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Jaromír Šimša

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THE SECOND ORDER DIFFERENTIAL EQUATION WITH AN OSCILLATORY COEFFICIENT*

JAROMÍR ŠIMŠA, Brno

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Consider a scalar differential equation

$$\frac{d^2x}{dt^2} - x = p(t)\,x, \tag{1}$$

where the coefficient function $p(t)$, continuous for $t \geqq 0$, is "small" as $t \to \infty$. Under the smallness condition is meant, that the integral

$$\int^{\infty} p(t)\,t^q\,dt \tag{2}$$

converges (perhaps *relatively*) for some $q \geqq 0$. Then the following question arises: Are there two solutions $x_1(t)$ and $x_2(t)$ of the equation (1) satisfying

$$x_1(t) = (1 + o(t^{-q}))\,e^t, \qquad x_2(t) = (1 + o(t^{-q}))\,e^{-t} \tag{3}$$

as $t \to \infty$?

A classical theorem gives the positive answer to this question provided that the integral (2) converges *absolutely* (see [1], Th. 17.2). As shown in [2], in the case $q \geqq 1$, the asymptotic formulas (3) follow from *ordinary* convergence of the integral (2); in the case $0 \leqq q < 1$ the same assertion is true under a supplementary condition

$$\int^{\infty} t^{-q} \left| \int_t^{\infty} p(s)\,s^q\,ds \right| dt < \infty. \tag{4}$$

The aim of the present paper is to prove *essentiality* of the condition (4). Constructing an oscillatory coefficient function $p(t)$ we show that the mere condition of convergence of the integral (2) does not guarantee the validity of formulas (3).

Theorem. *Let q be a real number satisfying* $0 \leqq q < 1$. *Assume that a number r is chosen as follows*:

$$2q - 1 < r < 1 \qquad \textit{if } q \geqq 1/2, \tag{5}$$

or

$$0 < r < 1 - 2q \qquad \textit{if } q < 1/2. \tag{6}$$

Then there exists a real function $p(t)$ defined and continuous for $t \geqq 0$ such that the integral (2) *converges and the equation* (1) *has a solution $x(t)$ satisfying*

$$x(t) = \begin{cases} (1 + t^{-r} + o(t^{-r}))\, e^t & \text{if } q \geqq 1/2, \\ \exp(t - t^r + o(t^r)) & \text{if } q < 1/2 \end{cases} \tag{7}$$

as $t \to \infty$.

Proof. Let q and r be any numbers satisfying (5) or (6). Then there exist numbers q_1 and q_2 such that r is equal to $|q_1 + q_2 - 1|$ and it holds either $1/2 \leqq q < q_1 < q_2 < 1$ or $0 \leqq q < q_1 < q_2 < 1/2$. Having fixed the numbers q, r, q_1 and q_2 we put

$$\begin{aligned} \alpha_0 &= 2^{-q_1} . r, & \beta_0 &= 2^{-q^2}, \\ \alpha_n &= \alpha_0 . n^{-q_1} \quad \text{and} & \beta_n &= \beta_0 . n^{-q^2} \qquad \text{for } n = 1, 2, \dots \end{aligned} \tag{8}$$

Consider now a sequence of functions $F_n(\Delta)$ defined by

$$F_n(\Delta) = \frac{(2n + 1 - \Delta)^{1+q} - (2n + \Delta)^{1+q}}{(2n - \beta_n - \Delta)^{1+q} - (2n - 1 + \beta_n + \Delta)^{1+q}}$$

for $\Delta \in [0, 1/2 - \beta_n)$. Obviously, $F_n(\Delta) \in C[0, 1/2 - \beta_n)$ and $F_n(\Delta) \to +\infty$ as $\Delta \to 1/2 - \beta_n$. Therefore, the equation

$$F_n(\Delta) = \frac{2 + \alpha_n}{2 - \alpha_n} \tag{9}$$

has a solution $\Delta = \Delta_n$ on $(0, 1/2 - \beta_n)$ if the number $F_n(0)$ is smaller than the right hand side of (9). This condition written in the form

$$\alpha_n^{-1} . (F_n(0) - 1) < 2/(2 - \alpha_n)$$

is fulfilled for all sufficiently large n, since $2/(2 - \alpha_n) \to 1$ and, as we now verify, $\alpha_n^{-1} . (F_n(0) - 1) \to 0$ as $n \to \infty$. We have

$$\begin{aligned} \alpha_n^{-1} . (F_n(0) - 1) &= \alpha_n^{-1}\left(\frac{(2n + 1)^{1+q} - (2n)^{1+q}}{(2n - \beta_n)^{1+q} - (2n - 1 + \beta_n)^{1+q}} - 1\right) = \\ &= \alpha_n^{-1}\left(\frac{(1 + 2^{-1}n^{-1})^{1+q} - 1}{(1 - 2^{-1}n^{-1}\beta_n)^{1+q} - (1 - 2^{-1}n^{-1} + 2^{-1}n^{-1}\beta_n)^{1+q}} - 1\right) = \\ &= \alpha_n^{-1}\left(\frac{(1 + q)\, 2^{-1}n^{-1} + O(n^{-2})}{(1 + q)\, 2^{-1}n^{-1} - (1 + q)\, n^{-1}\beta_n + O(n^{-2})} - 1\right) = \\ &= \alpha_n^{-1}\left(\frac{1 + O(n^{-1})}{1 - 2\beta_n + O(n^{-1})} - 1\right) = \frac{2\alpha_n^{-1}\beta_n + \alpha_n^{-1}O(n^{-1})}{1 - 2\beta_n + O(n^{-1})}. \end{aligned}$$

The last ratio tends to zero as $n \to \infty$, because, by (8), $\alpha_n^{-1}\beta_n \to 0$ and $n\alpha_n \to \infty$ as $n \to \infty$. Thus the existence of the solution $\Delta = \Delta_n$ of (9) is established for all $n \geqq n_0$.

If we now define numbers a_n, b_n, c_n and $d_n (2n - 1 < a_n < b_n < 2n < c_n < d_n < < 2n + 1)$ by means of the formulas

$$a_n = 2n - 1 + \beta_n + \Delta_n, \qquad b_n = 2n - \beta_n - \Delta_n, \tag{10}$$
$$c_n = 2n + \Delta_n \quad \text{and} \quad d_n = 2n + 1 - \Delta_n \quad \text{for } n \geqq n_0,$$

then the property of Δ_n can be expressed as follows:

$$\frac{d_n^{1+q} - c_n^{1+q}}{b_n^{1+q} - a_n^{1+q}} = \frac{2 + \alpha_n}{2 - \alpha_n} \quad \text{for } n \geqq n_0. \tag{11}$$

Now we can define a function $u(t)$ of the class $C^1[0, \infty)$ by the following way (see Fig. 1): we put $u(t) = 0$ for $t \in [0, 2n_0 - 1)$, $u(t) = \alpha_n$ for $t \in [a_n, b_n]$ and $u(t) = -\alpha_n$ for $t \in [c_n, d_n]$, where $n = n_0, n_0 + 1, \ldots$ It remains to define $u(t)$ on the intervals $(2n - 1, a_n)$, (b_n, c_n) and $(d_n, 2n + 1)$. We can peiform it rather arbitrarily but for further considerations it is convenient to keep the following conditions: $0 \leqq u(t) \leqq \alpha_n$ for $t \in (2n - 1, a_n) \cup (b_n, 2n)$, $-\alpha_n \leqq u(t) \leqq 0$ for $t \in (2n, c_n) \cup (d_n, 2n + 1)$ and

$$\int_{2n-1}^{a_n} u(t)\,dt = \int_{b_n}^{2n} u(t)\,dt = -\int_{2n}^{c_n} u(t)\,dt = -\int_{d_n}^{2n+1} u(t)\,dt = \varepsilon_n, \tag{12}$$

where $\varepsilon_n > 0$ is a sufficiently small number such that

$$\sum_{n=n_0}^{\infty} (2n + 1)^q \varepsilon_n < \infty. \tag{13}$$

Note that for $t \in [2n - 1, 2n + 1]$ we have $|u(t)| \leqq \alpha_n = \alpha_0 n^{-q_1} \leqq 3^{q_1} \alpha_0 t^{-q_1}$, since $t \leqq 3n$. Thus the function $u(t)$ satisfies

$$|u(t)| \leqq 3^{q_1} \alpha_0 t^{-q_1}, \qquad 0 < t < \infty. \tag{14}$$

From the definition of $u(t)$, (10) and (12) it follows

$$\int_{2n-1}^{2n+1} u(t)\,dt = \alpha_n (b_n - a_n + c_n - d_n) = -2\alpha_n \beta_n < 0 \tag{15}$$

and

$$\int_{2n-1}^{2n} u(t)\,dt \leqq \alpha_n \int_{2n-1}^{2n} dt = \alpha_n. \tag{16}$$

Taking in account that the integral $\int^t u(s)\,ds$ is nondecreasing in t on $[2n - 1, 2n]$ and nonincreasing on $[2n, 2n + 1]$, we obtain from (15) and (16)

$$-2 \sum_{k=n_0}^{n} \alpha_k \beta_k \leqq \int_0^t u(s)\,ds \leqq \alpha_n - 2 \sum_{k=n_0}^{n-1} \alpha_k \beta_k, \tag{17}$$

where $t \in [2n - 1, 2n + 1]$ and $n = n_0, n_0 + 1, \ldots$

In what follows we shall use some properties of a series

$$\sum_{k=1}^{\infty} k^{-\gamma}, \qquad \gamma = \text{const.} > 0. \tag{18}$$

Namely, the series (18) diverges for $\gamma < 1$ and it holds

$$n^{\gamma-1} \sum_{k=1}^{n} k^{-\gamma} \to (1-\gamma)^{-1} \quad \text{as} \quad n \to \infty. \tag{19}$$

For $\gamma > 1$ the series (18) converges and it holds

$$n^{\gamma-1} \sum_{k=n}^{\infty} k^{-\gamma} \to (\gamma-1)^{-1} \quad \text{as} \quad n \to \infty. \tag{20}$$

Easy proofs of (19) and (20) are based on a comparison of the series (18) to an area between the curve $x = t^{-\gamma}$ and the t-axis. They are omitted here.

Returning to (17) we distinguish two cases: $q < 1/2$ and $q \geqq 1/2$. If $q < 1/2$, then, by (17), the integral of $u(t)$ diverges to $-\infty$, since $\alpha_k \beta_k = \alpha_0 \beta_0 k^{-q_1-q_2}$ and $q_1 + q_2 = 1 - r < 1$. Note that for $t \in [2n-1, 2n+1]$ we have $t = (2 + o(1))n$. Thus the inequality (17) implies that the value of an expression

$$t^{-r} \int_0^t u(s)\, ds$$

lies between two values of the common form

$$-2\alpha_0\beta_0(2^{-r} + o(1))\, n^{-r}\Big(\sum_{k=1}^{n} k^{r-1} + O(1)\Big),$$

which has, by (8) and (19), a limit equal to -1. Consequently,

$$t^{-r} \int_0^t u(s)\, ds \to -1 \quad \text{as} \quad t \to \infty. \tag{21}$$

If $q \geqq 1/2$, then by (17), the integral of $u(t)$ converges, since $\alpha_k \beta_k = \alpha_0 \beta_0 k^{-q_1-q_2}$ and $q_1 + q_2 = 1 + r > 1$. Let us rewrite (17) in a form

$$-\alpha_n - 2\sum_{k=n}^{\infty} \alpha_k \beta_k \leqq \int_t^{\infty} u(s)\, ds \leqq -2 \sum_{k=n+1}^{\infty} \alpha_k \beta_k.$$

This means that the value of an expression

$$t^r \int_t^{\infty} u(s)\, ds$$

lies between two values of the common form

$$-2\alpha_0\beta_0(2^r + o(1))\, n^r\Big(\sum_{k=n}^{\infty} k^{-r-1} + O(n^{-q_1})\Big),$$

which has, by (8) and (20), a limit equal to -1. Consequently,

$$t^r \int_t^\infty u(s)\,ds \to -1 \quad \text{as} \quad t \to \infty. \tag{22}$$

Now we define functions $p(t)$ and $x(t)$ by

$$p(t) = u'(t) + 2u(t) + u^2(t) \qquad (0 \leqq t < \infty) \tag{23}$$

and

$$x(t) = C \exp\{t + \int_0^t u(s)\,ds\} \qquad (0 \leqq t < \infty),$$

where a constant C is chosen as follows:

$$C = \begin{cases} -\int_0^\infty u(t)\,dt & \text{if} \quad q \geqq 1/2, \\ 1 & \text{if} \quad q < 1/2. \end{cases}$$

Then the function $x(t)$ is a solution of (1) for $p(t)$ from (23) and, by (21) and (22), it satisfies (7). To finish the proof we must now show that the integral (2) converges for our function $p(t)$ from (23). To this purpose it will be shown that there converge both integrals

$$\int^\infty u'(t)\,t^q\,dt \qquad \text{and} \qquad \int^\infty (2u(t) + u^2(t))\,t^q\,dt. \tag{24}$$

As to the first one, integrating by parts we obtain

$$\int_t^T u'(s)\,s^q\,ds = u(s)\,s^q\,|_t^T - q \int_t^T u(s)\,s^{q-1}\,ds.$$

From (14) we have $u(t)\,t^q - 0$ as $n \to \infty$ and

$$\int^\infty |\,u(t)\,|\,t^{q-1}\,dt \leqq 3^{q_1}\alpha_0 \int^\infty t^{q-q_1-1}\,dt < \infty,$$

since $q < q_1$. Thus the first integral in (24) converges.

The second integral in (24) converges if a function

$$U(t) = \int_0^t (2u(s) + u^2(s))\,s^q\,ds \tag{25}$$

has a finite limit $U(\infty)$. The function $u(t)$ has been defined in such a manner that $U(t)$ is nondecreasing on $[2n-1, 2n]$ and nonincreasing on $[2n, 2n+1]$ for any $n = 1, 2, \ldots$ Thus we have

$$\min\{U(2n-1), U(2n+1)\} \leqq U(t) \leqq U(2n) \tag{26}$$

for $t \in [2n-1, 2n+1]$

The equality (11) may be written as follows: $U(b_n) - U(a_n) + U(d_n) - U(e_n) = 0$, and thus

$$U(2n+1) - U(2n-1) = [U(2n+1) - U(d_n)] + [U(c_n) - U(2n)] + \\ + [U(2n) - U(b_n)] + [U(a_n) - U(2n-1)]. \tag{27}$$

Further, the estimate (14) enables to bound the integrand in (25):

$$|\,2u(t) + u^2(t)\,|\,t^q \leqq 3\,|\,u(t)\,|\,t^q \leqq 3^{1+q_1}\alpha_0 t^{q-q_1}. \tag{28}$$

From (12), (27) and (28) we have

$$|U(2n+1) - U(2n-1)| \leqq 3(2n+1)^q \{ \int_{d_n}^{2n+1} + \int_{2n}^{c_n} + \int_{b_n}^{2n} + \int_{2n-1}^{a_n} \} |u(s)| \, ds =$$

$$= 12(2n+1)^q \varepsilon_n ,$$

which, with respect to (13), gives

$$\sum_{n=1}^{\infty} |U(2n+1) - U(2n-1)| < \infty .$$

Consequently, the sequence $\{U(2n+1)\}$ has a finite limit:

(29) $$U(2n+1) \to L = const. \neq \infty \qquad \text{as } n \to \infty .$$

It holds also

(30) $$U(2n) \to L \qquad \text{as } n \to \infty ,$$

because, by (28), the difference $U(2n) - U(2n-1)$ tends to zero:

$$0 \leqq U(2n) - U(2n-1) \leqq 3^{1+q_1} \alpha_0 \int_{2n-1}^{2n} t^{q-q_1} \, dt = o(1), \qquad \text{since} \qquad q_1 > q .$$

From (26), (29) and (30) we can see that $U(t) \to L$ as $t \to \infty$. The proof of Theorem is complete.

Remark. The proved Theorem suggests that in the case $q < 1$ the remainders $o(t^{-q})$ in (3) must be replaced by $o(t^{1-2q})$. We hope to prove this conjecture on another occasion.

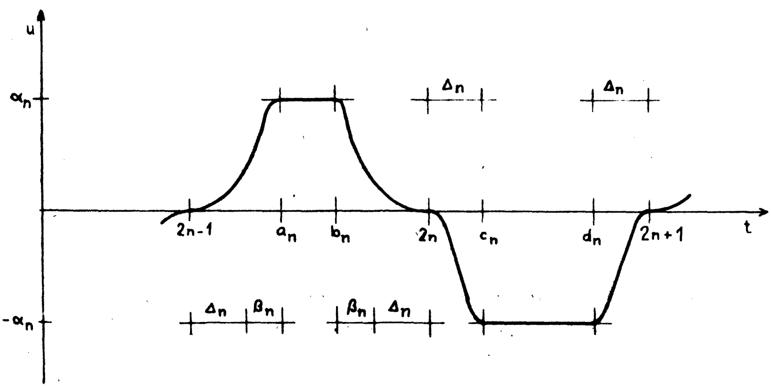

Fig. 5.

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J. Šimša
662 95 Brno, Janáčkovo nám. 2a
Czechoslovakia