any $i = 1, 2, \ldots, k$.
As the map f is the LF-mapping then

$$\begin{aligned}(f(\bar{\eta}_1^B, \ldots, \bar{\eta}_k^B)) \mid B \cap B' &= (f(\bar{\eta}_1^{B'}, \ldots, \bar{\eta}_k^{B'})) \mid B \cap B' = \\ = (f(\bar{\eta}_1^B, \ldots, \bar{\eta}_k^B) \mid B) \mid B \cap B' &= (f(\bar{\eta}_1^{B'}, \ldots, \bar{\eta}_k^{B'}) \mid B) \mid B \cap B'.\end{aligned}$$

From here and from definition of the sheaf follows that there exists one and only one element $f_{\mathfrak{B}}(\xi_1, \ldots, \xi_k) \in \mathfrak{N}_{k+1}(V)$ such that

$$f_{\mathfrak{B}}(\xi_1, \ldots, \xi_k) \mid B = f(\bar{\eta}_1^B, \ldots, \bar{\eta}_k^B) \mid B,$$

for any $B \in \mathfrak{B}$.

Now, let us put

(6) $$f_V(\xi_1, \ldots, \xi_k) := f_{\mathfrak{B}}(\xi_1, \ldots, \xi_k),$$

for an arbitrary $(\xi_1, \ldots, \xi_k) \in \mathfrak{N}_1(V) \times \ldots \times \mathfrak{N}_k(V)$.

We shall show next that the definition (6) does not depend on the choice of the covering $\mathfrak{B}$ of the set V.

To this end let us take other open covering $\mathfrak{A}$ of V such that for any $A \in \mathfrak{A}$ there exists point $(\eta_1^A, \ldots, \eta_k^A) \in \mathfrak{N}_1(U) \times \ldots \times \mathfrak{N}_k(U)$ such that

$$\xi_i \mid A = \bar{\eta}_i^A \mid A,$$

for $i = 1, 2, \ldots, k$.

By definition (6) we have

(7) $$f_{\mathfrak{A}}(\xi_1, \ldots, \xi_k) \mid A = f(\bar{\eta}_1^A, \ldots, \bar{\eta}_k^A) \mid A,$$

for any $A \in \mathfrak{A}$.

Now, let $\mathfrak{A} \vee \mathfrak{B} = \{A \cap B : A \in \mathfrak{A} \wedge B \in \mathfrak{B}\}$. Of course $\mathfrak{A} \vee \mathfrak{B}$ is an open covering of V, refinement of a covering $\mathfrak{A}$ and $\mathfrak{B}$. From (6) and (7) as well as definition of LF-mapping it follows

$$\begin{aligned}f_{\mathfrak{A}}(\xi_1, \ldots, \xi_k) \mid A \cap B &= f(\bar{\eta}_1^A, \ldots, \bar{\eta}_k^A) \mid A \cap B = \\ = f(\bar{\eta}_1^B, \ldots, \bar{\eta}_k^B) \mid A \cap B &= f_{\mathfrak{B}}(\xi_1, \ldots, \xi_k) \mid A \cap B,\end{aligned}$$

for all $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$ such that $A \cap B \neq 0$.

From here and definition of the sheaf we obtain

$$f_{\mathfrak{A}}(\xi_1, \ldots, \xi_k) = f_{\mathfrak{B}}(\xi_1, \ldots, \xi_k),$$

for any $(\xi_1, \ldots, \xi_k) \in \mathfrak{N}_1(V) \times \ldots \times \ldots \mathfrak{N}_k(V)$.

The verification that f_V is LF-mapping satisfying the condition

$$f_V(\eta_1 \mid V, \ldots, \eta_k \mid V) = f(\eta_1, \ldots, \eta_k) \mid V,$$

for all $(\eta_1, \ldots, \eta_k) \in \mathfrak{N}_1(U) \times \ldots \times \mathfrak{N}_k(U)$ is not difficult.

Lemma 4. *Let $\mathfrak{B}$ be an open covering of U and*

$$\{f^B : \mathfrak{N}_1(B) \times \ldots \times \mathfrak{N}_k(B) \to \mathfrak{N}_{k+1}(B)\}_{B \in \mathfrak{B}},$$

family of LF-mappings such that

$$f^B \mid B \cap B' = f^{B'} \mid B \cap B',$$

for all $B, B' \in \mathfrak{B}$ such that $B \cap B' \neq \emptyset$. Then there exists one and only one LF-mapping

$$f : \mathfrak{N}_1(U) \times \ldots \times \mathfrak{N}_k(U) \to \mathfrak{N}_{k+1}(U),$$

such that

$$f \mid B = f^B,$$

for any $B \in \mathfrak{B}$.

Proof: Let $(f^B)_{B \in \mathfrak{B}}$ be a family of LF-mappings of the form

$$f^B : \mathfrak{N}_1(B) \times \ldots + \mathfrak{N}_k(B) \to \mathfrak{N}_{k+1}(B),$$

satisfying the condition

$$f^B \mid B \cap B' = f^{B'} \mid B \cap B', \tag{8}$$

for all $B, B' \in \mathfrak{B}$, $B \cap B' \neq \emptyset$. Let $(\eta_1, \ldots, \eta_k) \in \mathfrak{N}_1(U) \times \ldots \times \mathfrak{N}_k(U)$ and let us take under consideration the family

$$\{f^B(\eta_1 \mid B, \ldots, \eta_k \mid B)\}_{B \in \mathfrak{B}},$$

of the elements of $\mathfrak{C}(B)$-module $\mathfrak{N}_{k+1}(B)$.

From our assumption (8) it follows that

$$f^B(\eta_1 \mid B, \ldots, \eta_k \mid B) \mid B \cap B' = f^{B'}(\eta_1 \mid B', \ldots, \eta_k \mid B') \mid B \cap B',$$

for any $B, B' \in \mathfrak{B}$, $B \cap B' \neq \emptyset$. Now, from here and from the fact that $\mathfrak{N}_{k+1}$ is a sheaf it follows that there exists an element $f(\eta_1, \ldots, \eta_k) \in \mathfrak{N}_{k+1}(U)$ such that

$$f(\eta_1, \ldots, \eta_k) \mid B = f^B(\eta_1 \mid B, \ldots, \eta_k \mid B),$$

for any $B \in \mathfrak{B}$ and $(\eta_1, \ldots, \eta_k) \in \mathfrak{N}_1(U) \times \ldots \times \mathfrak{N}_k(U)$.

Hence there exists one and anly one LF-mapping

$$f \in LF(\mathfrak{N}_1(U), \ldots, \mathfrak{N}_k(U); \mathfrak{N}_{k+1}(U)),$$

such that

$$f \mid B = f^B,$$

for any $B \in \mathfrak{B}$.

Let $\mathfrak{N}_1, \ldots, \mathfrak{N}_k, \mathfrak{N}_{k+1}$, $k \in N$ be an arbitrary sheaves of $\mathfrak{C}(U)$-modules over a differential space (M, C). Let us denote by $\mathrm{LF}(\mathfrak{N}_1, \ldots, \mathfrak{N}_k; \mathfrak{N}_{k+1})$ the category whose objects are $\mathfrak{C}(U)$-modules $\mathrm{LF}\ (\mathfrak{N}_1(U), \ldots, \mathfrak{N}_k(U);\ \mathfrak{N}_{k+1}(U))$, $U \in \tau_C$, of the LF-mappings.

The above proved lemmas imply the theorem

Theorem 1. *The three-triple* $(LF(\mathfrak{N}_1, \ldots, \mathfrak{N}_k; \mathfrak{N}_{k+1}), F, \tau_C)$ *is a sheaf over a differential space, where* F *is a contravariant functor from the category* τ_C *into category* $LF(\mathfrak{N}_1, \ldots, \mathfrak{N}_k; \mathfrak{N}_{k+1})$.

For the arbitrary sheaves $\mathfrak{N}_1, \ldots, \mathfrak{N}_k, \mathfrak{N}_{k+1}$ over a differential space we shall denote by $LF(\mathfrak{N}_1, \ldots, \mathfrak{N}_k; \mathfrak{N}_{k+1})$ the sheaf $(LF(\mathfrak{N}_1, \ldots, \mathfrak{N}_k; \mathfrak{N}_{k+1}), F, \tau_C)$. This sheaf will be called the sheaf of LF-mappings.

Now we shall give some examples of the most important sheaves of LF-mappings over a differential spaces.

Of course, one of the fundamental sheaf of LF-mappings over a differential space is a sheaf of the tangent vector fields on a differential space which we denote by $\mathfrak{X}$.

Now, let $\mathfrak{N}_1, \ldots, \mathfrak{N}_k, \mathfrak{N}_{k+1}$, $k \in N$ be the sheaves of $\mathfrak{C}(U)$-modules over a differential space (M, C) and

$$\omega : \mathfrak{N}_1(U) \times \ldots \times \mathfrak{N}_k(U) \to \mathfrak{N}_{k+1}(U),$$

$U \in \tau_C$, $\mathfrak{C}(U)$-k-linear map. It is not difficult to show that ω is an LF-mapping. Consequently the triple

$$(L_{\mathfrak{C}}(\mathfrak{N}_1, \ldots, \mathfrak{N}_k; \mathfrak{N}_{k+1}), F, \tau_C),$$

is a sheaf of LF-mappings on a differential space, where F is a contravariant functor from the category $L_{\mathfrak{C}}(\mathfrak{N}_1, \ldots, \mathfrak{N}_k; \mathfrak{N}_{k+1})$ of $\mathfrak{C}(U)$-modules $L_{\mathfrak{C}(U)}(\mathfrak{N}_1(U), \ldots$ $\ldots, \mathfrak{N}_k(U); \mathfrak{N}_{k+1}(U))$ of $\mathfrak{C}(U)$-k-linear mappings into the category τ_C. This sheaf is also denoted by

$$L_{\mathfrak{C}}(\mathfrak{N}_1, \ldots, \mathfrak{N}_k; \mathfrak{N}_{k+1}),$$

and called the sheaf of smooth tensor fields over a differential space.

Evidently in the particular case when $\mathfrak{N}_1 = \mathfrak{N}_2 = \ldots = \mathfrak{N}_k = \mathfrak{X}$ we have a sheaf $L^k_{\mathfrak{C}}(\mathfrak{X}, \mathfrak{N}_{k+1})$ of $\mathfrak{C}$-k-forms on the differential space with a value in the $\mathfrak{C}$-module $\mathfrak{N}_{k+1}$. The sheaf of all exterior form on differential space is denoted usually by $\Lambda^k(\mathfrak{X}, \mathfrak{C})$.

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Z. Żekanowski
ul. Marszalkowka 140m 88
00-61 Warszawa
Poland