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ASYMPTOTIC PROPERTIES OF THE SOLUTIONS OF THE EQUATION $\dot{z} = f(t, z)$ WITH A COMPLEX-VALUED FUNCTION f

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1. INTRODUCTION

Three recent papers [1], [2], [3] are devoted to the study of the asymptotic behaviour of the solutions of an equation

$$\dot{z} = f(t, z), \qquad \cdot = \frac{\mathrm{d}}{\mathrm{d}t}, \tag{1.1}$$

where f is a continuous complex-valued function of a real variable t and a complex variable z. The subject matter of the present paper is concerned with this problem as well. The basic tool used here is the technique of Ljapunov-like functions. For convenience we suppose that

$$f(t, z) = G(t, z)\,[h(z) + g(t, z)],$$

where G is a real-valued function, and g, h are complex-valued functions. The function h is assumed to be holomorphic and the right-hand side of

$$\dot{z} = G(t, z)\,[h(z) + g(t, z)] \tag{1.2}$$

to be in a suitable meaning "close" to this function. The main results are stated in the fourth section. In Section 5 we shall establish some stability results for the trivial solution of (1.2) by means of the results of Section 4 and of [1], [2]. Notice that the asymptotic properties of the equation (1.2) was first investigated by M. Ráb in [4], [5], where he has discussed the asymptotic behaviour of the solutions of the Riccati differential equation

$$\dot{z} = q(t) - p(t)\,z^2$$

with complex-valued coefficients p, q.

2. NOTATION

In what follows we use the following notation:

C – Set of all complex numbers
N – Set of all positive integers
$\operatorname{Re} b$ – Real part of a complex number b
$\operatorname{Im} b$ – Imaginary part of a complex number b
$\bar{b}$ – Conjugate of b
$|b|$ – Absolute value of b
$\operatorname{Bd} \Gamma$ – Boundary of a set $\Gamma \subset C$
$\operatorname{Cl} \Gamma$ – Closure of a set $\Gamma \subset C$
$\operatorname{Int} \Gamma$ – Interior of a Jordan curve $z = z(t)$, $t \in [\alpha, \beta]$ whose points z form a set $\Gamma \subset C$; Γ will be called the *geometric image* of the Jordan curve $z = z(t)$, $t \in [\alpha, \beta]$
I – Interval $[t_0, \infty)$
Ω – Simply connected region in C such that $0 \in \Omega$
$C[\alpha, \infty)$ – Class of all continuous real-valued functions defined on the interval $[\alpha, \infty)$
$C(\Gamma)$ – Class of all continuous real-valued functions defined on the set Γ
$\tilde{C}(\Gamma)$ – Class of all continuous complex-valued functions defined on the set Γ
$\mathscr{H}(\Gamma)$ – Class of all complex-valued functions defined and holomorphic in the region Γ

3. PRELIMINARIES

First we recall, for the reader's convenience, the definition of a Ljapunov-like function $W(z)$ determining certain sets $\mathring{K}(\lambda)$, $K(\lambda)$, $K(\lambda_1, \lambda_2)$, which are very useful when describing the asymptotic behaviour of the solutions of (1.2). For more details we refer to [1]. Let $h(z) \in \mathscr{H}(\Omega)$ be a function such that $h'(0) \neq 0$ and $h(z) = 0 \Leftrightarrow z = 0$. Put

$$W(z) = |w(z)|,$$

where

$$w(z) = z \exp\left[\int_0^z r(z^*)\,dz^*\right]$$

and

$$r(z) = \begin{cases} \dfrac{zh'(0) - h(z)}{zh(z)} & \text{for } z \in \Omega,\ z \neq 0, \\[2ex] -\dfrac{h''(0)}{2h'(0)} & \text{for } z = 0. \end{cases}$$

Assuming that Ξ denotes the system of all simply connected regions $\Gamma \subset \Omega$ with the property $0 \in \Gamma$, we define

$$\lambda_0 = \sup_{\Gamma \in \Xi} \lim_{M \to \infty} \inf_{z \in \Gamma_M} W(z),$$

where

$$\Gamma_M = \{z \in \Gamma : \inf_{z^* \in \operatorname{Bd} \Gamma} |z - z^*| < M^{-1}\} \cap \{z \in \Gamma : |z| > M\}.$$

Evidently $0 < \lambda_0 \leqq \infty$.

For $0 < \lambda < \lambda_0$ define the sets $\hat{K}(\lambda) \subset \Omega$ in the following manner: choose $\Gamma \in \Xi$ so that

$$\lim_{M \to \infty} \inf_{z \in \Gamma_M} W(z) > \lambda$$

and put

$$\hat{K}(\lambda) = \{z \in \Gamma : W(z) = \lambda\}.$$

According to [1] this definition is correct, and denoting

$$\hat{K}(0) = \{0\},$$

$$K(\lambda) = \bigcup_{0 \leqq \mu < \lambda} \hat{K}(\mu) \qquad \text{for } 0 < \lambda \leqq \lambda_0,$$

$$K(\lambda_1, \lambda_2) = \bigcup_{\lambda_1 < \mu < \lambda_2} \hat{K}(\mu) \qquad \text{for } 0 \leqq \lambda_1 < \lambda_2 \leqq \lambda_0,$$

we have

Theorem 3.1. *$K(\lambda_0)$ is a simply connected region. Every set $\hat{K}(\lambda)$, where $0 < \lambda < \lambda_0$, is the geometric image of a certain Jordan curve, and*

$$\hat{K}(\lambda) = \{z \in K(\lambda_0) : W(z) = \lambda\},$$

$$\operatorname{Int} \hat{K}(\lambda) = \{z \in K(\lambda_0) : W(z) < \lambda\}.$$

Moreover,

$$K(\lambda) = \operatorname{Int} \hat{K}(\lambda) \qquad \textit{for } 0 < \lambda < \lambda_0,$$

$$K(\lambda_1, \lambda_2) = K(\lambda_2) - \operatorname{Cl} K(\lambda_1) \quad \textit{for } 0 < \lambda_1 < \lambda_2 \leqq \lambda_0,$$

and

$$K(0, \lambda) = K(\lambda) - \{0\} \qquad \textit{for } 0 < \lambda \leqq \lambda_0.$$

Suppose that $G(t, z)\,[h(z) + g(t, z)] \in \tilde{C}(I \times \Omega)$, $G \in C(I \times (\Omega - \{0\}))$, $g \in \tilde{C}(I \times (\Omega - \{0\}))$. The following five results will be necessary at various points, the first being proved in [3], the second one in [2] and the last three ones in [1]:

Lemma 3.2. *Assume $E(t) \in C[t_0, \infty)$, $0 < \gamma_n < \lambda_0$,*

$$\inf_{n \in N} \gamma_n = 0.$$

Suppose $g \in \tilde{C}(I \times \Omega)$, $G \in C(I \times \Omega)$. If

$$G(t, z) \operatorname{Re} \left\{ h'(0) \left[1 + \frac{g(t, z)}{h(z)} \right] \right\} \leqq E(t)$$

or

$$-G(t, z)\,\mathrm{Re}\left\{h'(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq E(t)$$

for $t \geqq t_0$, $z \in \mathring{K}(\gamma_n)$, $n \in N$, *then* $G(t, 0)\, g(t, 0) = 0$ *for* $t \geqq t_0$.

Theorem 3.3. *Assume* $0 < \gamma < \lambda_0$. *Suppose that there hold*

(3.1) $$G(t, z) > 0$$

and

(3.2) $$\mathrm{Re}\left[g(t, z)\frac{h'(0)}{h(z)}\right] < -\mathrm{Re}\, h'(0)$$

for $t \geqq t_0$, $z \in \mathring{K}(\gamma)$.

If a solution $z(t)$ *of* (1.2) *satisfies*

$$z(t_1) \in \mathrm{Cl}\, K(\gamma),$$

where $t_1 \geqq t_0$, *then* $z(t) \in K(\gamma)$ *for* $t > t_1$.

Theorem 3.4. *Assume* $\delta \geqq 0$, $\vartheta \leqq \lambda_0$. *Suppose there is a function* $E(t) \in C[t_0, \infty)$ *such that*

$$\sup_{t_0 \leqq s \leqq t < \infty} \int_s^t E(\xi)\,\mathrm{d}\xi = \varkappa < \infty,$$

$$\delta\, \mathrm{e}^{\varkappa} < \vartheta,$$

and

$$G(t, z)\,\mathrm{Re}\left\{h'(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq E(t)$$

for $t \geqq t_0$, $z \in K(\delta, \vartheta)$.

If a solution $z(t)$ *of* (1.2) *satisfies*

$$z(t_1) \in \mathrm{Cl}\, K(\gamma),$$

where $t_1 \geqq t_0$ *and* $0 < \gamma\, \mathrm{e}^{\varkappa} < \vartheta$, *then*

$$z(t) \in \mathrm{Cl}\, K(\beta) \qquad \textit{for } t \geqq t_1,$$

where $\beta = \mathrm{e}^{\varkappa} \max\,[\gamma, \delta]$.

Theorem 3.5. *Assume* $\delta_n \geqq 0$, $\vartheta \leqq \lambda_0$, $s_n \in I$ *for* $n \in N$, *and* $\vartheta < \infty$. *Suppose there are functions* $E_n(t) \in C[t_0, \infty)$ *such that:*

(i) *for* $n \in N$ *the following conditions are fulfilled:*

(3.3) $$\int_{t_0}^{\infty} E_n(s)\,\mathrm{d}s = -\infty,$$

$$\sup_{s_n \leqq s \leqq t < \infty} \int_s^t E_n(\xi)\,\mathrm{d}\xi = \varkappa_n < \infty,$$

$$\delta_n e^{\varkappa_n} < \vartheta;$$

(ii) *for* $t \geqq s_n$, $z \in K(\delta_n, \vartheta)$, $n \in N$ *the inequality*

$$G(t, z) \operatorname{Re}\left\{h'(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq E_n(t)$$

holds.

Denote

$$\delta = \inf_{n \in N} [\delta_n e^{\varkappa_n}].$$

If a solution $z(t)$ *of* (1.2) *satisfies*

(3.4) $$z(t_1) \in K(\vartheta e^{-\varkappa_1}),$$

where $t_1 \geqq s_1$, *then to any* ε, $\delta < \varepsilon < \lambda_0$, *there is a* $T = T(\varepsilon, t_1) > 0$ *independent of* $z(t)$ *such that*

$$z(t) \in K(\varepsilon)$$

for $t \geqq t_1 + T$.

Theorem 3.6. *Let the assumptions of Theorem* 3.5 *be fulfilled, except* (3.3) *is replaced by*

$$\int_s^{s+t} E_n(\xi) \, d\xi \to -\infty \qquad as\ t \to \infty,$$

uniformly for $s \in [s_n, \infty)$.

If a solution $z(t)$ *of* (1.2) *satisfies* (3.4), *where* $t_1 \geqq s_1$, *then to any* ε, $\delta < \varepsilon < \lambda_0$, *there exists a* $T = T(\varepsilon) > 0$ *independent of* t_1 *and* $z(t)$ *such that*

$$z(t) \in K(\varepsilon)$$

for $t \geqq t_1 + T$.

4. MAIN RESULTS

Suppose $G \in C(I \times \Omega)$, $g \in \tilde{C}(I \times \Omega)$, $h \in \mathscr{H}(\Omega)$. Assume that $h'(0) \neq 0$ and $h(z) = 0 \Leftrightarrow z = 0$. Let $W(z)$, λ_0, $\hat{K}(\lambda)$, $K(\lambda)$, $K(\lambda_1, \lambda_2)$ be defined as in Section 3. We start with the following

Theorem 4.1. *Assume* $0 < \vartheta \leqq \lambda_0$. *Suppose there is an* $E(t) \in C[t_0, \infty)$ *such that*

$$\sup_{t_0 \leqq t < \infty} \int_{t_0}^{t} E(s) \, ds = \varkappa < \infty,$$

and that

(4.1) $$G(t, z) \operatorname{Re}\left\{h'(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq E(t)$$

for $t \geqq t_0$, $z \in K(0, \vartheta)$.

If a solution $z(t)$ *of* (1.2) *satisfies*

$$z(t_1) \in \operatorname{Cl} K(\gamma),$$

where $t_1 \geqq t_0$ *and*

$$0 < \beta = \gamma e^{\varkappa} \exp\left[-\int_{t_0}^{t_1} E(s)\,\mathrm{d}s\right] < \vartheta,$$

then

(4.2) $$z(t) \in \mathrm{Cl}\, K(\beta)$$

for $t \geqq t_1$.

Proof. Put $\mathscr{M} = \{t \geqq t_1 : z(t) \in K(0, \vartheta)\}$, $\mathscr{M}_0 = \{t \geqq t_1 : z(t) \in K(\vartheta)\}$. For $t \in \mathscr{M}$ we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} W^2(z) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[w(z)\,\overline{w(z)}\right] = \\ &= 2\,\mathrm{Re}\left[w'(z)\,\overline{w(z)}\,\dot{z}\right] = 2\,\mathrm{Re}\left\{w(z)\,\overline{w(z)}\,[z^{-1} + r(z)]\,\dot{z}\right\} = \\ &= 2W^2(z)\,\mathrm{Re}\left[h'(0)\,h^{-1}(z)\,\dot{z}\right],\end{aligned}$$

where $z = z(t)$. Hence

$$\begin{aligned}\dot{W}(z) &= W(z)\,\mathrm{Re}\left[h'(0)\,h^{-1}(z)\,\dot{z}\right] = \\ &= G(t, z)\,W(z)\,\mathrm{Re}\left\{h'(0)\,h^{-1}(z)\,[h(z) + g(t, z)]\right\} = \\ &= G(t, z)\,W(z)\,\mathrm{Re}\left\{h'(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\}\end{aligned}$$

for $t \in \mathscr{M}$. This in combination with (4.1) gives

$$\dot{W}(z(t)) \leqq E(t)\,W(z(t)) \qquad \text{for } t \in \mathscr{M}.$$

Let $\tau \geqq t_1$ be such that $z(\tau) = 0$. Then

$$\begin{aligned}|\dot{W}(z(\tau))| &= \left|\lim_{t\to\tau} \frac{W(z(t))}{t - \tau}\right| = \\ &= \left|\lim_{t\to\tau} \frac{|z(t) \exp[\int_0^{z(t)} r(z^*)\,\mathrm{d}z^*]|}{t - \tau}\right| = \\ &= \lim_{t\to\tau}\left\{\left|\frac{z(t)}{t - \tau}\right|\,\left|\exp\left[\int_0^{z(t)} r(z^*)\,\mathrm{d}z^*\right]\right|\right\} = \\ &= |\dot{z}(\tau)| = |G(\tau, 0)\,g(\tau, 0)|.\end{aligned}$$

By Lemma 3.2 we have $G(\tau, 0)\,g(\tau, 0) = 0$ and therefore

$$\dot{W}(z(\tau)) = 0.$$

Thus

(4.3) $$\dot{W}(z(t)) \leqq E(t)\,W(z(t)) \qquad \text{for } t \in \mathscr{M}_0.$$

We claim that (4.2) holds for $t \geqq t_1$. If it is not the case, there exists a $t^* > t_1$ such that $z(t^*) \in K(\beta, \vartheta)$ and $z(t) \in K(\vartheta)$ for $t \in [t_1, t^*]$. The inequality (4.3) is equivalent to

$$\frac{d}{dt}\{W(z(t))\exp[-\int_{t_1}^{t} E(s)\,ds]\} \leqq 0, \qquad t \in \mathscr{M}_0.$$

Integration from t_1 to t^* yields

$$W(z(t^*))\exp[-\int_{t_1}^{t^*} E(s)\,ds] - W(z(t_1)) \leqq 0,$$

whence

$$W(z(t^*)) \leqq W(z(t_1))\exp[\int_{t_1}^{t^*} E(s)\,ds] \leqq$$

$$\leqq \gamma \exp[\varkappa - \int_{t_0}^{t_1} E(s)\,ds] \leqq \beta < W(z(t^*)).$$

This contradiction proves (4.2) for $t \geqq t_1$.

Now, we are going to establish two theorems stating the conditions under which $z(t) \to 0$ as $t - t_1 \to \infty$ is uniformly satisfied with respect to $z(t)$, or $z(t)$ and t_1.

Theorem 4.2. *Assume* $0 < \vartheta \leqq \lambda_0$ *and* $\vartheta < \infty$. *Suppose there exists an* $E(t) \in C[t_0, \infty)$ *such that*

(4.4) $$\int_{t_0}^{\infty} E(s)\,ds = -\infty,$$

and that

$$G(t, z)\,\mathrm{Re}\left\{h'(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq E(t)$$

for $t \geqq t_0$, $z \in K(0, \vartheta)$.

If a solution $z(t)$ *of* (1.2) *satisfies*

(4.5) $$z(t_1) \in K(\vartheta e^{-\varkappa} \exp[\int_{t_0}^{t_1} E(s)\,ds]),$$

where $t_1 \geqq t_0$ *and*

$$\varkappa = \sup_{t_0 \leqq t < \infty} \int_{t_0}^{t} E(s)\,ds,$$

then to any ε, $0 < \varepsilon < \lambda_0$, *there is a* $T = T(\varepsilon, t_1) > 0$ *independent of* $z(t)$ *such that*

$$z(t) \in K(\varepsilon)$$

for $t \geqq t_1 + T$.

Proof. Put $\mathscr{M}_0 = \{t \geqq t_1 : z(t) \in K(\vartheta)\}$. From the proof of Theorem 4.1 we have

(4.3) $$\dot{W}(z(t)) \leqq E(t)\,W(z(t)) \qquad \text{for } t \in \mathscr{M}_0.$$

It follows by Theorem 4.1 that $z(t) \in K(\vartheta)$ for $t \geqq t_1$.

Choose ε, $0 < \varepsilon < \lambda_0$. Without loss of generality we may suppose $\varepsilon < \vartheta$. Let $T = T(\varepsilon, t_1) > 0$ be such that

$$\int_{t_1}^{t} E(s)\,\mathrm{d}s < \ln \frac{\varepsilon}{2\vartheta}$$

for $t \geqq t_1 + T$.

We claim that $z(t) \in K(\varepsilon)$ for $t \geqq t_1 + T$. If this is not true, there exists a $t^* \geqq \geqq t_1 + T$ for which

$$z(t^*) \notin K(\varepsilon). \tag{4.6}$$

The inequality (4.3) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\{W(z(t)) \exp[-\int_{t_1}^{t} E(s)\,\mathrm{d}s]\} \leqq 0, \qquad t \in \mathcal{M}_0.$$

By integration over $[t_1, t^*]$ we obtain

$$W(z(t^*)) \exp[-\int_{t_1}^{t^*} E(s)\,\mathrm{d}s] - W(z(t_1)) \leqq 0.$$

Hence

$$W(z(t^*)) \leqq W(z(t_1)) \exp[\int_{t_1}^{t^*} E(s)\,\mathrm{d}s] \leqq \vartheta \frac{\varepsilon}{2\vartheta} = \frac{\varepsilon}{2} < \varepsilon,$$

which contradics (4.6), thus proves that $z(t) \in K(\varepsilon)$ for $t \geqq t_1 + T$.

Theorem 4.3. *Let the assumptions of Theorem 4.2 be fulfilled except (4.4) is replaced by*

$$\int_{s}^{s+t} E(\xi)\,\mathrm{d}\xi \to -\infty \qquad \textit{as } t \to \infty \tag{4.7}$$

uniformly for $s \in [t_0, \infty)$.

If a solution $z(t)$ of (1.2) satisfies (4.5), where $t_1 \geqq t_0$ and

$$\varkappa = \sup_{t_0 \leqq t < \infty} \int_{t_0}^{t} E(s)\,\mathrm{d}s,$$

then to any ε, $0 < \varepsilon < \lambda_0$, there is a $T = T(\varepsilon) > 0$ independent of t_1 and $z(t)$ such that

$$z(t) \in K(\varepsilon)$$

for $t \geqq t_1 + T$.

Proof. The proof runs as that of Theorem 4.2. In view of (4.7) we can find a $T = T(\varepsilon) > 0$ so that $t - t_1 \geqq T$ implies

$$\int_{t_1}^{t} E(\xi)\,\mathrm{d}\xi = \int_{t_1}^{t_1+(t-t_1)} E(\xi)\,\mathrm{d}\xi < \ln \frac{\varepsilon}{2\vartheta}.$$

5. APPLICATIONS TO THE STABILITY THEORY

First of all we give the short survey of stability concepts used in this section. Suppose that the equation (1.1) possesses the trivial solution, i.e. that $f(t, 0) = 0$ for $t \in I$.

The trivial solution of the equation (1.1) is called *stable*, if to any $\varepsilon > 0$ and to any $t_1 \geqq t_0$ there exists a $\delta = \delta(\varepsilon, t_1) > 0$ such that each solution $z(t)$ of (1.1) satisfying $|z(t_1)| < \delta$ fulfils the condition $|z(t)| < \varepsilon$ for $t \geqq t_1$. The trivial solution of the equation (1.1) is said to be *uniformly stable*, if to any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for any $t_1 \geqq t_0$ each solution $z(t)$ of (1.1) satisfying $|z(t_1)| < \delta$ fulfils the condition $|z(t)| < \varepsilon$ for $t \geqq t_1$.

The trivial solution of the equation (1.1) is called *asymptotically stable*, if it is stable and to any $t_1 \geqq t_0$ there exists a $\vartheta = \vartheta(t_1) > 0$ such that each solution $z(t)$ of (1.1) satisfying $|z(t_1)| < \vartheta$ fulfils

$$\lim_{t \to \infty} z(t) = 0.$$

The trivial solution of the equation (1.1) is said to be *uniformly asymptotically stable*, if it is uniformly stable and there exists a $\vartheta > 0$ so that to any $\varepsilon > 0$ there is a $T = T(\varepsilon) > 0$ such that for any $t_1 \geqq t_0$ each solution $z(t)$ of (1.1) satisfying $|z(t_1)| < \vartheta$ fulfils $|z(t)| < \varepsilon$ for $t \geqq t_1 + T$.

The validity of the following lemma can be verified easily and therefore the proof will be omitted.

Lemma 5.1. *If the solutions of* (1.1) *depend continuously on initial values, then the trivial solution of* (1.1) *is uniformly stable if and only if to any* $\varepsilon > 0$ *there exist* $\delta = \delta(\varepsilon) > 0$ *and* $\sigma = \sigma(\varepsilon) > 0$ *such that for any* $t_1 \geqq \sigma$ *each solution* $z(t)$ *of* (1.1) *satisfying* $|z(t_1)| < \delta$ *fulfils* $|z(t)| < \varepsilon$ *for* $t \geqq t_1$.

Suppose $G(t, z)[h(z) + g(t, z)] \in \tilde{C}(I \times \Omega)$, $G(t, z) \in C(I \times (\Omega - \{0\}))$, $g(t, z) \in \tilde{C}(I \times (\Omega - \{0\}))$ and $h(z) \in \mathscr{H}(\Omega)$. Assume that $h'(0) \neq 0$ and $h(z) = 0 \Leftrightarrow z = 0$, and that $W(z)$, λ_0, $\mathring{K}(\lambda)$, $K(\lambda)$, $K(\lambda_1, \lambda_2)$ are defined as before.

Theorem 4.1 has the following consequence:

Corollary 5.2. *If* $G \in C(I \times \Omega)$, $g \in \tilde{C}(I \times \Omega)$ *and the assumptions of Theorem* 4.1 *are satisfied, then the trivial solution of* (1.2) *is stable.*

Combining Theorem 3.4 and Lemma 5.1 we obtain

Corollary 5.3. *Suppose* $\delta_n \geqq 0$, $\vartheta_n \leqq \lambda_0$, $s_n \in I$ for $n \in N$. *Assume that the solutions of* (1.2) *depend continuously on initial values and that* (1.2) *possesses the trivial solution. If there are functions* $E_n(t) \in C[t_0, \infty)$ *such that*

$$\sup_{s_n \leqq s \leqq t < \infty} \int_s^t E_n(\xi)\, d\xi = \varkappa_n < \infty \qquad \textit{for } n \in N,$$

$$\delta_n e^{\varkappa_n} < \vartheta_n \qquad \textit{for } n \in N,$$

$$\inf_{n \in N} [\delta_n e^{\varkappa_n}] = 0,$$

and that

$$G(t, z) \operatorname{Re} \left\{ h'(0) \left[1 + \frac{g(t, z)}{h(z)} \right] \right\} \leqq E_n(t),$$

for $t \geqq s_n$, $z \in K(\delta_n, \vartheta_n)$, $n \in N$, then the trivial solution of (1.2) *is uniformly stable.*

If we suppose in addition that $s_n = t_0$ for $n \in N$, then the assumption concerning the continuous dependence on initial values is superfluous.

Proof. Without loss of generality it may be supposed that $\vartheta_n < \lambda_0$ for $n \in N$. Choose ε, $0 < \varepsilon < \lambda_0$. Let n be such a positive integer that

$$\delta_n e^{\varkappa_n} < \varepsilon.$$

If a solution $z(t)$ of (1.2) satisfies the condition $z(t_1) \in \mathrm{Cl}\, K(\gamma)$, where $t_1 \geqq s_n$ and

$$\gamma e^{\varkappa_n} = \min \left[(\delta_n e^{\varkappa_n} + \varepsilon)/2, (\delta_n e^{\varkappa_n} + \vartheta_n)/2 \right],$$

then

$$z(t) \in \mathrm{Cl}\, K(\beta), \qquad t \geqq t_1,$$

where

$$\beta = e^{\varkappa_n} \max [\gamma, \delta_n] = \gamma e^{\varkappa_n} < \varepsilon.$$

Using Lemma 5.1 and $\mathrm{Cl}\, K(\beta) \subset K(\varepsilon)$, we get the assertion of the first part of the corollary.

If $s_n = t_0$ for $n \in N$, it is not necessary to use Lemma 5.1 and the assumption concerning the continuous dependence on initial values is superfluous.

Theorem 3.3 together with Lemma 5.1 yields

Corollary 5.4. *Suppose* $0 < \gamma_n < \lambda_0$, $s_n \in I$ *for* $n \in N$ *and*

$$\inf_{n \in N} \gamma_n = 0.$$

Assume that the solutions of (1.2) *depend continuously on initial values and that* (1.2) *possesses the trivial solution. If the conditions* (3.1) *and* (3.2) *are fulfilled for $t \geqq s_n$, $z \in \tilde{K}(\gamma_n)$, $n \in N$, then the trivial solution of* (1.2) *is uniformly stable.*

If we suppose in addition that $s_n = t_0$ for $n \in N$, then the assumption concerning the continuous dependence on initial values is superfluous.

The following corollary is a consequence of Theorem 4.2:

Corollary 5.5. *If $G \in C(I \times \Omega)$, $g \in \tilde{C}(I \times \Omega)$ and the assumptions of Theorem* 4.2 *are satisfied, then the trivial solution of* (1.2) *is asymptotically stable.*

Corollary 5.3 in combination with Theorem 3.5 and Theorem 3.6 gives the following two results:

Corollary 5.6. *Let the hypotheses of Theorem* 3.5 *be fulfilled with* $\delta = 0$. *Assume that the solutions of* (1.2) *depend continuously on initial values and that* (1.2) *possesses the trivial solution. Then the trivial solution is asymptotically stable.*

If we suppose in addition that $s_n = t_0$ *for* $n \in N$, *then the assumption concerning the continuous dependence on initial values is superfluous.*

Corollary 5.7. *Let the assumptions of Theorem* 3.6 *be fulfilled with* $\delta = 0$. *Suppose that the solutions of* (1.2) *depend continuously on initial values and that* (1.2) *possesses the trivial solution. Then the trivial solution is uniformly asymptotically stable.*

If we suppose in addition that $s_n = t_0$ *for* $n \in N$, *then the assumption concerning the continuous dependence on initial values is superfluous.*

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