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COVERING k-GROUPS OF n-GROUPS

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1. **Introduction.** Covering groups and their special case — the free covering groups appeared originally in the fundamental Post's paper [10] on n-groups. Also there the problem of reducibility of n-groups, formerly considered by Dörnte in [1], was discussed. Problems involving these notions also appeared later in [4], [8], [11], [2] and many other papers on the theory of n-semigroups and n-groups. One can find there also some modifications of Post's construction of the free covering group (called by him the abstract containing ordinary group).

In our paper we introduce generalizations of these notions, i.e. the notion of covering k-groups and free covering k-groups. We also show a construction of the free covering k-group (for $k = 2$ it is simply the free covering group), different in detail from Post's construction. Furthermore, we arrive at a generalization of Post's Coset Theorem, proved by methods distinct from those employed by Post, and a connection between covering k-groups and reducibility of n-groups.

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Most of the results of the present paper was annouced in [5].

2. **Some notions and notation.** We will be interested mainly in inner properties of n-groups, thus by n-group we always mean a non-empty one (as it was originally understood by Dörnte in [1]). Introducing the empty n-group appears to be convenient, when considering the category of n-groups (see [7]).

We use the usual notations which may be found in papers on n-groups, in particular in [3].

Let $\mathfrak{G} = (G,f)$ be an n-group. If $m = u(n - 1) + 1$, the m-ary operation g given by

$$g(x_1, \ldots, x_m) = \underbrace{f(f(\ldots,f(f}_{u}(x_1, \ldots, x_n), x_{n+1}, \ldots, x_{2n-1}) \ldots), \ldots, x_m),$$

called the simple iteration of the operation f, will be denoted by $f_{(u)}$. In certain situations, when the arity of the simple iteration does not play a crucial role, or when it will differ depending on additional assumptions, we will write $f_{(\cdot)}$, to mean $f_{(u)}$ for some $u = 1, 2, \ldots$ It has been already shown by Dörnte (see [1]) that the m-groupoid $\mathfrak{G}_{(u)}$ is an m-group. We will call $\mathfrak{G}_{(u)}$ the m-group derived from the n-group $\mathfrak{G}$, and $\mathfrak{G}$ — the creating n-group of the m-group $\mathfrak{G}_{(u)}$. Further, if $\mathfrak{G} = (G,f)$ will be an m-group, we denote any of its creating n-groups by $\mathfrak{G}_{(u^{-1})} = (G, f_{(u^{-1})})$. To justify that symbol, note that for simple iterations the formula $(f_{(\alpha)})_{(\beta)} = f_{(\alpha\beta)}$ holds.

The skew element to x (in an n-group $\mathfrak{G} = (G,f)$) will be usually denoted by $\bar{x}$, instead of $\bar{x}^{(f)}$, when it is clear from the context, that $\bar{x}$ is the skew element to x with respect to the n-group operation f.

We use the symbol $\overset{q}{x}$ in the following meaning:

for $q > 0$
$$f_{(.)}(x_1, \ldots, x_r, \overset{q}{x}, x_{r+q+1}, \ldots, x_n) =$$
$$= f_{(.)}(x_1, \ldots, x_r, x_{r+1}, \ldots, x_{r+q}, x_{r+q+1}, \ldots, x_n),$$
if $x_{r+1} = \ldots = x_{r+q} = x$;

for $q = 0$
$$f_{(.)}(x_1, \ldots, x_r, \overset{0}{x}, x_{r+1}, \ldots, x_n) = f_{(.)}(x_1, \ldots, x_r, x_{r+1}, \ldots, x_n).$$

In the category of n-groups monomorphisms coincide with injective homomorphisms and epimorphisms with surjective homomorphisms (see [7]). For that reason we call injective and surjective homomorphisms shortly monomorphisms and epimorphisms, although we never consider the category of n-groups. Some formulations will be expressed, for clarity, in terms of embeddings, instead of monomorphisms.

Definition 1. Let $\tau\colon \mathfrak{G} \to \mathfrak{A}_{(s)}$ be an embedding of the n-group $\mathfrak{G} = (G,f)$ into the n-group $\mathfrak{A}_{(s)} = (A, g_{(s)})$, derived from a k-group $\mathfrak{A} = (A, g)$, where $n = s(k-1) + 1$. If $\tau(G)$ generates the k-group $\mathfrak{A}$, the pair $\langle \mathfrak{A}, \tau \rangle$ will be called a covering k-group of the n-group $\mathfrak{G}$.

Definition 2. If the covering k-group $\langle \mathfrak{A}, \tau \rangle$ of the n-group $\mathfrak{G}$ satisfies the additional condition:

for each homomorphism $h; \mathfrak{G} \to \mathfrak{B}_{(s)}$, where $\mathfrak{B}_{(s)} = (B, g_{(s)})$ is an n-group derived from an arbitrary k-group $\mathfrak{B} = (B, g)$, there exists a unique homomorphism $h^* : \mathfrak{A} \to \mathfrak{B}$ such that $h^*\tau = h$ then the pair $\langle \mathfrak{A}, \tau \rangle$ is called a free covering k-group of the n-group $\mathfrak{G}$.

It is an immediate consequence of the definition that the free covering k-group is determined uniquely up to an isomorphisms. If the pair $\langle \mathfrak{A}, \tau \rangle$, where $\tau\ ;\ \mathfrak{G} \to \mathfrak{A}_{(s)}$ is a covering k-group of the n-group $\mathfrak{G}$, then, for each $\alpha = 0, 1, \ldots$, it is also a covering k-group of the $(\alpha(n-1)+1)$-group $\mathfrak{G}_{(\alpha)}$ derived from the n-group $\mathfrak{G}$.

The set $Z_s = \{0, 1, \dots, s - 1\}$, where s is a positive integer, with the $(k + 1)$-ary operation $\varphi(l_1, \dots, l_{k+1}) \equiv l_1 + \dots + l_{k+1} + 1 \pmod{s}$ forms a $(k + 1)$-group. It is a cyclic $(k + 1)$-group of order s (see [10]). We will denote it by $\mathfrak{C}_{s,k+1} = (Z_s, \varphi)$.

For the sake of description of the construction of the free covering k-group of some n-group and for investigation of covering k-groups it will be convenient to treat $(k + 1)$-groups and $(n + 1)$-groups rather, than k-groups and n-groups. Henceforth throughout the whole paper we assume always $n = sk$, $s = mq$.

3. **Free covering $(k + 1)$-groups of $(n + 1)$-groups.** Let $\mathfrak{G} = (G,f)$ be arbitrary $(n + 1)$-group and $c \in G$ an arbitrary, but fixed element of it. Form the set $G^{*s} = G \times Z_s$ and define a $(k + 1)$-ary operation f^* in G^{*s} in the following manner: Let

$$x_1, \dots, x_{k+1} \in G, \qquad l_1, \dots, l_{k+1} \in Z_s$$

then

$$f^*((x_1, l_1), \dots, (x_{k+1}, l_{k+1})) =$$
$$= (f_{(.)}(x_1, c^{l_{1k}}, \dots, x_{k+1}, c^{l_{k+1}k}, \bar{c}, c^{n-1-\varphi(l_1, \dots, l_{k+1})k}), \varphi(l_1, \dots, l_{k+1})).$$

Theorem 1. *The $(k + 1)$-groupoid $\mathfrak{G}^{*s} = (G^{*s},f^*)$ with the mapping $\tau : G \to G^{*s}$ given by the formula $\tau(x) = (x, 0)$ is a free covering $(k + 1)$-group of the $(n + 1)$-group $\mathfrak{G} = (G,f)$.*

Proof. We first show that the $(k + 1)$-groupoid $\mathfrak{G}^{*s} = (G^{*s},f^*)$ forms a $(k + 1)$-group.

In fact, let

$$a_1, \dots, a_{2k+1} \in G, \qquad l_1, \dots, l_{2k+1} \in Z_s,$$

then

$$f^*((a_1, l_1), \dots, (a_{i-1}, l_{i-1}), f^*((a_i, l_i), \dots, (a_{i+k}, l_{i+k})),$$
$$(a_{i+k+1}, l_{i+k+1}), \dots, (a_{2k+1}, l_{2k+1})) =$$
$$= f^*((a_1, l_1), \dots, (a_{i-1}, l_{i-1}), (f_{(.)}(a_i, c^{l_ik}, \dots, a_{i+k}, c^{l_{i+k}k}, \bar{c}, c^{n-1-\varphi(l_i, \dots, l^{i+k})k}), \varphi(l_i, \dots, l_{i+k})), \dots, (a_{2k+1}, l_{2k+1})) =$$
$$= (f_{(.)}(a_1, c^{l_1k}, \dots, a_i, c^{l_ik}, \dots, a_{i+k}, c^{l_{i+k}k}, \bar{c}, c^{n-1-\varphi(l_i, \dots, l_{i+k})k},$$
$$c^{\varphi(l_i, \dots, l_{i+k})k}, \dots, a_{2k+1}, c^{l_{2k+1}k}, \bar{c}, c^{n-1-\varphi_{(2)}(l_1, \dots, l_{2k+1})k}),$$
$$, \varphi_{(2)}(l_1, \dots, l_{2k+1})) =$$
$$= (f_{(.)}(a_1, c^{l_1k}, \dots, a_i, c^{l_ik}, \dots, a_k, c^{l_kk}, a_{k+1}, c^{l_{k+1}k}, \dots, a_{2k+1}, c^{l_{2k+1}k}, \bar{c},$$

$$\overset{n-1-\varphi_{(2)}(l_1,\dots,l_{2k+1})k}{c}), \varphi_{(2)}(l_1,\dots,l_{2k+1}))$$

which proves that the operation f^* is associative, thus $\mathfrak{G}^{*s}$ is an $(n+1)$-semigroup.

Now, we check the solvability of equations. Consider the equation

$$(*) \qquad f^*((a_1, l_1), \dots, (a_i, l_i), (x, \alpha), (a_{i+1}, l_{i+1}), \dots, (a_{k+1}, l_{k+1})) = (b, t)$$

with the variable (x, α). Then

$$(f_{(.)}(a_1, \overset{l_1 k}{c}, \dots, a_i, \overset{l_i k}{c}, x, \overset{\alpha k}{c}, a_{i+1}, \overset{l_{i+1}k}{c}, \dots, a_{k+1}, \overset{l_{k+1}k}{c}, \bar{c},$$
$$, \overset{n-1-\varphi(l_1,\dots,l_i,\alpha,l_{i+1},\dots,l_{k+1})k}{c}, \varphi(l_1, \dots, l_i, \alpha, l_{i+1}, \dots, l_{k+1})) = (b, t)$$

which leads to the following system of equations:

$$\varphi(l_1, \dots, l_i, \alpha, l_{i+1}, \dots, l_{k+1}) = t,$$
$$f_{(.)}(a_1, \overset{l_1 k}{c}, \dots, a_i, \overset{l_i k}{c}, x, \overset{\alpha k}{c}, a_{i+1}, \overset{l_{i+1}k}{c}, \dots, a_{k+1}, \overset{l_{k+1}k}{c}, \bar{c},$$
$$, \overset{n-1-\varphi(l_1,\dots,l_i,\alpha,l_{i+1},\dots,l_{k+1})k}{c}) = b.$$

The first of these equations has exactly one solution for α in the $(k+1)$-group $\mathfrak{C}_{s,k+1}$, and the second has exactly one solution for x in the $(n+1)$-group $\mathfrak{G}$. Thus the original equation $(*)$ has exactly one solution in the $(k+1)$-semigroup $\mathfrak{G}^{*s}$, whence the $(k+1)$-semigroup $\mathfrak{G}^{*s}$ is a $(k+1)$-group.

Consider the mapping $\tau : G \to G^{*s}$ given by the formula $\tau(x) = (x, 0)$. Note, that

$$f^*_{(s)}(\tau(a_1), \dots, \tau(a_{n+1})) = f^*_{(s)}((a_1, 0), \dots, (a_{n+1}, 0)) =$$
$$= (f_{(.)}(a_1, \overset{0}{c}, \dots, a_{n+1}, \overset{0}{c}, \bar{c}, \overset{n-1-0}{c}), 0) = (f(a_1, \dots, a_{n+1}), 0) =$$
$$= \tau(f(a_1, \dots, a_{n+1})),$$

which proves $\tau : \mathfrak{G} \to (\mathfrak{G}^{*s})_{(s)}$ to be a homomorphism. Obviously, τ is even a monomorphism.

Let $(a, l) \in G^{*s}$. Then

$$f^*_{(l)}((a, 0), \underbrace{(c, 0), \dots, (c, 0)}_{lk}) = (f(a, \overset{lk}{c}, \bar{c}, \overset{n-1-lk}{c}), l) = (a, l),$$

thus the $(k+1)$-group $\mathfrak{G}^{*s}$ is generated by the set $\tau(G)$.

Take an arbitrary homomorphism $h : \mathfrak{G} \to \mathfrak{B}_{(s)}$, where $\mathfrak{B}_{(s)} = (B, g_{(s)})$ is an $(n + 1)$-group derived from an arbitrary $(k + 1)$-group $\mathfrak{B} = (B, g)$. Define the mapping $h^* : G^{*s} \to B$ by the formula $h^*(a, l) = g_{(l)}(h(a), \overset{lk}{h(c)})$. Then

$$h^*(f^*((a_1, l_1), \dots, (a_{k+1}, l_{k+1}))) =$$

$$= h^*(f_{(\cdot)}(a_1, \overset{l_1 k}{c}, \dots, a_{k+1}, \overset{l_{k+1} k}{c}, \bar{c}, \overset{n-1-\varphi(l_1, \dots, l_{k+1})k}{c}), \varphi(l_1, \dots, l_{k+1})) =$$

$$= g_{(\varphi(l_1, \dots, l_{k+1}))}(h(f_{(\cdot)}(a_1, \overset{l_1 k}{c}, \dots, a_{k+1}, \overset{l_{k+1} k}{c}, \bar{c}, \overset{n-1-\varphi(l_1, \dots, l_{k+1})^k}{c})),$$

$$\overset{\varphi(l_1, \dots, l_{k+1})k}{h(c)}) =$$

$$= g_{(\cdot)}(h(a_1)\ \overset{l_1 k}{h(c)}, \dots, h(a_{k+1}), \overset{l_{k+1} k}{h(c)}, \overline{h(c)}, \overset{n-1}{h(c)}) =$$

$$= g(g_{(l_1)}(h(a_1), \overset{l_1 k}{h(c)}), \dots, g_{(k+1)}(h(a_{k+1}), \overset{l_{k+1} k}{h(c)})) =$$

$$= g(h^*(a_1), \dots, h^*(a_{k+1})).$$

Note, that $\overline{h(c)}$ in the former calculation denoted the element skew to $h(c)$ in the $(n + 1)$-group $\mathfrak{B}_{(s)}$, since a homomorphism commutes with the operation of taking the skew element.

It follows that the mapping $h^* : \mathfrak{G}^{*s} \to \mathfrak{B}$ is a homomorphism. It is easy to see that $h^*\tau = h$. Moreover, h^* is a unique homomorphism for which that equality holds, since $\tau(G)$ generates the $(k + 1)$-group $\mathfrak{G}^{*s}$. ∎

To simplify notation, the symbol $\mathfrak{G}^{*\alpha}_{(\beta)}$ will stand always for $(\mathfrak{G}^{*\alpha})_{(\beta)}$.

It turns out that a free covering group contains all free covering $(k + 1)$-groups as its subsets.

Corollary 1. A free covering $(k + 1)$-group $\mathfrak{G}^{*s}$ of an $(n + 1)$-group $\mathfrak{G}$ is isomorphic to the sub-$(k + 1)$-group of the $(k + 1)$-group $\mathfrak{G}^{*n}_{(k)}$ derived from the free covering group $\mathfrak{G}^{*n}$, consisting of elements of the form (a, lk) where $l = 0, 1, \dots, s-1$.

Proof. Define a mapping $w : G^{*s} \to G^{*n}$ by the formula $w(x, l) = (x, lk)$. It is easy to see that this mapping is injective. Furthermore,

$$w(f^*((a_1, l_1), \dots, (a_{k+1}, l_{k+1}))) =$$

$$= w(f_{(\cdot)}(a_1, \overset{l_1 k}{c}, \dots, a_{k+1}, \overset{l_{k+1} k}{c}, \bar{c}, \overset{n-1-\varphi(l_1, \dots, l_{k+1})k}{c}), \varphi(l_1, \dots, l_{k+1})) =$$

$$= (f_{(\cdot)}(a_1, \overset{l_1 k}{c}, \dots, a_{k+1}, \overset{l_{k+1} k}{c}, \bar{c}, \overset{n-1-\varphi(l_1, \dots, l_{k+1})k}{c}), \varphi(l_1, \dots, l_{k+1})k) =$$

$$= f^*_{(k)}((a_1, l_1 k), \ldots, (a_{k+1}, l_{k+1} k)) =$$
$$= f^*_{(k)}(w(a_1, l_1), \ldots, w(a_{k+1}, l_{k+1})),$$

which proves that the mapping $w : \mathfrak{G}^{*s} \to \mathfrak{G}^{*n}_{(k)}$ is a homomorphism. Each element of the $(k + 1)$-group $\mathfrak{G}^{*n}_{(k)}$ of the form (a, lk) is an element of the set $w(G^{*s})$. Thus the set of elements of the form (a, lk), where $l = 0, 1, \ldots, s - 1$, is a $(k + 1)$-group, isomorphic to $\mathfrak{G}^{*s}$. ∎

4. **The Coset Theorem.** Post proved the following characterization of covering groups: A group G' is a covering group of a certain n-group G iff there exists such an invariant subgroup G_0 of G' such that $G'/G_0 = Z_u$, where u is a divisor of $n - 1$. The following theorem is a generalization of that theorem to covering k-groups.

Theorem 2. *A $(k + 1)$-group $\mathfrak{A} = (A, g)$ is a covering $(k + 1)$-group of a certain $(n + 1)$-group if and only if for some $\alpha \in \{1, \ldots, q\}$ there exists an invariant sub-$(\alpha k + 1)$-group $\mathfrak{B} = (B, g_{(\alpha)})$ of the $(\alpha k + 1)$-group $\mathfrak{A}_{(\alpha)}$ such that $\mathfrak{A}_{(\alpha)}/\mathfrak{B} = (\mathfrak{C}_{q,k+1})_{(\alpha)}$, where the natural mapping $\zeta : A \to Z_q$ is a homomorphism of $\mathfrak{A}$ onto $\mathfrak{C}_{q,k+1}$. Then each subset of the form $\zeta^{-1}(l)$, where $l \in \mathfrak{C}_{q,k+1}$ is an element of order β, is an invariant sub-$(\beta k + 1)$-group of the $(\beta k + 1)$-group $\mathfrak{A}_{(\beta)}$. Moreover, for the element $l \in \mathfrak{C}_{q,k+1}$ of order q the pair $\langle \mathfrak{A}, \lambda \rangle$, where λ is the inclusion of $\zeta^{-1}(l)$ into A, is a covering $(k + 1)$-group of the $(n + 1)$-group $(\zeta^{-1}(l), g_{(s)})$.*

Proof. Let the pair $\langle \mathfrak{A}, \lambda \rangle$ be a covering $(k + 1)$-group of the $(n + 1)$-group $\mathfrak{D} = (G, f)$. Consider the free covering $(k + 1)$-group $\mathfrak{G}^{*s} = (G^{*s}, f^*)$ together with the embedding $\tau : \mathfrak{G} \to \mathfrak{G}^{*s}_{(s)}$. There exists a unique homomorphism $\lambda^* : \mathfrak{D}^{*s} \to \mathfrak{A}$ such that $\lambda^* \tau = \lambda$. Since, according to Definition 1, the subset $\lambda(G)$ generates the $(k + 1)$-group $\mathfrak{A}$, the homomorphism λ^* is a surjection. Let $\lambda^*(a_1, l) = \lambda^*(a_2, l)$. Then, according to Theorem 1, $g_{(l)}(\lambda(a_1), \overset{lk}{\lambda(c)}) = g_{(l)}(\lambda(a_2), \overset{lk}{\lambda(c)})$, hence $\lambda(a_1) = \lambda(a_2)$. The homomorphism λ being injective, this yields $a_1 = a_2$. Let us denote $W_l = \{(a, l) : a \in G\}$ for $l = 0, 1, \ldots, s - 1$. Thus the mapping $\lambda^* |_{W_l}$ is injective for $l = 0, 1, \ldots, s - 1$.

Now, let $\lambda^*(W_{l_1}) \cap \lambda^*(W_{l_2}) \neq \emptyset$. We show, that in consequence $\lambda^*(W_{l_1}) = \lambda^*(W_{l_2})$. In fact, let for some $a_1, a_2 \in G : \lambda^*(a_1, l_1) = \lambda^*(a_2, l_2)$ (we can assume $l_2 > l_1$). Then

$$g_{(l_1)}(\lambda(a_1), \overset{l_1 k}{\lambda(c)}) = g_{(l_2)}(\lambda(a_2), \overset{l_2 k}{\lambda(c)}),$$
$$g_{(l_1)}(\lambda(a_1), \overset{l_1 k}{\lambda(c)}) = g_{(l_2)}(\lambda(a_2), \overset{(l_2 - l_1) k}{\lambda(c)}, \overset{l_1 k}{\lambda(c)}),$$
$$\lambda(a_1) = g_{(l_2 - l_1)}(\lambda(a_2), \overset{(l_2 - l_1) k}{\lambda(c)}).$$

Take an arbitrary element $(x, l_2) \in W_{l_2}$. Then

$$\lambda^*(x, l_2) = g_{(l_2)}(\lambda(x), \overset{l_2k}{\lambda(c)}) =$$

$$= g_{(s+l_2)}(\lambda(x), \overset{n-2}{\lambda(a_2), \lambda(\bar{a}_2)}, \lambda(a_2), \overset{l_2k}{\lambda(c)}) =$$

$$= g_{(s+l_2)}(\lambda(x), \overset{n-2}{\lambda(a_2), \lambda(\bar{a}_2)}, \lambda(a_2), \overset{(l_2-l_1)k}{\lambda(c)}, \overset{l_1k}{\lambda(c)}) =$$

$$= g_{(s+l_1)}(\lambda(x), \overset{n-2}{\lambda(a_2), \lambda(\bar{a}_2)}, \lambda(a_1), \overset{l_1k}{\lambda(c)}) =$$

$$= g_{(l_1)}(\lambda(f(\overset{n-2}{x, a_2, \bar{a}_2}, a_1)), \overset{l_1k}{\lambda(c)}) = \lambda^*(f(\overset{n-2}{x, a_2, \bar{a}_2}, a_1), l_1).$$

Similarly, let $(x, l_1) \in W_{l_1}$. Then

$$\lambda^*(x, l_1) = g_{(l_1)}(\lambda(x), \overset{l_1k}{\lambda(c)}) =$$

$$= g_{(s+l_1)}(\lambda(x), \overset{n-2}{\lambda(a_1), \lambda(\bar{a}_1)}, \lambda(a_1), \overset{l_1k}{\lambda(c)}) =$$

$$= g_{(s+l_2)}(\lambda(x), \overset{n-2}{\lambda(a_1), \lambda(\bar{a}_1)}, \lambda(a_2), \overset{(l_2-l_1)k}{\lambda(c)}, \overset{l_1k}{\lambda(c)}) =$$

$$= g_{(l_2)}(\lambda(f(\overset{n-2}{x, a_1, \bar{a}_1}, a_2)), \overset{l_2k}{\lambda(c)}) = \lambda^*(f(\overset{n-2}{x, a_1, \bar{a}_1}, a_2), l_2).$$

The last equalities show that $\lambda^*(W_{l_1}) = \lambda^*(W_{l_2})$.

Let for some $l_1, l_2 \in \{0, 1, \ldots, s-1\}$ the equality $\lambda^*(W_{l_1}) = \lambda^*(W_{l_2})$ holds. Thus there exist elements $(a_1, l_1) \in W_{l_1}$ and $(a_2, l_2) \in W_{l_2}$ such that $\lambda^*(a_1, l_1) = \lambda^*(a_2, l_2)$, whence

$$g_{(l_1)}(\lambda(a_1), \overset{l_1k}{\lambda(c)}) = g_{(l_2)}(\lambda(a_2), \overset{l_2k}{\lambda(c)}).$$

Take an arbitrary positive integer j. From the last equality it follows, that

$$g_{(j)}(g_{(l_1)}(\lambda(a_1), \overset{l_1k}{\lambda(c)}), \overset{jk}{\lambda(c)}) = g_{(j)}(g_{l_2})(\lambda(a_2), \overset{l_2k}{\lambda(c)}), \overset{jk}{\lambda(c)});$$

$$g_{(j+l_1)}(\lambda(a_1), \overset{(j+l_1)k}{\lambda(c)}) = g_{(j+l_2)}(\lambda(a_2), \overset{(j+l_2)k}{\lambda(c)}).$$

Thus

$$g_{(j+l_1)}(\lambda(a_1), \overset{\varepsilon_1 sk}{\lambda(c)}, \overset{r_1k}{\lambda(c)}) = g_{(j+l_2)}(\lambda(a_2), \overset{\varepsilon_2 sk}{\lambda(c)}, \overset{r_2k}{\lambda(c)}),$$

where

$$0 \leqq (j + l_1) - \varepsilon_1 s < s, \qquad 0 \leqq (j + l_2) - \varepsilon_2 s < s,$$
$$r_1 = j + l_1 - \varepsilon_1 s, \qquad r_2 = j + l_2 - \varepsilon_2 s.$$

Hence

$$\lambda^*(f_{(\varepsilon_1)}(a_1, \overset{\varepsilon_1 n}{c}), r_1) = \lambda^*(f_{(\varepsilon_2)}(a_2, \overset{\varepsilon_2 n}{c}), r_2), \qquad \text{i.e.} \qquad \lambda^*(W_{r_1}) = \lambda^*(W_{r_2}).$$

It is easy to see that the last equality implies $\lambda^*(W_{l_1}) = \lambda^*(W_{l_2})$.

Let q be the least positive integer, for which $\lambda^*(W_{q-1}) = \lambda^*(W_{s-1})$. Suppose that $q < s$. Let $s = mq + r$ where $r < q$, then $\lambda^*(W_{(q-1)+s^1}) = \lambda^*(W_{(s-1)+s^1})$, whence $\lambda^*(W_q) = \lambda^*(W_0)$. Hence $\lambda^*(W_{s-1}) = \lambda^*(W_{mq+r-1}) = \lambda^*(W_{q-1+r})$(since $\lambda^*(W_{\gamma q}) = \lambda^*(W_{(\gamma-1)q+q}) = \lambda^*(W_{(\gamma-1)q})$). From the last equality, by the definition of q, it follows that $r = 0$. Thus q is always a divisor of s ($s = mq$). Let for some l_1 and l_2, such that $l_1 \leqq l_2 < q$, the equality $\lambda^*(W_{l_1}) = \lambda^*(W_{l_2})$ holds. Then $\lambda^*(W_{0+l_1}) = \lambda^*(W_{(l_2-l_1)+l_1})$, $\lambda^*(W_0) = \lambda^*(W_{l_2-l_1})$, whence $l_2 - l_1 = 0$ (by $l_2 - l_1 < q$). This proves that the mapping $\lambda^*| W_0 \cup W_1 \cup W_{q-1}$ is injective. The set A can be thus decomposed into pairwise disjoint cosets $\lambda^*(W_0), \lambda^*(W_1), \dots, \lambda^*(W_{q-1})$. Moreover, $g(\lambda^*(W_{l_1}), \dots, \lambda^*(W_{l_{k+1}})) = \lambda^*(f^*(W_{l_1}, \dots, W_{l_{k+1}})) \subset \lambda^*(W_{\varphi(l_1, \dots, l_{k+1})})$ which proves that the decomposition of the $(k + 1)$-group $\mathfrak{A}$ into cosets is compatible with the operation g. Denote that congruence relation by Θ. Thus $\mathfrak{A}/\Theta = \mathfrak{C}_{q,k+1}$. Let ζ be the natural mapping of the $(k + 1)$-group $\mathfrak{A}$ onto the quotient $(k + 1)$-group $\mathfrak{A}/\Theta$. Take an arbitrary element $l \in \mathfrak{C}_{q,k+1}$ of order β. Then the order of $l \in (\mathfrak{C}_{q,k+1})_{(\beta)}$ equals one. The mapping $\zeta : A \to Z_q$ is a homomorphism of the $(\beta k + 1)$-group $\mathfrak{A}_{(\beta)}$ onto the $(\beta k + 1)$-group $(\mathfrak{C}_{q,k+1})_{(\beta)}$.Thus the subset $W_l = \zeta^{-1}(l)$ is an invariant sub-$(\beta k + 1)$-group $\mathfrak{A}_{(\beta)}$ as the inverse image of an invariant element of order one. Let $l \in \mathfrak{C}_{q,k+1}$ be an element of order q. Then l is a generator of the cyclic $(k + 1)$-group $\mathfrak{C}_{q,k+1}$. Hence the set W_l generates the $(k + 1)$-group $\mathfrak{A}$, which proves that the pair $\langle \mathfrak{A}, \lambda \rangle$, where λ is the inclusion of W_l into A, is a covering $(k + 1)$-group of the $(n + 1)$-group $(W_l, g_{(s)})$.

Conversely, let the $(\alpha k + 1)$-group $\mathfrak{B} = (B, g_{(\alpha)})$ be an invariant sub-$(\alpha k + 1)$-group of the $(\alpha k + 1)$-group $\mathfrak{A}_{(\alpha)}) = A, g_{(\alpha)}$ such that $\mathfrak{A}_{(\alpha)}/\mathfrak{B} = (\mathfrak{C}_{q,k+1})_{(\alpha)}$. In addition, assume that the natural mapping $\zeta : A \to Z_q$ is a homomorphism of the $(k + 1)$-group $\mathfrak{A}$ onto the $(k + 1)$-group $\mathfrak{C}_{q,k+1}$. As it has been already shown, the set $W_l = \zeta^{-1}(l)$, where $l \in \mathfrak{C}_{q,k+1}$ is an element of order q, generates the $(k + 1)$-group $\mathfrak{A}$ and is a sub-$(qk + 1)$-group of the $(qk + 1)$-group $\mathfrak{A}_{(q)}$. The pair $\langle \mathfrak{A}, \lambda \rangle$, where λ is the inclusion of W_l into A, is in consequence a covering $(k + 1)$-group of the $(n + 1)$-group $(W_l, g_{(s)})$. ∎

The just proved theorem indicates a following strict connection between the embedding $\lambda : G \to A$ and the epimorphism ζ: $A \to Z_q$ — each covering $(k + 1)$-group $\langle \mathfrak{A}, \lambda \rangle$ determines a unique natural mapping $\zeta : \mathfrak{A} \to \mathfrak{C}_{q,k+1}$ and conversely, each epimorphism $\zeta : \mathfrak{A} \to \mathfrak{C}_{q,k+1}$ determines a unique inclusion $\lambda : \zeta^{-1}(0) \to A$, where the pair $\langle \mathfrak{A}, \lambda \rangle$ is a covering $(k + 1)$-group of the $(n + 1)$-group $(\zeta^{-1}(0), g_{(q)})$. Henceforth we shall use interchangeably the symbols $\langle \mathfrak{A}, \lambda \rangle$, where $\lambda : G \to A$ or $\langle \mathfrak{A}, \zeta \rangle$, where $\zeta : A \to Z_q$ or $\langle \mathfrak{A}, \lambda, \zeta \rangle$, to denote the covering $(k + 1)$-group $\mathfrak{A}$ of the $(n + 1)$-group $\mathfrak{G}$. Define the notion of index of a covering $(k + 1)$-group. That notion, in case of $k = 1$ (i.e. for covering groups) has been introduced by Post (see [10], p. 240).

Definition 3. *A covering $(k + 1)$-group $\langle \mathfrak{A}, \zeta \rangle$ has index q*, if $\zeta : A \to Z_q$.
It is quite plain from the definition that the index of a covering $(k + 1)$-group of $(n + 1)$-group is always a divisor of n/k.

Corollary 2. A covering $(k + 1)$-group $\langle \mathfrak{A}, \lambda \rangle$ of an $(n + 1)$-group $\mathfrak{G}$ is a free covering $(k + 1)$-group of $\mathfrak{G}$ if and only if the index of $\langle \mathfrak{A}, \lambda \rangle$ is equal to $s = n/k$.

Proof. Use the same notation as in Theorem 2. If $q = s$, then $\lambda^* : \mathfrak{G}^{*s} \to \mathfrak{A}$ is a monomorphism, thus an isomorphism, since λ^* is always surjective. Hence the $(k + 1)$-groups $\mathfrak{A}$ and $\mathfrak{G}^{*s}$ are isomorphic if and only if $q = s$. ∎

In a special case one can derive a theorem, related to the Theorem 2, which is exactly analogous to the mentioned above Post's Coset Theorem.

Theorem 3. *A $(k + 1)$-group $\mathfrak{A} = (A, g)$ contains an invariant sub-$(k + 1)$-group $\mathfrak{B} = (B, g)$ such that $\mathfrak{A}/\mathfrak{B} = \mathfrak{C}_{q,k+1}$ if and only if the greatest common divisor of q and k equals to 1 and $\langle \mathfrak{A}, \zeta \rangle$ is a covering $(k + 1)$-group of the $(n + 1)$-group $\mathfrak{G} = (\zeta^{-1}(0), g_{(s)})$. Then $B = \zeta^{-1}(l)$, where l is the unique element of the $(k + 1)$-group $\mathfrak{C}_{q,k+1}$ such that $q \mid lk + 1$.*

Proof. Let $\mathfrak{B} = (B, g)$ be an invariant sub-$(k + 1)$-group of the $(k + 1)$-group $\mathfrak{A} = (A, g)$, such that $\mathfrak{A}/\mathfrak{B} = \mathfrak{C}_{q,k+1}$ and ζ is the natural mapping of $\mathfrak{A}$ onto $\mathfrak{A}/\mathfrak{B}$. As is known from [10], each quotient $(k + 1)$-group which is determined by an invariant sub-$(k + 1)$-group, is derived from a group. But (see [10], p. 286) a cyclic $(k + 1)$-group of order q is derived from a certain group if and only if the greatest common divisor (abbreviated in the sequel by g.c.d.) of q and k equals to 1. By Theorem 2 the pair $\langle \mathfrak{A}, \zeta \rangle$ is a covering $(k + 1)$-group of the $(n + 1)$-group $\mathfrak{G} = (\zeta^{-1}(0), g_{(s)})$. Let $l = \zeta(B)$. The element l, corresponding to the invariant sub-$(k + 1)$-group, is an element of order one (see [10], p. 231). Hence $q \mid lk + 1$.

Conversely, let the pair $\langle \mathfrak{A}, \zeta \rangle$, where $\zeta : A \to Z_q$, be a covering $(k + 1)$-group of the $(n + 1)$-group $\mathfrak{G} = (\zeta^{-1}(0), g_{(s)})$. Assume that g.c.d. $(q, k) = 1$. Then in the cyclic $(k + 1)$-group there exists a unique element l of order one (see [10], p. 304). Hence, by Theorem 2, the set $B = \zeta^{-1}(l)$ is an invariant sub-$(k + 1)$-group of the $(k + 1)$-group $\mathfrak{A}$. The invariant sub-$(k + 1)$-group determines a unique congruence relation Θ on the $(k + 1)$-group $\mathfrak{A}$ such that B is an equivalence class of Θ. Simultaneously, the homomorphism $\zeta : \mathfrak{A} \to \mathfrak{C}_{q,k+1}$ determines also a congruence relation Θ'

on $\mathfrak{A}$ such that $B = \zeta^{-1}(l)$ is an equivalence class of Θ'. Hence $\mathfrak{A}/\mathfrak{B} = \mathfrak{C}_{q,k+1}$ (see [9]).

It is known from [10], p. 241, that there exists a strict connection between covering groups and reducibility of n-groups.

Corollary 3. If the $(k+1)$-group $\mathfrak{A} = (A, g)$ with the embedding $\lambda : \mathfrak{G} \to \mathfrak{A}_{(mq)}$ is a covering $(k+1)$-group of index q of the $(n+1)$-group $\mathfrak{G}$, then the same $(k+1)$-group $\mathfrak{A}$ with the embedding $\lambda : \mathfrak{G}_{(m-1)} \to \mathfrak{A}_{(q)}$ is simultaneously a free covering $(k+1)$-group of the $(qk+1)$-group $\mathfrak{G}_{(m-1)}$ which is a creating $(qk+1)$-group of the $(n+1)$-group $\mathfrak{G}$.

Proof. Let $\langle \mathfrak{A}, \lambda, \zeta \rangle$ be a covering $(k+1)$-group of index q of the $(n+1)$-group $\mathfrak{G}$. According to Theorem 2, $\langle \mathfrak{A}, \lambda, \zeta \rangle$ is a covering $(k+1)$-group of the $(qk+1)$-group $(\zeta^{-1}, g_{(q)})$ (since the element $0 \in \mathfrak{C}_{q,k+1}$, being a generator of $\mathfrak{C}_{q,k+1}$, is an element of order q). Simultaneously, the $(n+1)$-group $(\zeta^{-1}(0), g_{(q)})_{(m)}$ is isomorphic to the $(n+1)$-group $\mathfrak{G}$. Hence $\mathfrak{G}_{(m-1)} = (\zeta^{-1}(0), g_{(q)})$ is a creating $(qk+1)$-group of the $(n+1)$-group $\mathfrak{G}$. The $(k+1)$-group $\mathfrak{A}$ with the embedding $\lambda : \mathfrak{G}_{(m-1)} \to \mathfrak{A}_{(q)}$ is a covering $(k+1)$-group of index q of $\mathfrak{G}_{(m-1)}$. Thus, in view of Corollary 2, $\langle \mathfrak{A}, \lambda \rangle$ is a free covering $(k+1)$-group of $\mathfrak{G}_{(m-1)}$. ∎

Also the converse theorem is true.

Corollary 4. If the $(n+1)$-group $\mathfrak{G} = (G, f)$ is derived from the $(qk+1)$-group $\mathfrak{G}_{(m-1)}$, then the free covering $(k+1)$-group $\langle \mathfrak{G}^{*q}_{(m-1)}, \tau \rangle$ of the $(qk+1)$-group $\mathfrak{G}_{(m-1)}$ is also a covering $(k+1)$-group of index $\dot{Z}$ of $\mathfrak{G}$.

From Corollaries 3 and 4 one obtains the following generalization of Post's result (see [10], p. 241).

Corollary 5. An $(n+1)$-group $\mathfrak{G}$ posseses a covering $(k+1)$-group of index q if and only if the $(n+1)$-group $\mathfrak{G}$ is derivated from some $(qk+1)$-group $\mathfrak{G}_{(m-1)}$.

Proof. If the $(n+1)$-group $\mathfrak{G}$ posseses a covering $(k+1)$-group of index q, then in view of Corollary 3, $\mathfrak{G}$ is derivated from some $(qk+1)$-group $\mathfrak{G}_{(m-1)}$.

Conversely, if the $(n+1)$-group $\mathfrak{G}$ is derived from a $(qk+1)$-group $\mathfrak{G}_{(m-1)}$, then in view of Corollary 4, the free covering $(k+1)$-group $(\mathfrak{G}_{(m-1)})^{*q}$ of $\mathfrak{G}_{(m-1)}$ is also a covering $(k+1)$-group of index q of $\mathfrak{G}$. ∎

5. The category of covering $(k+1)$-groups of $(n+1)$-groups

As we already mentioned, a covering $(k+1)$-group $(\mathfrak{A}', \lambda, \zeta\rangle$ of index q of the $(n+1)$-group $\mathfrak{A}$ is determined by the pair of mappings: $\lambda : A \to A'$ and $\zeta : A' \to Z_q$. Hence it seems to be natural to define a morphism in the category of covering $(k+1)$-groups as a triple of homomorphisms.

Definition 4. Let $\langle \mathfrak{A}', \lambda_A, \zeta_A \rangle$ and $\langle \mathfrak{B}', \lambda_B, \zeta_B \rangle$ be covering $(k+1)$-groups of indices q_A and q_B of $(n+1)$-groups $\mathfrak{A} = (A, f)$ and $\mathfrak{B} = (B, f)$, respectively. A triple h', h, ζ of homomorphisms $h' : \mathfrak{A}' \to \mathfrak{B}'$, $h : \mathfrak{A} \to \mathfrak{B}$, $\xi : \mathfrak{C}_{q_A,k+1} \to \mathfrak{C}_{q_B,k+1}$ where

$\xi(0) = 0$ will be a morphism $\langle h', h, \xi\rangle : \langle \mathfrak{A}', \lambda_A, \zeta_A\rangle \to \langle \mathfrak{B}', \lambda_B, \zeta_B\rangle$ if the following diagram

$$\begin{tikzcd} Z_{q_A} \arrow[r, "\xi"] & Z_{q_B} \\ A \arrow[u, "\zeta_A"] \arrow[r, "h'"] & B \arrow[u, "\zeta_B"'] \\ A \arrow[u, "\lambda_A"] \arrow[r, "h"] & B \arrow[u, "\lambda_B"'] \end{tikzcd}$$

is commutative.

The following theorem indicates a connection between the homomorphisms h', h, ξ and simplifies the question of commutativity of the former diagram. First we prove

Lemma 1. *Let $\zeta_1 : G' \to Z_{q_1}$ and $\zeta_2 : G' \to Z_{q_2}$ be epimorphisms of a $(k+1)$-group $\mathfrak{D}' = (G', g)$ onto the cyclic $(k+1)$-groups $\mathfrak{C}_{q_1,k+1}$ and $\mathfrak{C}_{q_2,k+1}$. If $\zeta_2^{-1}(0) \supset \zeta_1^{-1}(0)$, then there exists a unique epimorphism $\zeta : \mathfrak{C}_{q_1,k+1} \to \mathfrak{C}_{q_2,k+1}$ such that $\xi\zeta_1 = \zeta_2$.*

Proof. For $a', b' \in G'$ let $\zeta_1(a') = \zeta_1(b') = l \in Z_{q_1}$. Then $a' = g_{(\cdot)}(a_1, \ldots, a_{lk+1})$, $b' = g_{(\cdot)}(b_1, \ldots, b_{lk+1})$ where $a_i \in \zeta_1^{-1}(0)$, $b_i \in \zeta_1^{-1}(0)$, $i = 1, \ldots, lk+1$. Hence $\zeta_2(a') = \zeta_2(g_{(\cdot)}(a_1, \ldots, a_{lk+1})) = \varphi_{(l)}(\zeta_2(a_1), \ldots, \zeta_2(a_{lk+1})) = \varphi_{(l)}(0, \ldots, 0) = \varphi_{(l)} \times \times(\zeta_2(b_1), \ldots, \zeta_2(b_{lk+1})) = \zeta_2(g_{(\cdot)}(b_1, \ldots, b_{lk+1})) = \zeta_2(b')$. Thus in view of the Isomorphism Theorem, there exists a unique epimorphism $\zeta : \mathfrak{C}_{q_1,k+1} \to \mathfrak{C}_{q_2,k+1}$ such that $\xi\zeta_1 = \zeta_2$. ∎

Theorem 4. *Consider the following diagrams*

$$\begin{tikzcd} \mathfrak{C}_{q_A,k+1} \arrow[r, "\xi"] & \mathfrak{C}_{q_B,k+1} \\ A' \arrow[u, "\zeta_A"] \arrow[r, "h'"] & B' \arrow[u, "\zeta_B"'] \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[r, "h'"] & B' \\ A \arrow[u, "\lambda_A"] \arrow[r, "h"] & B \arrow[u, "\lambda_B"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathfrak{C}_{q_A,k+1} \arrow[r, "\xi"] & \mathfrak{C}_{q_B,k+1} \\ A \arrow[u, "\zeta_A\lambda_A"] \arrow[r, "h"] & B \arrow[u, "\zeta_B\lambda_B"'] \end{tikzcd}$$

where $\langle \mathfrak{A}', \lambda_A, \zeta_A\rangle$ and $\langle \mathfrak{B}', \lambda_B, \zeta_B\rangle$ are covering $(k+1)$-groups of indices q_A and q_B of $(n+1)$-groups $\mathfrak{A} = (A, f)$ and $\mathfrak{B} = (B, f)$, respectively, and $h' : \mathfrak{A}' \to \mathfrak{B}'$, $\xi : \mathfrak{C}_{q_A,k+1} \to \mathfrak{C}_{q_B,k+1}$ where $\xi(0) = 0$, $h : \mathfrak{A}_{(m_A^{-1})} \to \mathfrak{B}_{(m_A^{-1})}$ where $\mathfrak{A}_{(m_A^{-1})}$ and $\mathfrak{B}_{(m_A^{-1})}$ are creating $(q_A k + 1)$-groups of the $(n+1)$-groups $\mathfrak{A}$ and $\mathfrak{B}$ determined by $\langle \mathfrak{A}', \lambda_A, \zeta_A\rangle$ and $\langle \mathfrak{B}', \lambda_B, \zeta_B\rangle$. Then the existence of any pair of homomorphisms h', h, ξ and commutativity of the respective diagram implies the existence of the third morphism and commutativity of the remaining two diagrams.

Proof. Let homomorphisms h', h be given in such a way, that the respective diagram is commutative. Let $a' \in \zeta_A^{-1}(0)$. By Lemma 1 there exists a unique epimorphism $\xi : \mathfrak{C}_{q_A,k+1} \to \zeta_B h(\mathfrak{A}')$ such that $\xi\zeta_A = \zeta_B h'$. Note, that $\xi(0) = \xi\zeta_A(a') = \zeta_B h'(a') = 0$. Thus the homomorphism $\xi : \mathfrak{C}_{q_A,k+1} \to \mathfrak{C}_{q_B,k+1}$ is an epimorphism. From the definition of ξ it is evident, that ξ does not depend on the choice of the homomorphisms h' and h.

Now, let homomorphisms h' and ξ be given in such a way, that the respective diagram is commutative. According to Corollary 3, $\langle \mathfrak{A}', \lambda_A, \zeta_A \rangle$ is a free covering $(k+1)$-group of the $(q_A k + 1)$-group $\mathfrak{A}_{(m_A^{-1})}$ and $\langle \mathfrak{B}', \lambda_B, \zeta_B \rangle$ is a covering $(k+1)$-group of the $(q_A k + 1)$-group $\mathfrak{B}_{(m_A^{-1})}$. Let $a \in A$. Then $\zeta_B h'\lambda_A(a) = \xi\zeta_A\lambda_A(a) = \xi(0) = 0$, whence $h'\lambda_A(A) \subset \lambda_B(B)$. Define a mapping $h : A \to B$ by the formula $h(x) = \lambda_B^{-1} h'\lambda_A(x)$. From the definition of h it is clear that $h : \mathfrak{A}_{(m_A^{-1})} \to \mathfrak{B}_{(m_A^{-1})}$ and $\lambda_B h = h'\lambda_A$.

Finally, let homomorphisms h and ξ be given in such a way, that the respective diagram is commutative. As we already mentioned, $\langle \mathfrak{A}', \lambda_A, \zeta_A \rangle$ is a free covering $(k+1)$-group of the $(q_A k + 1)$-group $\mathfrak{A}_{(m_A^{-1})}$. Thus there exists an homomorphism $h' : \mathfrak{A}' \to \mathfrak{B}'$ such that $h'\lambda_A = \lambda_B h$. For the homomorphisms h and h' there exists an epimorphism $\xi : \mathfrak{C}_{q_A,k+1} \to \mathfrak{C}_{q_B,k+1}$ such that $\xi'\zeta_A = \zeta_B h'$ and $\xi'(0) = 0$. It follows from the last equality, that $\xi' = \xi$. Hence $\xi\zeta_A = \zeta_B h'$. ∎

Proposition 1. Consider the following diagrams

$$\begin{tikzcd}
\mathfrak{C}_{q_A,k+1} \arrow[r, "\xi_1"] & \mathfrak{C}_{q_D,k+1} \arrow[r, "\xi_2"] & \mathfrak{C}_{q_B,k+1} \\
A' \arrow[u, "\zeta_A"] \arrow[r, "h_1'"] & D' \arrow[u, "\zeta_D"] \arrow[r, "h_2'"] & B' \arrow[u, "\zeta_B"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[r, "h_1'"] & D' \arrow[r, "h_2'"] & B' \\
A \arrow[u, "\lambda_A"] \arrow[r, "h_1"] & D \arrow[u, "\lambda_D"] \arrow[r, "h_2"] & B \arrow[u, "\lambda_B"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathfrak{C}_{q_A,k+1} \arrow[r, "\xi_1"] & \mathfrak{C}_{q_D,k+1} \arrow[r, "\xi_2"] & \mathfrak{C}_{q_B,k+1} \\
A \arrow[u, "\zeta_A\lambda_A"] \arrow[r, "h_1"] & D \arrow[u, "\zeta_D\lambda_D"] \arrow[r, "h_2"] & B \arrow[u, "\zeta_B\lambda_B"]
\end{tikzcd}$$

where $\langle \mathfrak{A}', \lambda_A, \zeta_A \rangle$, $\langle \mathfrak{B}', \lambda_B, \zeta_B \rangle$ are covering $(k+1)$-groups of indices q_A, q_B of $(n+1)$-groups $\mathfrak{A}, \mathfrak{B}$, respectively, and $\lambda_D : D \to D'$ is an embedding of a $(q_D k + 1)$-group $\mathfrak{D}$ into a $(q_D k + 1)$-group $\mathfrak{D}'_{(q_D)}$ derived from $\mathfrak{D}'$, and, in addition, $\lambda_D(D)$ generates $\mathfrak{D}'$ and q_D is the least positive integer for which $\lambda_D(D)$ is a sub-$(q_D k + 1)$-group of $\mathfrak{D}'_{(q_D)}$, ζ_D is an epimorphism of $\mathfrak{D}'$ onto $\mathfrak{C}_{q_D,k+1}$, $h'_1 : \mathfrak{A}' \to \mathfrak{D}'$, $h'_2 : \mathfrak{D}' \to \mathfrak{B}'$, $\xi_1 : \mathfrak{C}_{q_A,k+1} \to \mathfrak{C}_{q_D,k+1}$, $\zeta_2 : \mathfrak{C}_{q_D,k+1} \to \mathfrak{C}_{q_B,k+1}$ where $\xi_1(0) = 0$, $\xi_2(0) = 0$, $h_1 : \mathfrak{A}_{(m_A^{-1})} \to \mathfrak{D}_{(m_D m_A^{-1})}$, $h_2 : \mathfrak{D} \to \mathfrak{B}_{(m_D^{-1})}$. Then the existence of any two pairs of the three pairs of homomorphisms h_1', h_2'; h_1, h_2; ξ_1, ξ_2 and commutativity of the respective diagram implies the existence of the third pair of homomorphisms and commutativity of the remaining two diagrams.

Proof. Let the pairs h_1', h_2' and h_1, h_2 be given in such a way, that the respective diagram is commutative. From the definition of the covering $(k + 1)$-group and from Theorem 2 and the accompanying corollaries it follows that the pair $\langle \mathfrak{D}', \lambda_D \rangle$ is a free covering $(k + 1)$-group of the $(q_D k + 1)$-group $\mathfrak{D}$. Thus there exists the epimorphism $\zeta_D : \mathfrak{D}' \to \mathfrak{C}_{q_D, k+1}$ for which $\zeta_D^{-1}(0) = \lambda_D(D)$. Further, from Theorem 4 there exists a pair of epimorphisms $\xi_1 : \mathfrak{C}_{q_A, k+1} \to \mathfrak{C}_{q_D, k+1}$, $\xi_2 : \mathfrak{C}_{q_D, k+1} \to \mathfrak{C}_{q_B, k+1}$ for which the respective diagrams are commutative.

Now, let the pairs h_1', h_2' and ξ_1, ξ_2 be given in such a way, that the respective diagram is commutative. Let $D = \zeta_D^{-1}(0)$ and $\lambda : D \to D'$ be the inclusion of D into D'. From Theorem 2 it follows, that $\langle \mathfrak{D}', \lambda_D, \zeta_D \rangle$ is a free covering $(k + 1)$-group of the $(q_D k + 1)$-group $\mathfrak{D}$. Theorem 4 asserts the existence of homomorphisms $h_1 : \mathfrak{A}_{(m_A^{-1})} \to \mathfrak{D}_{(m_D m_A^{-1})}$ and $h_2 : \mathfrak{D} \to \mathfrak{B}_{(m_D^{-1})}$ for which the respective diagrams are commutative.

Finally, let the pairs h_1, h_2 and ξ_1, ξ_2 be given in such a way, that the respective diagram is commutative. Let $\langle \mathfrak{D}', \lambda_D, \zeta_D \rangle$ be a free covering $(k + 1)$-group of. the $(q_L k + 1)-$group $\mathfrak{D}$, Theorem 4 assures the existence of homomorphisms $h_1' :$ $: \mathfrak{A}' \to \mathfrak{D}'$ and $h_2' : \mathfrak{D}' \to \mathfrak{B}'$ for which the respective diagrams are commutative. ∎

A more detailed description of the category of covering $(k + 1)$-groups of $(n + 1)$-groups will appear in [6].

In case of $q_A = q_B$, one can draw from Proposition 1 the following

Corollary 6. If $\langle \mathfrak{A}', \lambda_A, \zeta_A \rangle$ and $\langle \mathfrak{B}', \lambda_B, \zeta_B \rangle$ are covering $(k + 1)$-groups of index q of $(n + 1)$-groups $\mathfrak{A}$ and $\mathfrak{B}$, respectively, $h_1' : \mathfrak{A}' \to \mathfrak{D}'$, $h_2' : \mathfrak{D}' \to \mathfrak{B}'$ where $\mathfrak{D}'$ is a certain $(k + 1)$-group, and $\zeta_B h_2' h_1' = \zeta_A$, then there exists an $(n + 1)$-group $\mathfrak{D}$, a mapping $\lambda_D : D \to D'$ and homomorphisms $h_1 : \mathfrak{A}_{(m-1)} \to \mathfrak{D}_{(m-1)}$, $h_2 : \mathfrak{D}_{(m-1)} \to$ $\to \mathfrak{B}_{(m-1)}$ such that $\langle \mathfrak{D}', \lambda_D, \zeta_D \rangle$ is a covering $(k + 1)$-group of index q of the $(n + 1)$-group $\mathfrak{D}$ and $\langle h_1', h_1, id_{Z_q} \rangle : \langle \mathfrak{A}', \lambda_A, \zeta_A \rangle \to \langle \mathfrak{D}', \lambda_D, \zeta_D \rangle$, $\langle h_2', h_2, id_{Z_q} \rangle :$ $: \langle \mathfrak{D}', \lambda_D, \zeta_D \rangle \to \langle \mathfrak{B}', \lambda_B, \zeta_B \rangle$.

Proof. The following diagram

$$\begin{tikzcd}
Z_q \arrow[r, "id_{Z_q}"] & Z_q \arrow[r, "id_{Z_q}"] & Z_q \\
A' \arrow[u, "\zeta_A"'] \arrow[r, "h_1'"] & D' \arrow[u, "\zeta_D"'] \arrow[r, "h_2'"] & B' \arrow[u, "\zeta_B"'] \\
A \arrow[u, "\lambda_A"'] & D \arrow[u, "\lambda_D"'] & B \arrow[u, "\lambda_B"']
\end{tikzcd}$$

where $\zeta_D = \zeta_B h_2'$, $D = \zeta_D^{-1}(0)$ is commutative. By Proposition 1, there exists homomorphisms $h_1 : \mathfrak{A}_{(m-1)} \to \mathfrak{D}_{(m-1)}$, $h_2 : \mathfrak{D}_{(m-1)} \to \mathfrak{B}_{(m-1)}$ fulfilling the demanded conditions. ∎

6. Some characterizations of $(n+1)$-groups derived from $(k+1)$-groups. In the first theorem of that paragraph, we give a condition for an $(n+1)$-group to be derived from some $(k+1)$-group. That condition is a modification (adjusted to our construction of the free covering group) of the condition proved by Post (see [10], p. 229).

Theorem 5. *An $(n+1)$-group $\mathfrak{G} = (G,f)$ is derived from some $(k+1)$-group $\mathfrak{G}_{(s-1)}$ if and only if for each element $c \in G$ there exists an element $d \in G$ such that*

1° $f(\overset{s}{d}, \overset{(k-1)s}{c}, x) = x$ for each $x \in G$;

2° $f(x_1, \dots, x_i, d, \overset{k-1}{c}, x_{i+1}, \dots, x_{n+1-k}) =$

$= f(x_1, \dots, x_i, \overset{k-1}{c}, d, x_{i+1}, \dots, x_{n+1-k}) = f(d, \overset{k-1}{c}, x_1, \dots, x_{n+1-k})$,

for each $x_1, \dots, x_{n+1-k} \in G$ and arbitrary $i = 1, \dots, n+1-k$. In addition, the $(k+1)$-ary operation g in the $(k+1)$-group $\mathfrak{G}_{(s-1)} = (G,g)$ can be given by the formula

$$g(x_1, \dots, x_{k+1}) = f(x_1, \dots, x_{k+1}, \overset{s-1}{d}, \overset{(k-1)(s-1)}{c}).$$

Proof. Let the $(n+1)$-group $\mathfrak{G} = (G,f)$ be derived from a $(k+1)$-group $\mathfrak{G}_{(s-1)} = (G,g)$, i.e. $g_{(s)} = f$. Take an arbitrary element $c \in G$. Let d be the element skew to c in the creating $(k+1)$-group $\mathfrak{G}_{(s-1)}$. Then

$$f(x_1, \dots, x_i, d, \overset{k-1}{c}, x_{i+1}, \dots, x_{n+1-k}) =$$
$$= g_{(s-1)}(x_1, \dots, x_i, g(d\ \overset{k-1}{c}, x_{i+1}), x_{i+2}, \dots, x_{n+1-k}) =$$
$$= g_{(s-1)}(x_1, \dots, x_i, x_{i+1}, \dots, x_{n+1-k}).$$

Similarly

$$f(x_1, \dots. x_i, \overset{k-1}{c}, d, x_{i+1}, \dots, x_{n+1-k}) =$$
$$= g_{(s-1)}(x_1, \dots, x_i, g(\overset{k-1}{c}, d, x_{i+1}), \dots, x_{n+1-k}) =$$
$$= g_{(s-1)}(x_1, \dots, x_i, x_{i+1}, \dots, x_{n+1-k}),$$

which proves that c and d fulfill the condition 2°. Hence

$$f(\overset{s}{d}, \overset{(k-1)s}{c}, x) = f(\underbrace{d, \overset{k-1}{c}, \dots, d\ \overset{k-1}{c},}_{s}\ x) =$$

$$= g_{(s-1)}(\underbrace{d, \overset{k-1}{c}, \ldots, d, \overset{k-1}{c}}_{s-1}, g(d, \overset{k-1}{c}\, x)) = g_{(s-1)}(\underbrace{d, \overset{k-1}{c}, \ldots, d, \overset{k-1}{c}}_{s-1}, x) =$$

$$= \ldots = g(d, \overset{k-1}{c}, x) = x,$$

that is, also the condition 1° is fulfilled. Moreover

$$g(x_1, \ldots, x_{k+1}) = g_{(s)}(x_1, \ldots, x_{k+1}, \underbrace{d, \overset{k-1}{c}, \ldots, d, \overset{k-1}{c}}_{s-1}) =$$

$$= f(x_1, \ldots, x_{k+1}, \underbrace{d, \overset{k-1}{c}, \ldots, d\ \overset{k-1}{c}}_{s-1}) = f(x_1, \ldots, x_{k+1}, \overset{s-1}{d}, \overset{(k-1)(s-1)}{c}).$$

Conversely, let in the $(n+1)$-group $\mathfrak{G} = (G, f)$ for each element $c \in G$ there exists an element $d \in G$ such that the condition 1^0 and 2^0 are fulfilled. Define a $(k+1)$-ary operation g in the set G by the formula:

$$g(x_1, \ldots, x_{k+1}) = f(x_1, \ldots, x_{k+1}, \overset{s-1}{d}, \overset{(k-1)(s-1)}{c}).$$

Then

$$g(x_1, \ldots, x_i, g(x_{i+1}, \ldots, x_{i+1+k}), \ldots, x_{2k+1}) =$$

$$= f(x_1, \ldots, x_i, f(x_{i+1}, \ldots, x_{i+1+k}, \overset{s-1}{d}, \overset{(k-1)(s-1)}{c}), x_{i+2+k}, \ldots, x_{2k+1}, \overset{s-1}{d}, \overset{(k-1)(s-1)}{c}) =$$

$$= f_{(2)}(x_1, \ldots, x_i, x_{i+1}, \ldots, x_{i+1+k}, \overset{s-1}{d}, \overset{(k-1)(s-1)}{c}), x_{i+2+k}, \ldots, x_{2k+1}, \overset{s-1}{d}, \overset{(k-1)(s-1)}{c}) =$$

$$= f_{(2)}(x_1, \ldots, x_{2k+1}, \overset{2(s-1)}{d}, \overset{2(k-1)(s-1)}{c}),$$

thus the operation g is associative.

$$= f_{(2)}(x_1, \ldots, x_{2k+1}, \overset{2(s-1)}{d}, \overset{2(k-1)(s-1)}{c}),$$

thus the operation g is associative.

Let $g(a_1, \ldots, a_{i-1}, x, a_i, \ldots, a_k) = b$ for fixed $a_1, \ldots, a_k \in G$. Hence

$$f(a_1, \ldots, a_{i-1}, x, a_i, \ldots, a_k, \overset{s-1}{d}, \overset{(k-1)(s-1)}{c}) = b.$$

The last equation has a unique solution in the $(n+1)$-group $\mathfrak{G} = (G,f)$. This proves that the set G with the just defined operation g is a $(k+1)$-group. In addition

$$g_{(s)}(x_1, \dots, x_{n+1}) = g_{(s-1)}(g(x_1, \dots, x_{k+1}), x_{k+2}, \dots, x_{n+1}) =$$

$$= g_{(s-1)}(f(x_1, \dots, x_{k+1}, \overset{s-1}{d}, \overset{(k-1)(s-1)}{c}), x_{k+2}, \dots, x_{n+1}) =$$

$$= g_{(s-2)}(f(f(x_1, \dots, x_{k+1}, \overset{s-1}{d}, \overset{(k-1)(s-1)}{c}), x_{k+2}, \dots, x_{2k+1}, \overset{s-1}{d}, \overset{(k-1)(s-1)}{c}), \dots, x_{n+1}) =$$

$$= g_{(s-2)}(f_{(2)}(x_1, \dots, x_{2k+1}, \overset{2(s-1)}{d}, \overset{2(k-1)(s-1)}{c}), \dots, x_{n+1}) =$$

$$= g_{s-2)}(f(x_1, \dots, x_{2k+1}, \overset{s-2}{d}, \overset{(k-1)(s-2)}{c}), \dots, x_{n+1}) = \dots =$$

$= f(x_1, \dots, x_{n+1})$, whence $g_{(s)} = f$, which proves the $(k+1)$-group $\mathfrak{G}_{(s-1)} = (G, g)$ to be creating $(k+1)$-group of the $(n+1)$-group $\mathfrak{G}$ ∎.

In some particular cases the covering $(k+1)$-groups have a very simple form.

Theorem 6. *If the $(n+1)$-group $\mathfrak{G} = (G,f)$ is derived from the $(k+1)$-group $\mathfrak{G}_{(s-1)} = (G, g)$, then $\langle \mathfrak{G}_{(s-1)} \times \mathfrak{C}_{s,\,k+1}, \lambda \rangle$, where $\lambda : G - G \times Z_s$ is given by the formula $\lambda(x) = (x, 0)$, is a free covering $(k+1)$-group of $\mathfrak{G}$.*

Proof. Let the $(k+1)$-group $\mathfrak{G}_{(s-1)} = (G, g)$ be a creating $(k+1)$-group of the $(n+1)$-group $\mathfrak{G} = (G,f)$, i.e. $g_{(s)} = f$. Form the direct product $\mathfrak{G}_{(s-1)} \times \mathfrak{C}_{s,k+1} = (G \times Z_s, g)$. The mapping $\lambda : \mathfrak{G} \to (\mathfrak{G}_{(s-1)} \times \mathfrak{C}_{s,k+1})_{(s)}$ is a homomorphism, since

$$\lambda(f(a_1, \dots, a_{n+1})) = (f(a_1, \dots, a_{n+1}), 0) =$$
$$= (g_{(s)}(a_1, \dots, a_{n+1}), \varphi_{(s)}(0, \dots, 0)) = g_{(s)}((a_1, 0), \dots, (a_{n+1}, 0)) =$$
$$= g_{(s)}(\lambda(a_1), \dots, \lambda(a_{n+1})).$$

Let the $(k+1)$-group $\mathfrak{G}^{*s} = (G^{*s}, f^*)$ with the embedding $\tau : \mathfrak{G} \to \mathfrak{G}^{*s}_{(s)}$ and the fixed element $c \in G$ be a free covering $(k+1)$-group of $\mathfrak{D}$. Then there exists a unique homomorphism $\lambda^* : \mathfrak{G}^{*s} \to \mathfrak{G}_{(s-1)} \times \mathfrak{C}_{s,k+1}$ such that $\lambda^* \tau = \lambda$, defined as in Theorem 1. The $(n+1)$-group $\mathfrak{G}$ is derived from the $(k+1)$-group $\mathfrak{G}_{(s-1)}$, thus, by Theorem 5, one can choose an element $d \in G$ to the given element $c \in G$ in such a way that the conditions 1° and 2° are satisfied. Let $(a, l) \in G^{*s}$. Then

$$\lambda^*(a, l) = g_{(l)}(\lambda(a), \overset{lk}{\lambda(c)}) = g_{(l)}((a, 0), \underbrace{(c, 0), \dots, (c, 0)}_{lk}) =$$

$$= (g_{(l)}(a, \overset{lk}{c}), \varphi_{(l)}(\overset{lk+1}{0})) = (g_{(s)}(a, \overset{lk}{c}, \underbrace{\overset{k-1}{c}, d, \ldots, \overset{k-1}{c}, d}_{s-1}), l) =$$

$$= (f(a, \overset{lk}{c}, \overset{(k-1)(s-l)}{c}, \overset{s-l}{d}), l) = (f(a, \overset{n+l-s}{c}, \overset{s-l}{d}), l).$$

For each $b \in G$ and $l \in Z_s$ the equation $f(x, \overset{n+l-s}{c}, \overset{s-l}{d}) = b$ has a unique solution in the $(n+1)$-group $\mathfrak{G}$. Hence for each element $(b, l) \in G \times Z_s$ there exists a unique element $(a, l) \in G^{*s}$ such that $\lambda^* (a, l) = (b, l)$. This proves the homomorphism $\lambda^* : \mathfrak{G}^{*s} \to \mathfrak{G}_{(s-1)} \times \mathfrak{C}_{s,k+1}$ to be an isomorphism. ∎

That theorem enables to give some other necessary and sufficient conditions for $(n+1)$-group to be derived from a $(k+1)$-group. We first prove

Lemma 2. *If $\mathfrak{A} = (A, f)$ is an $(n+1)$-group derived from a $(k+1)$-group $\mathfrak{A}_{(s-1)} = (A, g)$ and $h : \mathfrak{A} \to \mathfrak{B}$ is an epimorphism onto an $(n+1)$-group $\mathfrak{B}$, then $\mathfrak{B}$ is also derived from a certain $(k+1)$-group $\mathfrak{B}_{(s-1)}$.*

Proof. Take an arbitrary element $c \in B$. Since $h : \mathfrak{A} \to \mathfrak{B}$ is a surjection, there exists an element $c' \in A$ such that $h(c') = c$. The $(n+1)$-group $\mathfrak{A}$ is derived from the $(k+1)$-group $\mathfrak{A}_{(s-1)}$, thus an element $d' \in A$ can be chosen to the element c' in such a way, that c' and d' satisfy the conditions 1° and 2° of Theorem 5. Let $d = h(d')$. We show, that c and d also satisfy 1° and 2°. In fact, take an arbitrary element $x \in B$. There exists $x' \in A$ such that $h(x') = x$. Then

$$f(\overset{s}{d}, \overset{(k-1)s}{c}, x) = f(\overset{s}{h(d')}, \overset{(k-1)s}{h(c')}, h(x')) = h(f(\overset{s}{d'}, \overset{(k-1)s}{c'}, x')) = h(x') = x,$$

which proves that c and d satisfy 1°. A similar reasoning proves that 2° is also satisfied. Hence the $(n+1)$-group $\mathfrak{B}$ is derived from a certain $(k+1)$-group $\mathfrak{B}_{(s-1)}$. ∎

Proposition 2. An $(n+1)$-group $\mathfrak{G}$ is derivated from a $(k+1)$-group $\mathfrak{G}_{(s-1)}$ if and only if there exists an epimorphism $\varrho_G : \mathfrak{D}^{*s}_{(s)} \to \mathfrak{D}$ such that $\varrho_G \tau_G = id_G$. Moreover, the mapping ϱ_G can be chosen in such a way, that $\varrho_G ; \mathfrak{G}^{*s} \to \mathfrak{G}_{(s-1)}$.

Proof. According to Theorem 6 the direct product $\mathfrak{G}_{(s-1)} \times \mathfrak{C}_{s,k+1}$ with the embedding $\lambda : \mathfrak{G} \to (\mathfrak{G}_{(s-1)} \times \mathfrak{C}_{s,k+1})_{(s)}$ given by the formula $\lambda(x) = (x, 0)$ is a free covering $(k+1)$-group of the $(n+1)$-group $\mathfrak{G}$. The projection $\varrho_G : \mathfrak{G}_{(s-1)} \times \mathfrak{C}_{s,k+1} \to \mathfrak{G}_{(s-1)}$ is obviously a homomorphism, for which $\varrho_G \lambda = id_G$.

Conversely, let $\varrho_G : \mathfrak{G}^{*s}_{(s)} \to \mathfrak{G}$ be an epimorphism for which $\varrho_G \tau_G = id_G$. Since ϱ_G is an epimorphism and the $(n+1)$-group $\mathfrak{G}^{*s}_{(s)}$ is derivated from the $(k+1)$-group $\mathfrak{G}^{*s}$ by Lemma 2 $\mathfrak{G}$ is derived from some $(k+1)$-group $\mathfrak{G}_{(s-1)}$. Let $h : \mathfrak{G} \to (\mathfrak{G}_{(s-1)})_{(s)}$ be the identity. There exists a homomorphism $h^* : \mathfrak{G}^{*s} \to \mathfrak{G}_{(s-1)}$ such that $h^* \tau_G = h = id_G$. ∎

Proposition 3. An $(n+1)$-group $\mathfrak{G} = (G,f)$ is derived from a $(k+1)$-group $\mathfrak{G}_{(s-1)}$ if and only if there exists a mapping $\eta : G^{*s} \to G^{*s}$, where $\langle \mathfrak{G}^{*s}, \tau, \zeta \rangle$ is a free covering $(k+1)$-group of $\mathfrak{G}$, satisfying the following conditions:

1° $\zeta\eta(x) \equiv \zeta(x) - 1 \pmod{s}$ for each $x \in G^{*s}$;

2° $\eta(f^*_{(s)}(x_1, \ldots, x_{n+1}) = f^*_{(s)}(x_1, \ldots, x_{i-1}, \eta(x_i), x_{i+1}, \ldots, x_{n+1})$ for each $x_1, \ldots, \ldots, x_{n+1} \in G^{*s}$ and arbitrary $i = 1, \ldots, n+1$;

3° $\underbrace{\eta \ldots \eta}_{s}(x) = x$ for each $x \in G^{*s}$.

Moreover, such a mapping η is already an automorphism of $\mathfrak{G}^{*s}_{(s)}$.

Proof. If the $(n+1)$-group $\mathfrak{G} = (G,f)$ is derived from a $(k+1)$-group $\mathfrak{G}_{(s-1)} = (G, g)$, then by Theorem 6 the direct product $\mathfrak{G}_{(s-1)} \times \mathfrak{C}_{s,k+1}$ is a free covering $(k+1)$-group of $\mathfrak{G}$. It is easily verified that the mapping $\eta : G \times Z_s \to G \times Z_s$ given by the formula $\eta(a, l) = (a, \sigma(l))$, where $a \in G$, $l \in_{Z_s}$, $\sigma = (s-1, s-2, \ldots, 1, 0)$ is a cyclic permutation of order s, satisfies 1°, 2° and 3°.

Conversely, let the mapping $\eta : G^{*s} \to G^{*s}$ satisfy the conditions 1°, 2°, 3°. Define a mapping $\varrho : G^{*s} \to G$ by the formula $\varrho(x) = \tau^{-1}\underbrace{\eta \ldots \eta}_{l}(x)$ for $x \in W_l = \zeta^{-1}(l)$. Let $x_1, \ldots, x_{n+1} \in G^{*s}$, where $x_i \in W_{l_i}$ for $i = 1, \ldots, n+1$. Then

$$\varrho(f^*_{(s)}(x_1, \ldots, x_{n+1})) = \tau^{-1}\underbrace{\eta \ldots \eta}_{\varphi_{(s)}(l_1, \ldots, l_{n+1})}(f^*_{(s)}(x_1, \ldots, x_{n+1})) = \tau^{-1}\underbrace{\eta \ldots \eta}_{l_1 + \ldots + l_{n+1}}(f^*_{(s)}(x_1, \ldots, x_{n+1})) =$$

$$= \tau^{-1}(f^*_{(s)}(\underbrace{\eta \ldots \eta}_{l_1}(x_1), \ldots, \underbrace{\eta \ldots \eta}_{l_{n+1}}(x_{n+1}))) =$$

$$= f(\tau^{-1}\underbrace{\eta \ldots \eta}_{l_1}(x_1), \ldots, \tau^{-1}\underbrace{\eta \ldots \eta}_{l_{n+1}}(x_{n+1})) = f(\varrho(x_1), \ldots, \varrho(x_{n+1})).$$

Thus $\varrho : \mathfrak{G}^{*s}_{(s)} \to \mathfrak{G}$ is a homomorphism, and even an epimorphism. Consequently, by Lemma 2, the $(n+1)$-group $\mathfrak{G}$ is derived from some $(k+1)$-group $\mathfrak{G}_{(s-1)}$.

If the mapping $\eta : G^{*s} \to G^{*s}$ satisfies the condition $\underbrace{\eta \ldots \eta}_{s} = id$ then $\underbrace{\eta \ldots \eta}_{sk} = id$, whence $\underbrace{\eta \ldots \eta}_{n+1} = \eta$. Let $x_1, \ldots, x_{n+1} \in G^{*s}$. Then

$$\eta(f^*_{(s)}(x_1, \ldots, x_{n+1})) = \underbrace{\eta \ldots \eta}_{n+1}(f^*_{(s)}(x_1, \ldots, x_{n+1})) = f^*_{(s)}(\eta(x_1), \ldots, \eta(x_{n+1})),$$

which proves that $\eta : \mathfrak{G}^{*s}_{(s)} \to \mathfrak{G}^{*s}_{(s)}$ is a homomorphism. Also, the condition $\underbrace{\eta \ldots \eta}_{s} = id$ implies that η is injective and surjective. Hence η is an automorphism. ∎

In a special case, when g.c.d. $(s, k) = 1$ then in the cyclic $(k + 1)$-group $\mathfrak{C}_{s,k+1}$ there exists a unique element of order one. That fact admits to extract some additional relations between the covering $(k + 1)$-group and the creating $(k + 1)$-group of the $(n + 1)$-group.

Corollary 7. If the $(n + 1)$-group $\mathfrak{G} = (G,f)$ is derived from a $(k + 1)$-group $\mathfrak{G}_{(s-1)} = (G, g)$ and g.c.d. $(s, k) = 1$, then the sub-$(k + 1)$-group $\mathfrak{B} = (\zeta^{-1}(l), f^*)$ of the free covering $(k + 1)$-group $\langle \mathfrak{G}^{*s}, \tau, \zeta \rangle$, where $l \in \mathfrak{C}_{s,k+1}$ is the element of order one, is isomorphic to $\mathfrak{G}_{(s-1)}$.

Proof. By Theorem 6, the direct product $\langle \mathfrak{G}_{(s-1)} \times \mathfrak{C}_{s,k+1}, \lambda \rangle$ is a free covering $(k + 1)$-group of the $(n + 1)$-group $\mathfrak{G} = (G,f)$. In view of Theorem 2, the set $W_l = G \times \{l\}$, where $l \in \mathfrak{C}_{s,k+1}$ is the element of order one, is a sub-$(k + 1)$-group $\mathfrak{B} = (W_l, f^*)$ of the $(k + 1)$-group $\mathfrak{G}^{*s} = \mathfrak{G}_{(s-1)} \times \mathfrak{C}_{s,k+1}$. The $(k + 1)$-group $\mathfrak{G}_{(s-1)} \times \{l\}$ is isomorphic to $\mathfrak{G}_{(s-1)}$. ∎

Theorem 7. *Let g.c.d.* $(s, k) = 1$ *and* $\langle \mathfrak{G}^{*s}, \tau, \zeta \rangle$ *be a free covering* $(k + 1)$*-group of the* $(n + 1)$*-group* $\mathfrak{G}$*. Then the following conditions are equivalent:*

1° *the* $(n + 1)$*-group* $\mathfrak{G}$ *is derivated from the* $(k + 1)$*-group* $\mathfrak{G}_{(s-1)}$;

2° *all sub-*$(n + 1)$*-groups of the* $(n + 1)$*-group* $\mathfrak{G}^*_{(s)}$ *of the form* $(\zeta^{-1}(l), f^*_{(s)})$, *where* $l \in Z_s$, *are isomorphic;*

3° *the sub-*$(n + 1)$*-group of the* $(n + 1)$*-group* $\mathfrak{G}^{*s}_{(s)}$ *of the form* $(\zeta^{-1}(l), f^*_{(s)})$, *where* $l \in \mathfrak{C}_{s,k+1}$ *is the element of order one, is isomorphic to the* $(n + 1)$*-group* $\mathfrak{G}$.

Proof. We show that $1° \Rightarrow 2° \Rightarrow 3° \Rightarrow 1°$.

Let the $(n + 1)$-group $\mathfrak{G}$ be derivated from the $(k + 1)$-group $\mathfrak{G}_{(s-1)}$. By Theorem 6 the direct product $\mathfrak{G}_{(s-1)} \times \mathfrak{C}_{s,k+1}$ is a free covering $(k + 1)$-group of the $(n + 1)$-group $\mathfrak{G}$. The order of each element in the cyclic $(k + 1)$-group $\mathfrak{C}_{s,k+1}$ is a divisor of s, whence the one-element subsets $\{l\}$ are sub-$(n + 1)$-groups of the $(n + 1)$-group $(\mathfrak{C}_{s,k+1})_{(s)}$. In consequence, the $(n + 1)$-group $(\mathfrak{G}_{(s-1)})_{(s)}$ is isomorphic to each of the $(n + 1)$-groups $(\mathfrak{G}_{(s-1)} \times \{l\})_{(s)}$.

Now, we assume that all the sub-$(n + 1)$-groups of the form $(\zeta^{-1}(l), f^*_{(s)})$, where $l \in Z_q$, are isomorphic. As shown in Theorem 3, in a $(k + 1)$-group $\mathfrak{G}^{*s}$ there exists a sub-$(k + 1)$-group $\mathfrak{B} = (\zeta^{-1}(l_0), f^*)$ where $l_0 \in \mathfrak{C}_{s,k+1}$ is the element of order one. The $(n + 1)$-group $\mathfrak{B}_{(s)} = (\zeta^{-1}(l_0), f^*_{(s)})$ derived from the $(k + 1)$-group $\mathfrak{B}$ is, by assumption, isomorphic to the $(n + 1)$-group $(\zeta^{-1}(0), f^*_{(s)})$ which by itself is isomorphic to the $(n + 1)$-group $\mathfrak{G}$.

Finally, assume that the sub-$(n + 1)$-group $(\zeta^{-1}(l), f^*_s)$, where $l \in \mathfrak{C}_{s,k+1}$ is the element of order one, is isomorphic to the $(n + 1)$-group $\mathfrak{G}$. According to Theorem 2, the $(n + 1)$-group $(\zeta^{-1}(l), f^*_{(s)})$ is derived from the $(k + 1)$-group $(\zeta^{-1}(l), f^*)$, whence the $(n + 1)$-group $\mathfrak{G}$ isomorphic to the $(n + 1)$-group $(\zeta^{-1}(l), f^*_{(s)})$ is also derivated from some $(k + 1)$-group. ∎

It is known that usually there is no unique $(k + 1)$-group creating a given $(n + 1)$-group (see [1]), thus an $(n + 1)$-group can be derived from distinct nonisomorphic $(k+1)$-groups. Post proved (see [10]) that in case of $k=1$, the creating 2-group (i.e. the creating group) is determined uniquely up to an isomorphism. But the restriction to $k=1$ is only a sufficient condition. Here we give a certain condition, also sufficient to the uniqueness of the creating group, which includes the case considered by Post.

Proposition 4. If an $(n+1)$-group $\mathfrak{G}=(G,f)$ is derived from a certain $(k+1)$-group and γ is the least positive integer for which g.c.d. $(s/\gamma, k)=1$, then all the creating $(\gamma k+1)$-groups of $\mathfrak{G}$ are isomorphic.

Proof. Let γ be the least positive integer for which g.c.d. $(s/\gamma, k)=1$. The $(n+1)$-group $\mathfrak{G}$ is, by assumption, derived from some $(k+1)$-group $\mathfrak{G}_{(s-1)}$, thus it is also derived from the $(\gamma k+1)$-group $(\mathfrak{G}_{(s-1)})_{(\gamma)}$. In accordance to Corollary 7 the $(\gamma k+1)$-group $(\mathfrak{G}_{s-1})_{(\gamma)}$ is isomorphic to the sub-$(\gamma k+1)$-group $\mathfrak{B}=(\zeta^{-1}_{(l)}, f^*)$, where $l\in\mathfrak{C}_{s/\gamma,k+1}$ is the element of order one, of the free covering $(\gamma k+1)$-group $\mathfrak{G}^{*s/\gamma}=(G^{*s/\gamma}, f^*)$. The $(\gamma k+1)$-group $\mathfrak{B}$ is determined up to an isomorphism, consequently all the creating $(\gamma k+1)$-groups of the $(n+1)$-group $\mathfrak{G}$ are also isomorphic. ∎

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