Archivum Mathematicum

Vladimír Lešovský
Deviation of two curvatures

Archivum Mathematicum, Vol. 18 (1982), No. 4, 183--186

Persistent URL: http://dml.cz/dmlcz/107142

ARCH. MATH. 4, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XVIII: 183—186, 1982

DEVIATION OF TWO CURVATURES*

VLADIMÍR LEŠOVSKÝ, Brno

(Received November 17, 1981)

In this paper, we determine the deviation of curvatures of two (generalized) connections in terms of the deflection of both connections in question.

Let **FM** denote the category of fibered manifolds and their morphisms. Let $\pi : Y \to X$ be a fibered manifold and let $Y \oplus TX$ denote the fiber product over X. A (generalized) connection Γ on Y means any section $\Gamma : Y \to J^1Y$ (the first jet prolongation of Y), [3], [6], [8]. Such a connection Γ on Y can be considered as a section $Y \oplus TX \to TY$ linear with respect to TX. In local fiber coordinates x^i, y^α on Y, the equations of Γ are

$$\Gamma \equiv \mathrm{d}y^\alpha = F_i^\alpha(x, y)\,\mathrm{d}x^i,$$

with arbitrary smooth functions F_i^α, [3]. Let VY be the vertical tangent bundle of Y considered as a fibered manifold over X. The vertical prolongation $V\Gamma$ of Γ, [2], [3], [9], is a connection on VY with the following equations

$$V\Gamma \equiv \mathrm{d}y^\alpha = F_i^\alpha(x, y)\,\mathrm{d}x^i, \qquad \mathrm{d}\eta^\alpha = \partial_\beta F_i^\alpha \eta^\beta\,\mathrm{d}x^i,$$

where $\partial_\beta F_i^\alpha = \dfrac{\partial F_i^\alpha}{\partial y^\beta}$ and $\eta^\alpha = \mathrm{d}y^\alpha$ are the induced coordinates on VY, [3].

A 2-fibered manifold is a quintuple $U \xrightarrow{q} Y \xrightarrow{\pi} X$, where $q : U \to Y$ and $\pi : Y \to X$ are fibered manifolds. If $q : U \to Y$ is a vector bundle, $U \xrightarrow{q} Y \xrightarrow{\pi} X$ is called a semi-vector bundle. Let a projectable connection $\bar{\Gamma} : U \to J^1U$ over $\Gamma : Y \to J^1Y$ be given. If all maps $\bar{\Gamma}|_{U_y}$ of vector space U_y into $(J^1U)_{\Gamma(y)}$ are linear for every $y \in Y$, then $\bar{\Gamma}$ is called a semi-linear connection, [3]. Obviously, $V\Gamma$ is a semi-linear connection.

Having another fibered manifold $p : W \to Z$ with a connection $\Delta : W \oplus TZ \to TW$ and an FM-morphism $\varphi : W \to Y$ over $f : Z \to X$ we define the deflection $\mu(\Delta, \Gamma, \varphi) : W \oplus TZ \to VY$ of connections Δ and Γ with respect to φ by

$$\mu(\Delta, \Gamma, \varphi) = T\varphi \circ \Delta - \Gamma \circ (\varphi \oplus Tf).$$

* (Delivered at the Joint Czech-Polish-G.D.R. Conference in Differential Geometry and its Applications, September 1980, Nové Město na Moravě, Czechoslovakia.)

One sees easily that the vectors $(T\varphi)(\Delta(w, \xi))$ and $\Gamma(\varphi(w), Tf(\xi))$ are over the same vector on the base manifold TX for every $(w, \xi) \in W \oplus TZ$. If the mapping μ vanishes, the connections Δ and Γ are called φ-related. In local fiber coordinates z^p, w^λ on W let the equations of Δ be

$$\Delta \equiv dw^\lambda = G^\lambda_p(z, w)\, dz^p$$

and let

$$x^i = f^i(z), \qquad y^\alpha = f^\alpha(z, w),$$

be the equations of φ, then the equations of $\mu(\Delta, \Gamma, \varphi)$ are

$$\eta^\alpha = (\partial_p f^\alpha + \partial_\lambda f^\alpha G^\lambda_p - F^\alpha_i\, \partial_p f^i)\, dz^p.$$

We recall, that the curvature of Γ is a section $\Omega_\Gamma : Y \oplus \wedge^2 TX \to VY$, [1], [4]. In local coordinates

$$\Omega_\Gamma \equiv \eta^\alpha = (\partial_j F^\alpha_i + F^\beta_j\, \partial_\beta F^\alpha_i)\, dx^i \wedge dx^j,$$

[6].

For a manifold Z, a vector bundle $p : E \to X$, a linear connection Γ on E and a linear morphism

$$\varphi : \wedge^k TZ \to E \qquad \text{over} \qquad f : Z \to X \tag{1}$$

we define the exterior differential $d_\Gamma \varphi : \wedge^{k+1} TZ \to E$ by the following way. We construct the induced vector bundle $f^*E \to Z$,

$$f^*E = \{(z, e) \in Z \times E;\, fz = pe\},$$

and the induced connection $f^*\Gamma$ on f^*E. The connection $f^*\Gamma$ is the unique connection on f^*E for which the deflection $\mu(f^*\Gamma, \Gamma, p_E)$ vanishes, where p_E is the natural projection $f^*E \to E$. The linear morphism φ can be considered as a section $\tilde{\varphi} : Z \to f^*E \otimes \wedge^k T^*Z$ and we take its exterior differential $d_{f^*\Gamma}\tilde{\varphi} : Z \to f^*E \otimes \otimes \wedge^{k+1} T^*Z$, that can be interpreted as a mapping $d_\Gamma \varphi : \wedge^{k+1} TZ \to E$, [7]. In local fiber coordinates x^i, e^α on E and in local coordinates z^p on Z, let

$$x^i = f^i(z), \qquad e^\alpha = \varphi^\alpha_{p_1 \dots p_k}(z)\, dz^{p_1} \wedge \dots \wedge dz^{p_k},$$

be the equations of φ and let

$$de^\alpha = \Gamma^\alpha_{\beta i}(x)\, e^\beta\, dx^i,$$

be the equations of Γ. Then

$$d_\Gamma \varphi \equiv e^\alpha = (\partial_p \varphi^\alpha_{p_1 \dots p_k} - \Gamma^\alpha_{\beta i} \varphi^\beta_{p_1 \dots p_k}\, \partial_p f^i)\, dz^p \wedge dz^{p_1} \wedge \dots \wedge dz^{p_k}.$$

Let $W \to Z$ be a fibered manifold with a connection Δ on W. Let $U \to Y \to X$ be a semi-vector bundle with a semi-linear connection $\bar{\Gamma}$ on U over Γ on Y. For any linear morphism $\psi : W \oplus \wedge^k TZ \to U$ over $\varphi : W \to Y$ and over $f : Z \to X$, we define its exterior differential

$$d_{(\Delta,\bar{\Gamma})}\psi : W \oplus \Lambda^{k+1}TZ \to U,$$

with respect to Δ and Γ by the following way, [3].

For every $w \in W$ we take a section ϱ tangent to $\Delta(w)$. Therefore $\psi \circ \varrho$ has the form (1). Now we have the former case, when we consider the induced bundle from the vector bundle U with respect to the mapping $\varphi \circ \varrho$ with the relevant induced connection and we construct

$$d_{(\varphi\circ\varrho)^*\bar{\Gamma}}(\psi \circ \varrho)\,|_z : \Lambda^{k+1}T_zZ \to U_{\varphi(w)}.$$

In local fiber coordinates x^i, y^α, u^ϱ on U and z^p, w^λ on W, let $x^i = f^i(z)$, $y^\alpha = \varphi^\alpha(z, w)$, $u^\varrho = \psi^\varrho_{p_1\dots p_k}(z, w)\, dz^{p_1} \wedge \dots \wedge dz^{p_k}$ be the equations of ψ and let

$$dw^\lambda = G^\lambda_p(z, w)\, dz^p \qquad \text{or} \qquad dy^\alpha = F^\alpha_i(x, y)\, dx^i, \qquad du^\varrho = \Gamma^\varrho_{\sigma i}(x, y)\, u^\sigma\, dx^i$$

be the equations of Δ or Γ, respectively. Then

$$d_{(\Delta,\bar{\Gamma})}\psi \equiv u^\varrho = (\partial_p\psi^\varrho_{p_1\dots p_k} + G^\lambda_p\,\partial_\lambda\psi^\varrho_{p_1\dots p_k} - \Gamma^\varrho_{\sigma i}\psi^\sigma_{p_1\dots p_k}\,\partial_p f^i)\, dz^p \wedge dz^{p_1} \wedge \dots \wedge dz^{p_k}.$$

By direct calculation, we prove the following assertion.

Theorem. *Let Γ or Δ be a connection on $Y \to X$ or $W \to Z$ with curvature Ω_Γ or Ω_Δ, respectively, and let $\varphi : W \to Y$ be an* FM-*morphism over $f : Z \to X$. Then it holds*

$$V\varphi \circ \Omega_\Delta - \Omega_\Gamma \circ (\varphi \oplus \Lambda^2 Tf) = -d_{(\Delta,V\Gamma)}\mu(\Delta, \Gamma, \varphi).$$

One interesting example is the case when Δ and Γ are related connections, then the diagram commutes:

$$\begin{array}{ccc} VW & \xrightarrow{\quad V\varphi \quad} & VY \\ \uparrow{\scriptstyle \Omega_\Delta} & & \uparrow{\scriptstyle \Omega_\Gamma} \\ W \oplus \Lambda^2 TZ & \xrightarrow{\varphi \oplus \Lambda^2 Tf} & Y \oplus \Lambda^2 TX, \end{array}$$

see [5].

REFERENCES

[1] Dekrét, A.: *Horizontal structures on fibered manifolds*, Math. Slovaca, 27 (1977), 257—265.

[2] Dekrét, A.: *On connections and covariant derivate on fibre manifolds*, Proceedings of the Czechoslovakian-GDR-Polish Scientific School on Differential Geometry, IM PAN, Warszawa 1979, 40—55.

[3] Kolář, I.: *Connections in 2-fibered manifolds*, to appear in: Arch. Math., Brno.

[4] Kolář, I.: *Higher order torsion of spaces with Cartan connection*, Cahiers Topologie Géom. Différentielle, 12 (1971), 137—146.

[5] Kolář, I.: *Induced connections and connection morphisms,* Proceedings of the Czechoslovakian-GDR-Polish Scientific School on Differential Geometry, IM PAN, Warszawa 1979, 125—134.
[6] Kolář, I.: *On generalized connections,* to appear in Beitr. Algebra Geom.
[7] Koszul, J. L.: *Fibre bundles and differential geometry,* Tata Institute of Fundamental Research, Bombay 1960.
[8] Libermann, P.: *On "Fibre parallelism" and locally reductive spaces,* Global Analysis Appl., Internat. Sem. Course, Trieste 1972, Vol. III., (1974), 13—23.
[9] Vilms, J.: *Curvature of nonlinear connections,* Proc. AMS, **19** (1968), 1125—1129.

V. Lešovský
602 00 Brno, Botanická 54
Czechoslovakia