Miroslav Bartušek

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Archivum Mathematicum, Vol. 17 (1981), No. 3, 125--136

Persistent URL: http://dml.cz/dmlcz/107101

ARCH. MATH. 3, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XVII: 125—136, 1981

ON ASYMPTOTIC PROPERTIES OF OSCILLATORY SOLUTIONS OF THE SYSTEM OF DIFFERENTIAL EQUATIONS OF FOURTH ORDER

MIROSLAV BARTUŠEK, Brno

(Received February 2, 1980)

Consider the system of differential equations

$$y_i' = f_i(t, y_1, y_2, y_3, y_4), \qquad i \in N_4 \tag{1}$$

where $f_i \in C^\circ(D)$, $D = \{(t, x_1, x_2, x_3, x_4) : t \in [0, \infty), y_i \in R, i \in N_4\}$,

$$f_i(t, x_1, x_2, x_3, x_4)\, x_{i+1} > 0 \qquad \text{for } x_{i+1} \neq 0, \; i \in N_3,$$

$$f_4(t, x_1, x_2, x_3, x_4)\, x_1 < 0 \qquad \text{for } x_1 \neq 0, \tag{2}$$

$N_n = \{1, 2, \ldots, n\}$, $R = (-\infty, \infty)$, $C^\circ(D)$ is the set of all continuous functions on D. Let $N = \{1, 2, \ldots\}$, $R_+ = [0, \infty)$.

The special case of (1) is the differential equation of the fourth order

$$y^{(4)} = f(t, y, y', y'', y'''), \tag{3}$$

where $f \in C^\circ(D)$ and $f(t, x_1, x_2, x_3, x_4)\, x_1 < 0$ for $x_1 \neq 0$.

Definition 1. Let $Y = (y_i)_1^4$ be a non-trivial solution of (1), defined on the interval $[a, b)$, $0 \leq a < b \leq \infty$. Y is said to be oscillatory if for every $i \in N_4$ there exists a sequence $(t_k)_{k=1}$ of zeros of y_i, $\lim t_k = b$ such that

$$\sup \{\sum_{i=1}^{4} |y_i(t)| ; t_k \leq t < b\} > 0 \qquad \text{for } k \in N$$

holds. Y is said to be strongly oscillatory if there exist sequences $(t_k^i)_{k=1}^\infty$, $i \in N_4$, $t_k^i \in [a, b)$ such that

$$t_k^i < t_{k+1}^i, \; y_i(t_k^i) = 0, \qquad y_i(t) \neq 0 \qquad \text{for } t \in (t_1^i, b),$$
$$t \neq t_k^i, \; k \in N, \; i \in N_4.$$

In the present paper we shall study the strongly oscillatory solutions. Especially some sufficient conditions are given under which all components of such solutions are unbounded.

Asymptotic behaviour of oscillatory solutions of (3) under the assumption

$$f(t, x_1, x_2, x_3, x_4)\, x_1 \geqq 0$$

was studied in [2] and for the differential equation of the third order in [1] and [2]. However, about the behaviour of oscillatory solutions of (1) with the property (2) we know very few.

In all the work we shall suppose that strongly oscillatory solutions exist.

First, the following Lemma will be proved that shows the character of the oscillatory solutions.

Lemma. *Let $(y_i)_1^4$ be a solution of (1), defined on $[a, b)$ and let $(t_k^1)_0^\infty$ be the sequence of consecutive zeros of y_1,*

$$\lim_{k\to\infty} t_k^1 = b, \qquad \text{i.e.} \qquad t_k^1 < t_{k+1}^1,\ y_1(t_k^1) = 0,\ \ y(t) \neq 0$$

for $t \in (t_0^1, b)$, $t \neq t_k$, $k \in N$.

(I) *If at least one of the two inequalities*

$$y_2(t_0^1)\, y_3(t_0^1) < 0, \qquad y_2(t_0^1)\, y_4(t_0^1) > 0 \tag{4}$$

is not valid, then there exist sequences $(t_k^i)_{k=2}^\infty$, $i = 2, 3, 4$ such that for $k = 2, 3, \ldots$

$$t_k^1 < t_k^4 < t_k^3 < t_k^2 < t_{k+1}^1, \qquad y_i(t_k^1) \neq 0, \qquad y_i(t_k^i) = 0,$$

$$y_i(t)\, y_1(t) < 0 \quad \text{for} \quad t \in (t_k^1, t_k^i), \quad y_i(t)\, y_1(t) < 0 \quad \text{for} \quad t \in (t_k^i, t_{k+1}^1), \tag{5}$$

$i = 2, 3, 4.$

(ii) *If (4) is valid, then there exist sequences $(t_k^i)_{k=0}^\infty$ $i = 2, 3, 4$ such that either for $k \in N$*

$$t_k^1 < t_k^2 < t_k^3 < t_k^4 < t_{k+1}^1, \qquad y_i(t_k^i) = 0, \qquad y_i(t_k^1) \neq 0,$$

$$(-1)^i y_i(t)\, y_1(t) > 0 \qquad \text{for } t \in (t_k^1, t_k^i),$$

$$(-1)^i y_i(t)\, y_1(t) < 0 \qquad \text{for } t \in (t_k^i, t_{k+1}^1),\ i = 2, 3, 4 \tag{6}$$

holds or there exists a number $k_0 \in \{0, 1, 2, \ldots\}$ such that (6) holds for $k \leqq k_0$ and (5) holds for $k \geqq k_0 + 2$.

Proof. Put for the simplicity $t_k^1 = t_k$, $k = 0, 1$. According to the Rolle's Theorem and (1), (2) there exist numbers $x_2 \in (t_0, t_1)$ and $x_5 \in (t_1, t_2)$ such that

$$y_1'(x_2) = y_2(x_2) = 0, \qquad y_1'(x_5) = y_2(x_5) = 0. \tag{7}$$

Next, it follows from the assumptions of the lemma and (1), (2) that

$$y_2(t_0)\, y_1(x_2) \geqq 0, \qquad y_2(t_1)\, y_1(x_2) \leqq 0, \tag{8}$$

$$y_4(t)\, y_1(x_2) \qquad \text{is decreasing on } (t_0, t_1). \tag{9}$$

Consider some cases.

1° $\qquad y_i(t_0)\,y_1(x_2) \geqq 0, \qquad y_4(t_0)\,y_1(x_2) > 0, \qquad i = 2, 3.$

Suppose that $y_4(t) \neq 0$ for $t \in (t_0, t_1)$. Then according to (9) $y_4(t) > 0$ for $t \in (t_0, t_1]$ and we have from (1) and (2) successively for $i = 4, 3, 2, 1$ that $y_i(t)\,y_1(x_2) > 0$, $y_i(t)\,y_1(x_2)$ is increasing, $t \in (t_0, t_1)$ which contradicts to $y_1(t_1) = 0$. Thus, with respect to (9) there exist the only number $x_4 \in (t_0, t_1)$ such that $y_4(x_4) = 0$, $y_i(t)\,y_1(x_2) > 0$ for $t \in [t_0; x_4)$, $i \in N_4$, $y_4(t)\,y_1(x_2) < 0$ for $t \in (x_4, t_1]$.

The existence of the only numbers x_3, x_2 with the properties $t_0 < x_{i+1} < x_i < t_1$, $y_i(x_i) = 0$, $y_i(t)\,y_1(x_2) > 0$ for $t \in [t_0, x_i)$, $y_i(t)\,y_1(x_2) < 0$ for $t \in (x_i, t_1]$ can be proved successively for $i = 3, 2$ in the same procedure. From this t_1 is the simple zero of y_1 and $\operatorname{sgn} y_1(x_4) = -\operatorname{sgn} y_1(x_5)$. Thus

$$y_i(t_1)\,y_1(x_5) \geqq 0, \qquad y_4(t_1)\,y_1(x_5) > 0, \qquad i = 2, 3$$

and we have the same situation as at the beginning at t_0. The repeating of the considerations shows that (5) is valid. The statement (5) is valid in the cases

2° $\qquad y_i(t_0)\,y_1(x_2) > 0, \qquad y_4(t_0)\,y_1(x_2) \leqq 0, \qquad i = 2, 3$

3° $\qquad y_2(t_0)\,y_1(x_2) \geqq 0, \qquad y_3(t_0)\,y_1(x_2) \leqq 0, \qquad y_4(t_0)\,y_1(x_2) < 0,$
$\qquad |y_2(t_0)| + |y_3(t_0)| > 0,$

too, as in both cases $y_1(t)\,y_1(x_2) > 0$ in some right neighbourhood of $t = t_0$ and this situation was met in 1° on the intervals $[x_4, x_3)$ and $[x_3, x_2]$.

4° $\qquad y_i(t_0)\,y_1(x_2) \geqq 0, \qquad y_2(t_0)\,y_3(t_0) = 0, \qquad y_4(t_0)\,y_1(x_2) \leqq 0,$
$\qquad i = 2, 3.$

From this and from (9) $y_3(t)\,y_1(x_2)$ is decreasing on $(t_0, t_1]$. If $y_3(t_0) = 0$, then $y_2(t)\,y_1(x_2)$ is decreasing on $(t_0, t_2]$ and according to (7) we get at t_1 the same situation as in the case 1° for t_0. If $y_3(t_0)\,y_1(x_2) > 0$, $y_2(t_0) = 0$, then in some right neighbourhood of t_0 the following relation is valid (see (9))

$$y_i(t)\,y_1(x_2) > 0, \qquad y_4(t)\,y_1(x_2) < 0, \qquad t \in (t_1, t_1 + \varepsilon), \qquad \varepsilon > 0,\ i \in N_3$$

and we get the situation that was studied in 1° on the interval (x_4, x_3). Thus the statement (4) is valid in this case, too.

5° $\qquad y_2(t_0) = 0, \qquad y_3(t_0)\,y_1(x_2) < 0.$

It follows from (1) and (2) that successively for $i = 3, 2, 1$ $y_i(t)\,y_1(x_2)$ are negative on $(t_0, \bar{t})$ for a suitable $\bar{t} < t_2$ that contradicts the assumptions of the Lemma. This case is inadmissible.

6° $\qquad y_2(t_0)\,y_1(x_2) > 0, \qquad y_3(t_0)\,y_1(x_2) < 0, \qquad y_4(t_0) = 0.$

In virtue of (9) $y_i(t)\,y_1(x_2) < 0$ for $t \in (t_0, t_1]$, $i = 3, 4$ holds and according to (8) $y_2(t_1)\,y_1(x_2) < 0$, thus we have at $t = t_1$ the case 1°.

The last possible case is

$$7^\circ \qquad y_2(t_0)\,y_1(x_2) > 0, \qquad y_3(t_0)\,y_1(x_2) < 0, \qquad y_4(t_0)\,y_1(x_2) > 0.$$

If $y_4(t)\,y_1(x_2) > 0$ for $t \in [t_0, t_1)$, then with respect to (8) we get in t_1 one of the cases 2°, 3°, 4° or 6° with the exception of $y_2(t_1) = 0$. When $y_2(t_1) = 0$, then we get the case (5) or in some right neighbourhood of t_1 we have the same situation as in 1° on the interval $[t_1, x_3]$. Now, let there exist a zero x_4 of y_4 in the interval (t_0, t_1). Then according to (9) $y_4(t_1)\,y_1(x_2) < 0$. If $y_2(t_1) = 0$, then we get the same situation in some left neighbourhood of t_1 as in 1°, $t \in [t_0, x_3]$. According to (8) in the other cases we get at t_1 the cases 1° or 7°.

Now the statement of Lemma follows from the above considerations. Lemma is proved.

Definition 2. Let $(y_i)_1^4$ be the strongly oscillatory solution of (1). It is said to be the first kind if there exists an integer k_0 such that (5) holds for $k = k_0$. $(y_i)_1^4$ is said to be of the second kind if (6) holds.

Lemma 2. *Let* $(y_i)_1^4$ *be an arbitrary solution of* (1) *and let there exist functions* $\varphi_i \in C^1(R)$, $i \in N_4$ *such that* $\varphi_i(s)\,s > 0$ *for* $s \neq 0$, $\varphi_i'(s) \geqq 0$ *for* $s \in R$ *and on* D

$$(10) \qquad \varphi_2(x_2)\,\varphi_3'(x_3)\,f_3(t, x_1, x_2, x_3, x_4) = \varphi_1'(x_1)\,\varphi_4(x_4)\,f_1(t, x_1, x_2, x_3, x_4)$$

holds. Then the function $F(t) = \varphi_2(y_2(t))\,\varphi_3(y_3(t)) - \varphi_1(y_1(t))\,\varphi_4(y_4(t))$ *is non decreasing on* $[a, b)$.

Proof. We have according to (1) and (2)

$$\begin{aligned} F'(t) = {} & \varphi_2'(y_2)\,\varphi_3(y_3)\,f_2(t, y_1, y_2, y_3, y_4) + \\ (11) \qquad & + \varphi_2(y_2)\,\varphi_3'(y_3)\,f_3(t, y_1, y_2, y_3, y_4) - \varphi_1'(y_1)\,\varphi_4(y_4)\,f_1(t, y_1, y_2, y_3, y_4) - \\ & - \varphi_1(y_1)\,\varphi_4'(y_4)\,f_4(t, y_1, y_2, y_3, y_4) \geqq 0. \end{aligned}$$

Remark 1. The condition (10) is fulfilled e.g.

a) if there exist functions $g_i \in C^\circ(R)$ and $h_i \in C^\circ(R)$, $i \in N_4$ such that $f_1(t, x_1, x_2, x_3, x_4) = a(t) \prod_{i=1}^{4} g_i(x_i)$, $f_3(t, x_1, x_2, x_3, x_4) = Ma(t) \prod_{i=1}^{4} h_i(x_i)$, $g_j(s) > 0$, $h_k(s) > 0$, $g_2(0) = 0$, $h_4(0) = 0$, $g_2(s)\,s > 0$ and $h_4(s)\,s > 0$ for $s \neq 0$; $j = 1, 3, 4$; $k = 1, 2, 3$; $t \in [a, b)$, $x_i \in R$, $a > 0$. In this case we can put

$$\varphi_1(s) = \int_0^s \frac{h_1(t)}{g_1(t)}\,dt, \qquad \varphi_2(s) = \frac{g_2(s)}{h_2(s)}, \qquad \varphi_3(s) = \int_0^s \frac{g_3(t)}{h_3(t)}\,dt,$$

$$\varphi_4(s) = \frac{h_4(s)\,M}{g_4(s)}, \qquad s \in R.$$

b) for the equation (3) with $\varphi_i(s) = s$, $i \in N_4$.

Consequence. Let the assumptions of Lemma 2 be valid. Then every oscillatory solution of (1) defined on $[a, b)$ and fulfilling the condition $F(a) \geqq 0$ is strongly oscillatory of the first kind.

In the rest of the paper we shall deal only with the strongly oscillatory solutions of the first kind of (1). In all theorems $(y_i)_1^4$ means such a solution, defined on the interval $I = [a, b)$, $0 \leqq a < b \leqq \infty$ and denote by $(t_k^i)_{k=1}^\infty$, $i \in N_4$ the sequence of zeros of y_i with the properties (5), $\lim_{k \to \infty} t_k^i = b$. Then according to (1) and (2)

$$\begin{aligned}
&y_j'(t)\, y_1(t) > 0 \qquad \text{for } t \in (t_k^1, t_k^{j+1}),\\
(12) \qquad &y_j'(t)\, y_1(t) < 0 \qquad \text{for } t \in (t_k^{j+1}, t_{k+1}^1),\\
&y_4'(t)\, y_1(t) < 0 \qquad \text{for } t \in (t_k^1, t_{k+1}^1),\\
&y_4'(t_k^1) = 0, \qquad k \in N; j \in N_3.
\end{aligned}$$

Note, that $(|\, y_i(t_k^{i+1})\,|)_{k=1}^\infty$, $i \in N_4$ $(t_k^5 = t_k^1)$ is the sequence of absolute values of all local extremes of y_i on the interval $[t_1^4, b)$.

In the further considerations M_i will denote the suitable positive constant.

Lemma 3. *Let $i \in N_4$ and there exist continuous functions $H_1 : R_+^4 \to R_+$, $H_2 : R_+^4 \to$ $\to (0, \infty)$ such that $H_1(x_1, x_2, x_3, x_4)\, x_{i+1} > 0$ for $x_{i+1} \neq 0$,*

$$H_1(|\, x_1\,|, |\, x_2\,|, |\, x_3\,|, |\, x_4\,|) \leqq |\, f_i(t, x_1, x_2, x_3, x_4)\,|,$$
$$|\, f_{i+1}(t, x_1, x_2, x_3, x_4)\,| \leqq H_2(|\, x_1\,|, |\, x_2\,|, |\, x_3\,|, |\, x_4\,|)$$

in D $(x_5 = x_1)$. Let an interval $\Delta = [t_1, t_2]$ be given such that y_{i+1} has a zero in Δ and let $v_j = \max_{0 \leqq t_j \leqq t_j} |\, y_j(t)\,|$. Then

$$\int_0^{v_{i+1}} \min_{\substack{0 \leqq x_j \leqq v_j \\ j \neq i+1}} H_1(x_1, x_2, x_3, x_4)\, \mathrm{d}x_{i+1} \leqq 2v_i \max_{0 \leqq x_j \leqq v_j} H_2(x_1, x_2, x_3, x_4);$$

$j \in N_4$.

Proof. It follows from the assumtpions that there exist numbers t_3, t_4 such that $y_{i+1}(t_3) = 0$, $|\, y_{i+1}(t_4)\,| = v_{i+1}$ and y_{i+1} do not change the sign in the interval $[t_3, t_4] \subset [t_1, t_2]$ (if $t_4 < t_3$ the proof is similar). Then

$$\int_0^{v_{i+1}} \min_{\substack{0 \leqq x_j \leqq v_j \\ j \neq i+1}} H_1(x_1, x_2, x_3, x_4)\, \mathrm{d}x_{i+1} \leqq \int_{t_3}^{t_4} H_1(|\, y_1(t)\,|, |\, y_2(t)\,|, |\, y_3(t)\,|,$$

$$|\, y_4(t)\,|)\, |\, y_{i+1}'(t)\,|\, \mathrm{d}t \leqq \int_{t_3}^{t_4} |\, y_i'(t)\,|\ |\, y_{i+1}'(t)\,|\, \mathrm{d}t \leqq 2v_i \max_{0 \leqq x_j \leqq v_j} H_2(x_1, x_2, x_3, x_4).$$

The lemma is proved.

Theorem 1. *Let the assumptions of Lemma 2 be valid and let there exist continuous functions $g_m : R_+^2 \to R_+$, $g_2 : R_+^3 \to R_+$, $m = 1, 3$; $G_k : R_+ \to (0, \infty)$, $G_3 : R_+^2 \to$ $\to (0, \infty)$, $k = 1, 2$ such that $g_i = 0$ iff the first argument is equal to zero, g_i are non-*

decreasing with respect to the first argument, $i \in N_4$ and

$$(13)\qquad \begin{aligned} &g_1(|x_2|, |x_1|) \leqq |f_1(t, x_1, x_2, x_3, x_4)| \leqq G_1(|x_1|, |x_2|),\\ &g_2(|x_3|, |x_1|, |x_2|) \leqq |f_2(t, x_1, x_2, x_3, x_4)| \leqq G_2(|x_1|, |x_3|),\\ &g_3(|x_1|, |x_2|) \leqq |f_4(t, x_1, x_2, x_3, x_4)| \leqq G_3(|x_1|).\end{aligned}$$

Let $\varphi_4'(s) > 0$ for $s \in R$. Then $\limsup_{t \to b-} y_1(t) = \infty$ holds.

Proof. Put

$$F_1(t) = F(t) - \int_{t^1_1}^{t} g_3(|y_1(s)|, |y_2(s)|) |\varphi_1(y_1(s))| \varphi_4'(y_4(s))\, ds,$$

where F is defined in Lemma 2. Then according to Lemma 1, (2) and (13) $F(t_1^1) = F_1(t_1^1) > 0$,

$$F_1' \geqq |\varphi_1(y_1)| \varphi_4'(y_4) \{-f_4(t, y_1, y_2, y_3, y_4) \operatorname{sgn} y_1 - g_3(|y_1|, |y_2|)\} \geqq 0$$

and thus F, F_1 are positive, non-decreasing.

Suppose that the statement of the theorem is not valid. Then

$$(14)\qquad |y_1(t_k^2)| \leqq M < \infty, \qquad k \in N.$$

We shall consider two cases.

$$(15)\qquad 1^0 \lim_{t \to b-} F(t) = M_1 < \infty.$$

It follows from the definition of F_1 that

$$(16)\qquad \lim_{k \to \infty} \int_{t_k^1}^{t_{k+1}^1} g_3(|y_1(t)|, |y_2(t)|) \varphi_4'(y_4(t)) |\varphi_1(y_1(t))|\, dt = 0$$

holds.

First, we prove that the sequence $(|y_2(t_{k+1}^1)|)_1^\infty$ is bounded. Let this propositior be not valid. Then there exists an infinite subset $K_1 \subset N$ such that

$$(17)\qquad \lim_{k \to \infty} |y_2(t_{k+1}^1)| = \infty, \qquad k \in K_1$$

and from (15) and (17) we have

$$(18)\qquad |y_3(t_{k+1}^1)| \leqq M_2, \qquad k \in K_1.$$

According to Lemma 3 ($\Delta = [t_k^2, t_{k+1}]$, $i = 1$)

$$\int_0^{|y_2(t_{k+1}^1)|} \min_{0 \leqq x_1 \leqq M} g_1(s, x_1)\, ds \leqq 2 \, . \, |y_1(t_k^2)| \max_{\substack{0 \leqq x_1 \leqq M\\ 0 \leqq x_3 \leqq M_2}} G_2(x_1, x_3) \leqq M_3$$

that contradicts (17). Thus

$$(19)\qquad |y_2(t_{k+1}^1)| \leqq M_4, \qquad k \in N$$

and according to (16), (12), (15) and Lemma 1

$$\int_{t_k^1}^{t_{k+1}^1} g_3(|y_1(t)|, |y_2(t)|)\,\varphi_4'(y_4(t))\,|\varphi_1(y_1(t))|\,\mathrm{d}t \geqq$$

$$\geqq M_5 \int_{t_k^2}^{t_{k+1}^1} g_3(|y_1(t)|, |y_2(t)|)\,\varphi_4'(y_4(t))\,|\varphi_1(y_1(t))|\;|y_1'(t)|\,\mathrm{d}t \geqq$$

$$\geqq M_5 \int_0^{|y_1(t_k^2)|} |\varphi_1(s \operatorname{sgn} y_1(t_k^2))| \min_{0 \leqq x_2 \leqq M_4} g_3(s, x_2)\,\varphi_4'(y_4(t(s)))\,\mathrm{d}s, \qquad k \in N,$$

$$M_5 = [\max_{\substack{0 \leqq x_1 \leqq M \\ 0 \leqq x_2 \leqq M_4}} G_1(x_1, x_2)]^{-1} > 0,$$

(20) $$\lim_{k \to \infty} y_1(t_k^2) = 0, \qquad \lim_{k \to \infty} |y_4(t_k^3)| = \infty, \qquad k \in N.$$

Now suppose that $|y_2(t_k^3)|$ does not converge to zero for $k \to \infty$, $k \in N$. Then there exists an infinite set $K_2 \subset N$ such that (see (15))

(21) $$|y_2(t_k^3)| \geqq M_6 > 0, \qquad |y_3(t_k^4)| \leqq M_7 < \infty, \qquad k \in K_2$$

and according to (20), (12) and Lemma 1

$$|y_1(t_k^3)| - |y_1(t_k^4)| = \int_{t_k^4}^{t_k^3} |y_1'(t)|\,\mathrm{d}t \geqq M_8 \int_{t_k^4}^{t_k^3} g_1(|y_2(t)|, |y_1(t)|)\,|y_2'(t)|\,\mathrm{d}t \geqq$$

$$\geqq M_8 \int_{|y_2(t_k^4)|}^{|y_2(t_k^3)|} \min_{0 \leqq x_1 \leqq M} g_1(s, x_1)\,\mathrm{d}s, \qquad M_8 = [\max_{\substack{0 \leqq x_1 \leqq M \\ 0 \leqq x_3 \leqq M_7}} G_2(x_1, x_3)]^{-1}.$$

It follows from this, (20) and (21) that there exists an integer k_0 such that

$$|y_2(t_k^4)| \geqq \frac{M_6}{2}, \qquad k \in K_2, k \geqq k_0$$

and

$$|y_1(t_k^3)| - |y_1(t_k^4)| \geqq \int_{t_k^4}^{t_k^3} g_1(|y_2(t)|, |y_1(t)|)\,\mathrm{d}t \geqq$$

$$\geqq \min_{0 \leqq x_1 \leqq M} g_1\left(\frac{M_6}{2}, x_1\right)(t_k^3 - t_k^4),$$

(23) $$\lim_{k \to \infty} (t_k^3 - t_k^4) = 0, \qquad k \in K_2.$$

Then

(24) $$|y_4(t_k^3)| = \int_{t_k^4}^{t_k^3} |y_4'(t)|\,\mathrm{d}t \leqq \int_{t_k^4}^{t_k^3} G_3(|y_1(t)|)\,\mathrm{d}t \leqq \max_{0 \leqq x_1 \leqq M} G_3(x_1)(t_k^3 - t_k^4)$$

that contradicts (20 and (23). Thus

$$\lim_{k\to\infty} y_2(t_k^3) = 0, \qquad |y_2(t_k^3)| \leqq M_9, \qquad k \in N \tag{25}$$

and according to (15) and Lemma 1

$$\lim_{k\to\infty} |y_3(t_k^4)| = \infty. \tag{26}$$

It follows from Lemma 1 and (12) that there exists a sequence $(\xi_k)_1^\infty$ such that

$$|\varphi_3(y_3(\xi_k))| = |\varphi_4(y_4(\xi_k))|, \qquad \xi_k \in (t_k^4, t_k^3)$$

holds and

$$\begin{aligned} F(\xi_k) &= \varphi_2(y_2)\,\varphi_3(y_3) - \varphi_1(y_1)\,\varphi_4(y_4)\,|_{t=\xi_k} = \\ &= |\varphi_4(y_4)|\,\left(|\varphi_2(y_2)| + |\varphi_1(y_1)|\right)|_{t=\xi_k}. \end{aligned} \tag{27}$$

From this and according to (20), (15) and (25) we have

$$\lim_{k\to\infty} |y_3(\xi_k)| = \lim_{k\to\infty} |y_4(\xi_k)| = \infty, \qquad |y_3(\xi_k)| \geqq M_{10} > 0, \qquad k \in N. \tag{28}$$

Then, by use of (25) and (12) successively

$$\begin{aligned} |y_2(\xi_k)| - |y_2(t_k^4)| &= \int_{t_k^4}^{\xi_k} |y_2'(t)|\,dt \geqq \int_{t_k^4}^{\xi_k} g_2(|y_3(t)|, |y_1(t)|, |y_2(t)|\,dt \geqq \\ &\geqq \min_{\substack{0 \leqq x_1 \leqq M \\ 0 \leqq x_2 \leqq M_9}} g_2(M_3, x_1, x_2)(\xi_k - t_k^4), \end{aligned}$$

$$\lim_{k\to\infty} (\xi_k - t_k^4) = 0, \tag{29}$$

$$\begin{aligned} |y_4(\xi_k)| &= \int_{t_k^4}^{\xi_k} |y_4'(t)|\,dt \leqq \int_{t_k^4}^{\xi_k} G_3(|y_1(t)|)\,dt \leqq \\ &\leqq \max_{0 \leqq x_1 \leqq M} G_3(x_1)(\xi_k - t_k^4), \end{aligned} \tag{30}$$

that contradicts (28) and (29). Thus (14) is false in the case 1°.

$$2^\circ \text{ Let } \lim_{t\to b-} F(t) = \infty. \tag{31}$$

Then according to (14) and Lemma 1

$$\lim_{k\to\infty} |y_4(t_k^3)| = \lim_{k\to\infty} |y_4(t_k^1)| = \infty. \tag{32}$$

Suppose that $\liminf_{k\to\infty} |y_2(t_k^3)| < \infty$. Then there exists an infinite subset $K_1 \subset N$ such that $|y_2(t_k^3)| \leqq M_9$, $k \in K_1$ and according to (14) and (31) the relations (27) and (28) hold and $\xi_k - t_k^4$ is bounded that contradicts to (30) and (28). Thus

$$\lim_{k\to\infty} |y_2(t_k^3)| = \infty. \tag{33}$$

If there exists an infinite set $K_1 \subset N$ such that $|y_2(t_k^4)| \geqq M_6$, $k \in K_1$ holds then (22) is valid and according to (13) $t_k^3 - t_k^4$ is bounded for $k \in K_1$ that contradicts to (24) and (32). Thus $\lim_{k\to\infty} y_3(t_k^4) = 0$ and from (33), (13) and from the estimation (22) made for the interval $[t_k^1, t_k^4]$ we get that $t_{k_i}^4 - t_k^1$ is bounded for $k \in N$. From this and from the estimation (30) made for the interval $[t_k^1, t_k^4]$ we can conclude that $y_4(t_k^1)$ is bounded for $k \in N$, too, which contradicts to (32). The theorem is proved.

Theorem 2. *Let the assumption of Lemma* 2 *be valid and* $\lim_{s\to\pm\infty} |\varphi_j(s)| = \infty, j = 1, 2,$ $\varphi_4'(s) > 0$ *for* $s \in R$. *Further' let there exist a positive constant* M *and continuous. non-decreasing functions* $g_i : R_+ \to R_+$ *and* $G_i : R_+ \to (0, \infty)$, $i \in N_4$ *such that* $g_i(0) = 0$, $g_i(s) > 0$ *for* $s > 0$, $\lim_{s\to\infty} g_2(s) = \infty$, $g_3(s) = MG_3(s)$, $s \in R$ *and*

$$(34) \qquad g_i(|x_{i+1}|) \leqq |f_i(t, x_1, x_2, x_3, x_4)| \leqq G_i(|x_{i+1}|), \qquad i \in N_4.$$

Then

$$(35) \qquad \limsup_{t\to b-} |y_i(t)| = \infty, \qquad i \in N_4.$$

Proof. As the assumptions of Theorem 1 are valid, there exists an infinite set $K \subset N$ such that

$$(36) \qquad \lim_{k\to\infty} |y_1(t_k^2)| = \infty, \qquad k \in K.$$

First, we state some simple estimations. According to (34), (12), (1) and Lemma 1 we have

$$(37) \qquad \int_0^{|y_2(t_{k+1}^1)|} G_1(s)\,ds = \int_{t_k^2}^{t_{k+1}^1} G_1(|y_2(t)|)\,|y_2'(t)|\,dt \geqq \int_{t_k^2}^{t_{k+1}^1} |y_1'(t)|\;|y_2'(t)| \times$$

$$\times dt \geqq \int_{t_k^2}^{t_{k+1}^1} g_2(|y_3(t)|)\,|y_1'(t)|\,dt \geqq g_2(|y_3(t_k^2)|)\,|y_1(t_k^2)|.$$

Similarly

$$(38) \qquad \int_0^{|y_3(t_k^2)|} G_2(s)\,ds \geqq g_3(|y_4(t_k^3)|)\,|y_2(t_k^3)|$$

and

$$(39) \qquad |y_4(t_k^2)| - |y_4(t_k^3)| = \int_{t_k^3}^{t_k^2} |y_4'(t)|\,dt \geqq g_4(|y_1(t_k^3)|)(t_k^2 - t_k^3).$$

It follows from Lemma 3 ($\Delta = [t_k^4, t_k^3]$, $i = 2$) that

$$(40) \qquad \int_0^{|y_3(t_k^4)|} g_2(s)\,ds \leqq 2\,|y_2(t_k^3)|\,G_3(|y_4(t_k^3)|).$$

Thus, according to (38)

(41) $$\int_0^{|y_3(t_k^4)|} g_2(s)\,\mathrm{d}s \leqq 2M^{-1} \int_0^{|y_3(t_k^2)|} G_2(s)\,\mathrm{d}s, \qquad k \in N.$$

First we prove that

(42) $$\lim_{t\to b_-} F(t) = \infty.$$

Suppose on the contrary that

(43) $$\lim_{t\to b_-} F(t) = M_1 < \infty.$$

Then it follows from (43) and (36) that

(44) $$\lim_{k\to\infty} y_4(t_k^2) = 0, \qquad k \in K.$$

Suppose that $|y_2(t_k^3)|$ does not tend to infinity for $k \to \infty$, $k \in K$. Then there exists an infinite set $K_1 \subset K$ such that $|y_2(t_k^2)|$ is bounded for $k \in K_1$ and we can prove the first relation of (22) in the same way as in Theorem 1 (we must use the interval $[t_k^1, t_k^2]$ instead of $[t_k^2, t_{k+1}^1]$) which is a contradiction to (36). Thus

$$\lim_{k\to\infty} |y_2(t_k^3)| = \infty, \qquad k \in K.$$

Similarly it can be seen that $\lim_{k\to\infty} |y_2(t_{k+1}^1)| = \infty$, $k \in K$ and by use of (43), (12) and Lemma 1 we have successively

$$|y_3(t_k^2)| \leqq M_2, \qquad k \in K,$$

$$|y_2(t_k^3)| = \int_{t_k^3}^{t_k^2} |y_2'(t)|\,\mathrm{d}t \leqq G_2(M_2)(t_k^2 - t_k^3),$$

(45) $$\lim_{k\to\infty} (t_k^2 - t_k^3) = \infty, \qquad \lim_{k\to\infty} |y_1(t_k^3)| = \infty, \qquad k \in K.$$

But (45) is a contradiction to (39) and (44). Thus (42) is valid.

Now, suppose that (35) is not valid for $i = 2$. Then

(46) $$|y_2(t)| \leqq M_3, \qquad t \in [t_k^1, t_{k+1}^1], \qquad k \in K$$

and according to (42)

(47) $$\lim_{k\to\infty} |y_3(t_k^4)| = \infty, \qquad k \in K.$$

As according to (37), (36) and (46) $\lim_{k\to\infty} y_3(t_k^2) = 0$, $k \in K$ we get the contradiction to (41) and (47). Thus $\limsup_{t\to b_-} y_2(t) = \infty$. Suppose that the statement (35) of the theorem is false for $i = 3$. Then

(48) $$|y_3(t)| \leqq M_4, \qquad t \in [a, b)$$

and by use of (42)

(49) $$\lim_{k\to\infty} |y_2(t_k^3)| = \lim_{k\to\infty} |y_2(t_{k+1}^1)| = \infty.$$

Similarly to (37) it can be proved that

$$\int_{|y_3(t_k^2)|}^{|y_3(t_{k+1}^1)|} G_2(s)\,\mathrm{d}s \geqq g_3(|y_4(t_k^2)|)\,|y_2(t_{k+1}^1)|$$

and from this, according to (48), (49), (42) and (12)

$$\lim_{k\to\infty} y_4(t_k^2) = \lim_{k\to\infty} y_4(t_k^3) = 0, \qquad \lim_{k\to\infty} |y_1(t_k^3)| = \infty, \qquad k\in K.$$

According to (39) the last relations give us $\lim_{k\to\infty}(t_k^2 - t_k^3) = 0$, $k \in K$ and successively for $i = 3, 2$

$$|y_i(t_k^{5-i})| = \int_{t_k^3}^{t_k^2} |y_i'(t)|\,\mathrm{d}t \geqq G_i(|y_{i+1}(t_k^{5-i})|)(t_k^2 - t_k^3),$$

$$\lim_{k\to\infty} |y_i(t_k^{5-i})| = 0, \qquad k\in K,$$

that contradicts to (49). Thus it follows from the proved part of the theorem that there exists an infinite set $K_1 \subset K$ such that for $k \in K_1$

(50) $$\lim_{k\to\infty} |y_2(t_k^3)| = \infty \qquad or \qquad \lim_{k\to\infty} |y_2(t_{k+1}^1)| = \infty$$

and

(51) $$\lim_{k\to\infty} |y_3(t_k^4)| = \infty \qquad or \qquad \lim_{k\to\infty} |y_3(t_{k+1}^1)| = \infty \qquad \text{hold.}$$

Suppose that (35) is false for $i = 4$. Then

(52) $$|y_4(t)| \leqq M_5, \qquad t\in[a, b)$$

and with respect to (42)

(53) $$\lim_{k\to\infty} |y_1(t_k^3)| = \infty, \qquad k\in K_1.$$

If $|y_3(t_{k+1}^1)|$ does not tend to infinity for $k \to \infty$, $k \in K_1$, then there exists an infinite set $K_2 \subset K_1$ such that (use (51) and Lemma 1)

$$|y_3(t_{k+1}^1)| \leqq M_6, \qquad \lim_{k\to\infty} |y_3(t_k^4)| = \infty, \qquad |y_3(t_k^2)| \leqq M_6, \qquad k\in K_2,$$

that contradicts to (41). Thus

(54) $$\lim_{k\to\infty} |y_3(t_{k+1}^1)| = \infty, \qquad k\in K_1.$$

Now, we shall prove that

(55) $$\lim_{k\to\infty} |y_2(t_{k+1}^1)| = \infty, \qquad k\in K_1.$$

Suppose on the contrary, that there exists an infinite set $K_2 \subset K_1$ such that

(56) $$|y_2(t^1_{k+1})| \leqq M_7, \qquad k \in K_2$$

is valid. Then according to (50) $\lim_{k\to\infty} |y_2(t^3_k)| = \infty$, $k \in K_2$ and it follows successively from (37), (36), (56) and (50) that

$$\lim_{k\to\infty} y_3(t^2_k) = 0, \qquad k \in K_2,$$

(57) $$|y_2(t^3_k)| = \int_{t^3_k}^{t^2_k} |y'_2(t)| \,\mathrm{d}t \leqq G_2(|y_3(t^2_k)|)(t^2_k - t^3_k),$$

$$\lim_{k\to\infty} (t^2_k - t^3_k) = \infty, \qquad k \in K_2 \qquad \text{holds.}$$

The last relation contradicts to (39) and (52). Thus (55) is valid. Similarly to (37) the estimation

$$\int_{|y_4(t^3_k)|}^{|y_4(t^2_k)|} G_3(s)\,\mathrm{d}s \leqq g_4(|y_1(t^3_k)|)\,|y_3(t^2_k)|,$$

can be proved and by virtue of (52), (53)

(58) $y_3(t^2_k)$ *is bounded for* $k \in K_1$.

Then it follows from (39), (52) and (53) that $t^2_k - t^3_k$ is bounded for $k \in K_1$ and (57), (58) and Lemma 1 give us that $y_2(t^3_k)$ and $y_2(t^4_k)$ are bounded for $k \in K_1$, too.

Finally, from the last conclusion and from (40) we have that $y_3(t^4_k)$, $k \in K_1$ is bounded. But the boundedness of $y_3(t^4_k)$ and $y_2(t^4_k)$ contradicts to (42). The theorem is proved.

Corollary. *Let there exist continuous non-decreasing functions* $g : R_+ \to R_+$ *and* $G : R_+ \to (0, \infty)$ *such that* $g(0) = 0$, $g(s) > 0$ *for* $s > 0$ *and*

$$g(|x_1|) \leqq |f(t, x_1, x_2, x_3, x_4)| \leqq G(|x_1|)$$

holds in D. Then for the solution y of (3)

$$\limsup_{t\to b_-} |y^{(i)}(t)| = \infty, \qquad i = 0, 1, 2, 3 \text{ holds.}$$

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M. Bartušek
662 95 Brno, Janáčkovo nám. 2a
Czechoslovakia