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THE RICCATI DIFFERENTIAL EQUATION WITH COMPLEX-VALUED COEFFICIENTS AND APPLICATION TO THE EQUATION $x'' + P(t)\,x' + Q(t)\,x = 0$

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Consider the Riccati differential equation

$$z' = q(t) - p(t)\,z^2, \tag{1}$$

where $q(t)$ and $p(t)$ are certain continuous complex functions of the real variable $t \in [t_0, \infty)$ and z is the complex variable.

The aim of the present paper is to study the asymptotic behavior of solutions of (1) supposing $q(t)$ is "close enough" to the zero and $p(t)$ to the complex constant different from the zero.

The basic idea is to consider (1) as a perturbation of

$$w' = -aw^2,$$

where $a \neq 0$ is a complex number. The results are presented in a general form using the Ljapunov function method and comprehend some results of [1], [2] (Theorem 1, 2). The equation (1) is studied by M. Ráb in [3], [4] under the assumption $q(t)$ is "close enough" to the non-zero complex constant.

The results will be applied to the differential equation

$$x'' + P(t)\,x' + Q(t)\,x = 0 \tag{2}$$

under the corresponding assumptions on functions $P(t)$, $Q(t)$. This idea is used in [5] supposing $\lim\limits_{t\to\infty} [P^2(t) - 4Q(t)]^{1/2} = \Lambda$, $\operatorname{Re} \Lambda^{1/2} > 0$. Some results concerning these problems are generalized in [6], [7], [8], [9].

1. PRELIMINARIES

Let R or K denote the sets of all real or complex numbers, respectively. If $z = u + iv$, $u, v \in R$, we denote $\operatorname{Re} z = u$, $\operatorname{Im} z = v$, $\bar{z} = u - iv$, $z = (z\bar{z})^{1/2}$.

In what follows we shall use "Ljapunov" functions $W(z)$, $W_j(z)$, $V_j(z)$, $j = 1, 2$

defined by

(3) $$W(z) = \operatorname{Re}\left[\frac{\bar{a}}{z}\right], \qquad z \in K \setminus \{0\},$$

(4) $$W_1(z) = \operatorname{Re}\left[\frac{(1+i)\bar{a}}{z}\right], \qquad W_2(z) = \operatorname{Re}\left[\frac{(1-i)\bar{a}}{z}\right], \qquad z \in K \setminus \{0\},$$

(5) $$V_j(z) = |z|^j, \qquad j = 1, 2,\ z \in K,$$

where $a \in K \setminus \{0\}$ is fixed.

Let $A \in K \setminus \{0\}$ and let γ be a real parametr, $\gamma \neq 0$. Then the equation

$$\gamma = \operatorname{Re}\left[\frac{A}{z}\right]$$

represents a pencil of circles not-involving the point $z = 0$, where the function $\operatorname{Re}\left[\frac{A}{z}\right]$ is not defined. The circle K_γ corresponding to the value γ has the center $\frac{A}{2\gamma}$ and the radius $r = \frac{|A|}{2|\gamma|}$. The straight-line $\operatorname{Re}[\bar{A}z] = 0$ being the axis of the pencil corresponds to the value $\gamma = 0$.

Define for the real function $U(z)$ the differentiation of $U(z)$ with respect to (1) as follows:

$$D_f U(t, z) = \frac{\partial U(z)}{\partial \operatorname{Re} z} \operatorname{Re} f(t, z) + \frac{\partial U(z)}{\partial \operatorname{Im} z} \operatorname{Im} f(t, z),$$

where $f(t, z) = q(t) - p(t) z^2$.

Then it holds

(6) $$D_f W(t, z) \geqq \operatorname{Re}[\bar{a}p(t)] - \frac{|a|\,|q(t)|}{|z^2|},$$

(7) $$D_f W_j(t, z) \geqq \operatorname{Re}[(1 \pm i)\bar{a}p(t)] - \frac{\sqrt{2}\,|a|\,|q(t)|}{|z^2|},$$

where $z \in K \setminus \{0\}$, $t \in [t_0, \infty)$.

Further for $j = 1$ or $j = 2$ it holds

(8) $$j|z|^{j-1}(-|q(t)| - |z| \operatorname{Re}[p(t)z]) \leqq D_f V_j(t, z) \leqq$$
$$\leqq j|z|^{j-1}(|q(t)| - |z| \operatorname{Re}[p(t)z])$$

where $z \in K \setminus \{0\}$ or $z \in K$, respectively.

Remark 1. Trajectories $w(t)$ of (3) satisfying the initial condition $w(t_0) = w_0 \neq 0$ have the following properties:

(i) If $\operatorname{Im}[aw_0] \neq 0$, then $\operatorname{Re}\left[\frac{i\bar{a}}{w(t)}\right] = \gamma$, where $\gamma \in R \setminus \{0\}$ is determined by the initial condition, for all $t \geqq t_0$ and $w(t) \to 0$ as $t \to \infty$;

(ii) if $\text{Im}[aw_0] = 0$, $\text{Re}[aw_0] > 0$, then $\text{Im}[aw(t)] = 0$ for all $t \geqq t_0$ and $w(t) \to 0$ as $t \to \infty$;

(iii) if $\text{Im}[aw_0] = 0$, $\text{Re}[aw_0] < 0$, then $\text{Im}[aw(t)] = 0$ for $t \in [t_0, \omega)$, where $\omega < \infty$, and $\lim_{t \to \omega-} |z(t)| = \infty$.

The following lemmas are necessary for our later considerations.

Lemma 1. *Let $t_* < t^*$ and let $z(t)$ be a solution of* (1). *Assume $a \in K \setminus \{0\}$.*

Suppose (i) for $t \in [t_*, t^*]$ it holds

$$\text{Re}[az(t)] > 0 \tag{9}$$

and

$$|z(t)| \geqq |z(t_*)|; \tag{10}$$

(ii) *for $t \in [t_*, t^*]$ and $z \in M = \{z : \text{Re}[az] > 0, |z| \geqq |z(t_*)|\}$ it holds*

$$D_f W_j(t, z) \geqq 0, \qquad j = 1, 2, \tag{11}$$

where $W_j(z)$ is defined by (4).

Then, it holds

$$|z(t)| < 2|z(t_*)| \qquad \text{for } t \in [t_*, t^*].$$

Proof. It follows from the assumptions (9), (10), (11) that there exist $\gamma(t)$, $\gamma(t) > 0$ and $j \in \{1, 2\}$ such that $W_j(z(t)) = \gamma(t)$ for $t \in [t_*, t^*]$. By definition $W_j(z)$ we obtain

$$\frac{|z(t)|}{2} \leqq r(t) \leqq \frac{|z(t)|}{\sqrt{2}},$$

where $r(t)$ is the radius of the circle. This together with (10), (11) implies the statement of Lemma 1.

Lemma 2. *Let the hypothesis of Lemma 1 be satisfied with the exception that* $\text{Re}[az(t)] > 0$ *and* $|z(t)| \geqq |z(t_*)|$ *are replaced by* $\text{Re}[az(t)] < 0$ *and* $|z(t)| \geqq |z(t^*)|$, *respectively. Then, it holds*

$$|z(t)| < 2|z(t^*)| \qquad \textit{for } t \in [t_*, t^*].$$

Proof. The proof is analogous to that of the previous lemma.

2. MAIN RESULTS

Theorem 1. *Suppose*

$$\lim_{t \to \infty} q(t) = 0, \tag{12}$$

$$\lim_{t \to \infty} p(t) = a, \tag{13}$$

(14) $$\operatorname{Re}[aq(t)] \geqq 0, \qquad q(t) \neq 0$$

and

(15) $$\operatorname{Re}[\bar{a}p(t)] > 0$$

for $t \geqq t_0$, *where* $a \in K \setminus \{0\}$.

Then every solution $z(t)$ *of* (1) *satisfying at* $t_1 \geqq t_0$ *the condition*

(16) $$\operatorname{Re}[az(t_1)] \geqq 0$$

exists for all $t \geqq t_1$ *and it holds*

(17) $$\lim_{t \to \infty} z(t) = 0.$$

Proof. Let $z = z(t)$ be any solution of (1) satisfying (16).

First, we are going to establish domains where there occurs $z(t)$. It follows from (13), (15) that there exist $A > 0$, $B > 0$ such that

$$\operatorname{Re}\left[\frac{p(t)}{a}\right] \geqq A, \qquad \left|\operatorname{Im}\left[\frac{p(t)}{a}\right]\right| \leqq B \qquad \text{for } t \geqq t_0.$$

Then, with respect to (14), it holds for $t \geqq t_0$

$$\operatorname{Re}[aq(t)] - \operatorname{Re}[ap(t)\, z^2] \geqq -A \operatorname{Re}[a^2 z^2] - B\,|\operatorname{Im}[a^2 z^2]|.$$

Define $\Omega = \{z : -A \,.\, \operatorname{Re}[a^2 z^2] - B\,|\operatorname{Im}[a^2 z^2]| > 0\}$. It is easy to see that $\Omega \neq \emptyset$, and if $w \in \Omega$, then $-\operatorname{Re}[a^2 w^2] > 0$. Hence

(18) $$\operatorname{Re}[aq(t)] - \operatorname{Re}[ap(t)\, z^2] > 0$$

for $z \in \Omega$, $t \geqq t_0$, in the case $z = 0$ is valid (18) or

$$\operatorname{Re}[aq(t)] - \operatorname{Re}[ap(t)\, z^2] \geqq 0, \qquad \operatorname{Im}[aq(t)] - \operatorname{Im}[ap(t)\, z^2] \neq 0$$

for $t \geqq t_0$.

That implies (i) $\operatorname{Re}[az'(t)] > 0$ for $t \geqq t_1$ such that $z(t) \in \Omega$; (ii) $\operatorname{Re}[az'(t)] > 0$ or $\operatorname{Re}[az'(t)] \geqq 0$, $\operatorname{Im}[az'(t)] \neq 0$ for $t \geqq t_1$ such that $z(t) = 0$.

This together with (16) implies

(19) $$\operatorname{Re}[az(t)] \geqq 0, \qquad \operatorname{Re}[az(t)] = 0 \Leftrightarrow \operatorname{Im}[az(t)] = 0$$

for all $t \geqq t_1$ for which there exists $z(t)$.

Choose "Ljapunov" functions $W_j(z)$ defined by (4). Then there exists $\gamma(t) > 0$, $j \in \{1, 2\}$ such that $\gamma(t) = W_j(z(t))$ for $z(t) \neq 0$, $t \geqq t_1$. In view of (13), (15) we infer from (7) and (19) that $z(t)$ is bounded for all $t \geqq t_1$ for which there exists $z(t)$. From the fact that each limit point of the set $M = \{(t, z(t)), t \geqq t_1\}$ is on the boundary of the domain on which the right-hand side of (1) is continuous, it follows that $z(t)$ exists for all $t \geqq t_1$.

Now, it remains to prove (17). Let $\varepsilon > 0$ be arbitrary. From (12), (13) there follows the existence of $T = T(\varepsilon)$ such that for all $t \geqq T$ it holds

$$\operatorname{Re}\left[(1 \pm i)\,\bar{a}p(t)\right] \geqq \frac{2}{3}\,|a|^2,$$

$$|q(t)| \leqq \frac{|a|\,\varepsilon^2}{12}.$$

With respect to (7) we receive $D_f W_j(t, z) > 0$ for $t \geqq T$, $|z| \geqq \frac{\varepsilon}{2}$.

Put $J = \left\{t \geqq T : |z(t)| \geqq \frac{\varepsilon}{2}\right\}$. Suppose $J \neq \emptyset$. Then there exists $\tau = \tau(\varepsilon)$ such that $|z(\tau)| < \frac{\varepsilon}{2}$. We claim $|z(t)| < \varepsilon$ for all $t \geqq \tau$. If this were not true, there would exist a $t^* > \tau$ such that $|z(t^*)| \geqq \varepsilon$, and define $t_2 = \sup\left\{t \in [\tau, t^*] : |z(t)| < \frac{\varepsilon}{2}\right\}$. Clearly $t^* > t_2 > \tau$. Then,

$$|z(t_2)| = \frac{\varepsilon}{2}, \qquad |z(t)| \geqq \frac{\varepsilon}{2} \qquad \text{for } t \in [t_2, t^*].$$

Since $[t_2, t^*] \subset J$, we have $D_f W_j(t, z) > 0$, $j = 1, 2$, for $t \in [t_2, t^*]$ and $z \in M = \{z : |z| \geqq |z(t_2)|\}$. Using Lemma 1 we obtain

$$|z(t)| < 2\frac{\varepsilon}{2} = \varepsilon \qquad \text{for } t \in [t_2, t^*],$$

which contradicts $|z(t^*)| \geqq \varepsilon$. The proof is complete.

Theorem 2. *Let the assumptions of Theorem* 1 *be satisfied with the exception* (12) *is replaced by*

(20) $$\int_{t_0}^{\infty} |q(t)|\,\mathrm{d}t < \infty$$

and suppose in addition

(21) $$\operatorname{Im}\left[\bar{a}p(t)\right] \equiv 0 \qquad \textit{for } t \geqq t_0.$$

Then, the conclusion of Theorem 1 *is valid.*

Proof. Let $z = z(t)$ be any solution of (1) satisfying (16). To prove the boundedness and existence of $z(t)$ choose $V_1(z)$. In the proof of Theorem 1 we obtained (19) from (13), (14), (15) and (16). In addition it follows from (21)

$$\operatorname{Re}\left[p(t)\,z(t)\right] = \operatorname{Re}\left[\frac{p(t)}{a}\right]\operatorname{Re}\left[a z(t)\right],$$

thus with respect to (15) and (19) it holds

(22) $$\operatorname{Re}\left[p(t)\,z(t)\right] \geqq 0, \qquad \operatorname{Re}\left[p(t)\,z(t)\right] = 0 \Leftrightarrow z(t) = 0$$

for all $t > t_1$ for which there exists $z(t)$.

Integrating the second inequality of (8), where $z = z(t)$, from $t_2 \geqq t_1$ to t we get according to (20)

$$V_1(z(t)) \leqq V_1(z(t_2)) + \text{const}$$

for $t \geqq t_2$ such that $z(t) \neq 0$. From the same reason as in the previous proof it follows that $z(t)$ is defined for all $t \geqq t_1$.

First we are going to show $\liminf_{t \to \infty} | z(t) | = 0$. Suppose for the sake of argument, that there exists an $\varepsilon > 0$ such that $| z(t) | \geqq \varepsilon$ for $t \geqq t_2 \geqq t_1$. According the assumption (13) there exists $t_3 \geqq t_2$ such that $\operatorname{Re}[\bar{a}p(t)] \geqq \frac{2}{3} | a |^2$. Choosing the function $W(z)$ and integrating (6), where $z = z(t) \neq 0$, from $t_3 \geqq t_2$ to t we obtain

$$W(z(t)) \geqq W(z(t_3)) + \frac{2}{3} | a |^2 (t - t_3) - \frac{| a |}{\varepsilon^2} \int_{t_3}^{t} | q(s) | \, ds,$$

$W(z(t)) \to \infty$ for $t \to \infty$, a contradiction.

Now, let us prove (17). Choose the function $V_2(z)$. There exists a sequence $\{s_n\}$, $s_n \to \infty$ such that for arbitrary $\varepsilon > 0$ there exists $n_1 \in N$ such that $V_2(z(s_n)) < \frac{\varepsilon}{2}$ for $n \geqq n_1$. There exists a $L > 0$ such that $| z(t) | \leqq L$ for $t \geqq t_1$ and $n_2 \in N$ such that for $n \geqq n_2$ it holds

$$\int_{s_n}^{\infty} | q(s) | \, ds < \frac{\varepsilon}{4L}.$$

Let $n_3 = \max(n_1, n_2)$. Using (8) we get

$$V_2(z(t)) \leqq V_2(z(s_n)) + 2 \int_{s_n}^{t} | q(s) | \, | z(s) | \, ds - 2 \int_{s_n}^{t} | z(s) |^2 \operatorname{Re}[p(s) z(s)] \, ds,$$

for $t \geqq s_n$, $n \geqq n_3$ and with respect to (22)

$$V_2(z(t)) < \varepsilon \qquad \text{for } t \geqq s_n.$$

The proof is complete.

Theorem 3. *Let the assumptions of Theorem* 1 *be fulfilled.*

Let $z(t)$ be a complete solution of (1) *defined on $[t_1, \omega)$, where $t_1 \geqq t_0$.*

If $\omega = \infty$, then

$$\lim_{t \to \infty} z(t) = 0. \tag{23}$$

If $\omega < \infty$, then $\operatorname{Re}[az(t)] < 0$ *for $t \in [t_1, \omega)$ and*

$$\lim_{t \to \omega-} | z(t) | = \infty.$$

Proof. Let $z(t)$ be any solution of (1) defined on $[t_1, \omega)$. If $z(t)$ satisfies at $T \geqq t_1$ the condition $\operatorname{Re}[az(T)] \geqq 0$, then by Theorem 1 there hold $\omega = \infty$ and (23).

Now, let $\operatorname{Re}[az(t)] < 0$ be for $t \in [t_1, \omega)$. If $\omega < \infty$, then $\lim_{t \to \omega-} | z(t) | = \infty$.

Let $\omega = \infty$. Suppose by contradiction that (23) is not satisfied. Then, there exists a $K > 0$ such that $\limsup_{t\to\infty} | z(t) | \geqq 3K$. From (12), (13) it follows that there exists $T_1(K) \approx T_1 \geqq t_1$ such that

$$| q(t) | \leqq \frac{| a | K^2}{3}$$

$$\text{Re}\left[(1 \pm i)\,\bar{a}p(t)\right] \geqq \frac{2}{3} | a |^2$$

$$\text{Re}\left[\bar{a}p(t)\right] \geqq \frac{2}{3} | a |^2$$

for $t \geqq T_1$. From the definition of the superior limit it follows that there exists $T_2 \geqq T_1$ such that

$$| z(T_2) | \geqq 2K.$$

Using Lemma 2 it is not difficult to see that

(24) $$| z(t) | \geqq K \qquad \text{for } t \geqq T_2 .$$

Finally, choose the pencil of circles $W(z) = \gamma$, $\gamma < 0$ covering the half-plane $\text{Re}\,[az] < 0$. With respect to (24) there exists $\gamma_0 < 0$ so that $W(z(t)) \gneqq \gamma_0$ for $t \geqq T$. To each point of the domain $\text{Re}\,[az] < 0$, $W(z) \geqq \gamma_0$ there exists a unique circle $W(z) = \gamma$, $\gamma \in [\gamma_0, 0)$ passing through it.

According to (6) it holds

$$D_f W(t, z(t)) \geqq \frac{2}{3} | a |^2 - \frac{| a |^2 K^2}{3K^2} = \frac{1}{3} | a |^2 .$$

Integrating this inequality from $T \geqq T_2$ to t we get

$$W(z(t)) \geqq W(z(T)) + \frac{1}{3} | a |^2 (t - T) \to \infty \qquad \text{as } t \to \infty$$

which contradicts the fact that $\text{Re}\,[az(t)] < 0$ for $t \in [t_1, \infty)$.

Since in the case $\omega = \infty$ it holds (23) and the proof is complete.

Theorem 4. *Let the assumptions of Theorem 2 be fulfilled.*

Let $z(t)$ be a complete solution of (1) defined on $[t_1, \omega)$, where $t_1 \geqq t_0$.

Then, the conclusion of Theorem 3 is valid.

Proof. The scheme of the proof is in the main the same as that used in the proof of Theorem 3 and thus it will be omitted here.

Theorem 5. *Suppose in addition to the assumptions stated in Theorem 2 that* $\text{Re}\, p(t)$, $\text{Im}\, p(t)$ *are monotonic.*

Then, each solution $z(t)$ of (1) defined for all $t \geqq t_1 \geqq t_0$ satisfies for $\alpha \geqq 2$

(25) $$\int_{t_1}^{\infty} | z(t) |^{\alpha}\, dt < \infty .$$

Proof. According to Theorem 4 it holds $\lim_{t\to\infty} z(t) = 0$. Consider circles $V_1(z) = \gamma$, $\gamma > 0$. Put $\mathcal{M} = \{t \geqq t_1,\ z(t) \neq 0\}$, $\mathcal{M}_0 = [t_1, \infty)$. According to (8) for $t \in \mathcal{M}$ it holds

$$-|q(t)| - |z(t)| \operatorname{Re}[p(t)\, z(t)] \leqq D_f V_1(t, z(t)) = \\ = V_1'(z(t)) \leqq |q(t)| - |z(t)| \operatorname{Re}[p(t)\, z(t)].$$

Let $\tau \geqq t_1$ be such that $z(\tau) = 0$. Then

$$D^+ V_1'(z(\tau)) = \lim_{t\to\tau+} \frac{|z(t)|}{t-\tau} = |z'(\tau)| = |q(\tau)|,$$

$$D^- V_1'(z(\tau)) = \lim_{t\to\tau-} \frac{|z(t)|}{t-\tau} = -|q(\tau)|,$$

e.g. $V_1'(z(\tau))$ does not exist, as $q(t) \neq 0$ for $t \geqq t_0$. The set $\mathcal{M}_0 \setminus \mathcal{M}$ is, as known, at most countable.

Define

$$B(t) = \begin{cases} V_1'(z(t)) & t \in \mathcal{M} \\ 0 & t \in \mathcal{M}_0 \setminus \mathcal{M}. \end{cases}$$

For $t \in \mathcal{M}_0$ it holds

$$(26) \qquad -|q(t)| - |z(t)| \operatorname{Re}[p(t)\, z(t)] \leqq B(t) \leqq \\ \leqq |q(t)| - |z(t)| \operatorname{Re}[p(t)\, z(t)].$$

The function $B(t)$ is continuous on $\mathcal{M}$. Denote $\mathcal{M}_1 = \{t \geqq t_1 : B(t)$ is not continuous$\}$. Since $\mathcal{M}_1 \subset \mathcal{M}_0 \setminus \mathcal{M}$ is valid, $\mathcal{M}_1$ is at most countable and thus

$$\int_{t_1}^{t} B(s)\, ds = V_1(z(t)) - V_1(z(t_1)), \qquad t \geqq t_1.$$

Consequently integrating the inequality (26) we get

$$-\int_{t_1}^{t} |q(s)|\, ds - \int_{t_1}^{t} |z(s)| \operatorname{Re}[p(s)\, z(s)]\, ds \leqq V_1(z(t)) - V_1(z(t_1)) \leqq \\ \leqq \int_{t_1}^{t} |q(s)|\, ds - \int_{t_1}^{t} |z(s)| \operatorname{Re}[p(s)\, z(s)]\, ds.$$

From the proof of Theorem 2 it follows either that $\operatorname{Re}[p(t)\, z(t)] < 0$ for $t \geqq t_1$, or there exists $\tau \geqq t_1$ such that $\operatorname{Re}[p(t)\, z(t)] > 0$ for $t \geqq \tau$. Hence,

$$(27) \qquad \int_{t_1}^{\infty} |z(t)|\ |\operatorname{Re}[p(t)\, z(t)]|\, dt < \infty.$$

According to (13), (15) it follows from (27)

$$(28) \qquad \int_{t_1}^{\infty} \operatorname{Re}^2 [p(t)\, z(t)]\, dt < \infty.$$

Integration the equation (1) from t_1 to t, $t \to \infty$, we receive

$$\left| \int_{t_1}^{\infty} p(t)\, z^2(t)\, dt \right| < \infty.$$

Hence there exist integrals

$$\int_{t_1}^{\infty} \operatorname{Re} p(t) \operatorname{Re}[p(t)\, z^2(t)]\, dt, \qquad \int_{t_1}^{\infty} \operatorname{Im} p(t) \operatorname{Im}[p(t)\, z^2(t)]\, dt. \tag{29}$$

It holds $\operatorname{Re}[u] \operatorname{Re}[uz^2] - \operatorname{Re}^2[uz] = -|u|^2 \operatorname{Im}^2 z$, $\operatorname{Im}[u] \operatorname{Im}[uz^2] + \operatorname{Re}^2[uz] = |u|^2 \operatorname{Re}^2 z$. Using (28), (29) we get

$$\int_{t_1}^{\infty} |p(t)|^2 \operatorname{Im}^2 z(t)\, dt < \infty, \qquad \int_{t_1}^{\infty} |p(t|^2 \operatorname{Re}^2 z(t)\, dt < \infty,$$

therefore

$$\int_{t_1}^{\infty} |p(t)|^2 |z(t)|^2\, dt < \infty.$$

Thus, with respect to (13), (15) it holds

$$\int_{t_1}^{\infty} |z(t)|^2\, dt < \infty,$$

and with respect to (17) the inequality (25) is proved. The proof is complete.

Remark 2. Choose in the equation (1) the functions

$$p(t) \equiv 1, \qquad q_\alpha(t) = \frac{1}{\sqrt[\alpha]{t^2}} - \frac{1}{\alpha t \sqrt[\alpha]{t}} \qquad t \geqq t_0 > \frac{1}{\alpha},$$

where if $\alpha \geqq 2$ or $1 < \alpha < 2$, then the assumptions of Theorem 1 or Theorem 5, respectively, are fulfilled. Thus the solution $z(t) = \frac{1}{\sqrt[\alpha]{t}}$ for $t > \frac{1}{\alpha}$ does not satisfy (25).

This example shows the invalidity of the assertion of Theorem 5 under the assumptions of Theorem 1 and the invalidity of Theorem 5 for $1 < \alpha < 2$.

3. APPLICATIONS

Using some results concerning solutions of the Riccati differential equation we establish asymptotic behaviour of the equation

$$x'' + P(t)\, x' + Q(t)\, x = 0, \tag{30}$$

where $P(t)$ and $Q(t)$ are complex functions of the real variable $t \in J = [t_0, \infty)$ and x is the complex variable.

Remark 3. Let

(31) $$P(t) \in C^1(J), \qquad Q(t) \in C^0(J).$$

(i) If $x(t)$ is a solution of (30) on an interval $J_0 \subset J$ and $x(t) \neq 0$ on J_0, then the function

$$z(t) = x'(t)\, x^{-1}(t) + \frac{1}{2} P(t)$$

is a solution of the equation

(32) $$z' = \frac{1}{4} P^2(t) - Q(t) + \frac{1}{2} P'(t) - z^2$$

on J_0.

(ii) If $z(t)$ is a solution of (32) on $J_0 \subset J$ and $\beta \in J_0$ then the function

$$x(t) = \exp \int_\beta^t \left(z(s) - \frac{1}{2} P(s) \right) \mathrm{d}s$$

is a solution of (30) on J_0.

Successive corollaries imidiently follow from Theorem 1 – 5 and Remark 4.

Corollary 1. *Suppose* (31) *and*

(33) $$\lim_{t \to \infty} (P^2(t) - 4Q(t) + 2P'(t)) = 0,$$

(34) $$\operatorname{Re}[P^2(t) - 4Q(t) + 2P'(t)] \geqq 0, \qquad P^2(t) - 4Q(t) + 2P'(t) \neq 0.$$

Then each solution $x(t)$ *of* (30) *satisfing at* t_1 *initial conditions*

$$\operatorname{Re}\left[x'(t_1)\, x^{-1}(t_1) + \frac{1}{2} P(t_1) \right] \geqq 0, \qquad x(t_1) \neq 0,$$

exists for $t \geqq t_1$ *and it holds*

$$\lim_{t \to \infty} [2x'(t)\, x^{-1}(t) + (Pt)] = 0.$$

Corollary 2. *Let us assume* (31), (34) *and*

(35) $$\int_{t_0}^{\infty} | P^2(t) - 4Q(t) + 2P'(t) | \, \mathrm{d}t < \infty.$$

Then, the conclusion of Corollary 1 is valid.

Corollary 3. *Let us assume* (31), (33), (34) *and let* $x(t)$ *be a complete solution of* (30) *defined on* $[t_1, \omega)$, $t_1 \geqq t_0$.

If $\omega = \infty$, *then*

$$\lim_{t \to \infty} [2x'(t)\, x^{-1}(t) + P(t)] = 0.$$

If $\omega < \infty$, *then* $\operatorname{Re}[x'(t)\, x^{-1}(t) + \frac{1}{2} P(t)] < 0$ *for* $t \in [t_1, \omega)$ *and*

$$\lim_{t \to \omega -} \left| x'(t) x^{-1}(t) + \frac{1}{2} P(t) \right| = \infty.$$

Corollary 4. *Let us assume* (31), (34), (35) *and let* $x(t)$ *be a complete solution of* (30) *defined on* $[t_1, \omega)$, $t_1 \geqq t_0$.

Then, the conclusion of Corollary 3 is valid.

Corollary 5. *Let us suppose* (31), (34), (35).

Then, each solution $x(t)$ *of* (30) *defined for all* $t \geqq t_1 \geqq t_0$ *and* $x(t) \neq 0$, *satisfies for* $\alpha \geqq 2$

$$\int_{t_1}^{\infty} \left| x'(t) x^{-1}(t) + \frac{1}{2} P(t) \right|^{\alpha} \mathrm{d}t < \infty.$$

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