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Archivum Mathematicum, Vol. 17 (1981), No. 2, 95--112

Persistent URL: http://dml.cz/dmlcz/107098

ARCH. MATH. 2, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XVII: 95—112, 1981

IRREGULAR IMAGINARY FIELDS

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(Received March 3, 1980)

0. INTRODUCTION

In this paper the *irregular imaginary fields* (i.e. the imaginary subfields of the l^{th} cyclotomic fields, l is an odd prime, with the relative class number divided by l) are given for $l < 125\,000$ (except the l^{th} cyclotomic fields). This result was reached from *Wagstaff's* tables by means of computer.

The brief expository on the irregular class group is mentioned in Paragraph 1.

1. THE IRREGULAR CLASS GROUP

1.1. Notation. In the whole paper we shall denote by

l an odd prime,
$\zeta = \cos 2\pi/l + \mathrm{i} \sin 2\pi/l$,
$\mathbf{Q}$ the field of rational numbers,
$\mathbf{R}$ the field of real numbers,
r a primitive root modulo l^n for each positive integer n,
G the Galois group of $\mathbf{Q}(\zeta)/\mathbf{Q}$, hence the group G is cyclic of order $l - 1$,
s the generator of G such that $s(\zeta) = \zeta^r$,
B_n Bernoulli number in the "*even index*" notation, thus $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6, \dots$

Under a *field* K we understand a subfield K of the l^{th} cyclotomic field $\mathbf{Q}(\zeta)$ different from $\mathbf{Q}$.

The field K defines the prime l uniquely: If $L_1 \supseteq L_2$ are number fields, then according to *Kronecker* ([3], Proposition 2.8) the discriminant of L_2 divides the discriminant of L_1. From the *Minkowski's convex body theorem* we get that the discriminant of K differs from ± 1 ([3], Theorem 2.9). Further, the discriminant of $\mathbf{Q}(\zeta)$ is equal to $(-1)^{\frac{l-1}{2}} l^{l-2}$ ([3], Theorem 2.8).

By $G(K)$ we denote the Galois group of $\mathbf{Q}(\zeta)$ over K. *The field K is determined by l* *and by the order d of the group G(K) uniquely.* In this paper *we identify the field K with the pair* $[l, d]$ (d positive integer, $d \mid l-1$, $d \neq l-1$).

By the *irregular class group* $\Delta(K)$ of the field K we understand the *l-Sylow subgroup* of the ideal class group of K. For simplicity we put $\Delta = \Delta(\mathbf{Q}(\zeta))$. The Galois group G acts on Δ in the natural way.

The field K is called *real* if it is a subfield of $\mathbf{R}$. In the opposite case it is called *imaginary*.

Clearly, it holds

1.2. Proposition. *A field* $K = [l, d]$ *is real if and only if d is even.*

1.3. Theorem. (*Pollaczek* [4], *Satz* III *and* § 6). *The group* Δ *is the direct sum*

$$\Delta = \Sigma \Delta_i (i \in \mathscr{I})^{1})$$

of cyclic groups Δ_i *of orders* l^{m_i} (m_i *are positive integers*) *such that for each* $\delta \in \Delta_i$ *the equality*

$$s(\delta) = r^{T_i l^{m_i - 1}} \delta$$

holds, where T_i *are integers,* $2 \leqq T_i \leqq l-2$.

If $i \in \mathscr{I}$ *and* T_i *is odd, then*

$$B_{l^{m_i-1}(l-T_i-1)+1} \equiv 0 (\mathrm{mod}\ l^{m_i}).$$

The group $\Delta(K)$ is the set of all the invariant classes in Δ under $G(K)$ ([3], Note after Proposition 4.21), thus we have

1.4. Theorem. *Let* $K = [l, d]$ *be a field.*
Then

$$\Delta(K) = \{\delta \in \Delta : \sigma(\delta) = \delta \quad \textit{for each} \quad \sigma \in G(K)\} = $$
$$= \{\delta \in \Delta : s^{\frac{l-1}{d}}(\delta) = \delta\} = \Sigma \Delta_i (i \in \mathscr{I}(K)),$$

where $\mathscr{I}(K) = \{i \in \mathscr{I} : d \mid T_i\}$.

1.5. Notation. For a field $K = [l, d]$ put

$$\mathscr{I}^{+}(K) = \{i \in \mathscr{I}(K) : T_i \text{ even}\},$$
$$\mathscr{I}^{-}(K) = \{i \in \mathscr{I}(K) : T_i \text{ odd}\},$$
$$\Delta^{+}(K) = \Sigma \Delta_i (i \in \mathscr{I}^{+}(K)),$$
$$\Delta^{-}(K) = \Sigma \Delta_i (i \in \mathscr{I}^{-}(K)).$$

Then we obtain from 1.4.

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[1]) For $\mathscr{I} = \emptyset$ by $\Sigma \Delta_i$ $(i \in \mathscr{I})$ we understand the trivial group.

1.6. Proposition. *Let K be a field. Then*

$$\Delta^+(K) = \begin{cases} \Delta(K \cap \mathbf{R}) & \textit{in the case } K \cap \mathbf{R} \neq \mathbf{Q}, \\ 1 & \textit{in the case } K \cap \mathbf{R} = \mathbf{Q},^{2)} \end{cases}$$

$$\Delta(K) = \Delta^-(K) \oplus \Delta^+(K).$$

1.7. By $\mathscr{T}$ we denote the set of all odd integers T, $1 \leqq T \leqq l - 4$, with $l \mid B_{T+1}$. From *Vandiver* ([6]) we obtain (*Pollaczek's formulation* [4], *Satz IX*) that for each $T \in \mathscr{T}$ there exists a positive integer $\mathrm{h}(T)$ such that

$$B_{l^{\mathrm{h}(T)-1}T+1} \equiv 0 (\mathrm{mod}\ l^{\mathrm{h}(T)})$$

and

$$B_{l^{X-1}T+1} \not\equiv 0 (\mathrm{mod}\ l^X)$$

for each integer $X > \mathrm{h}(T)$. Then

$$\mathrm{card}\ \Delta^-(\mathbf{Q}(\zeta)) = l^a,$$

where $a = \Sigma \mathrm{h}(T)\ (T \in \mathscr{T})$.[3])

This relation can be generalized as follows:

1.8. Theorem. (*Carlitz* [2]). *Let $K = [l, d]$ be a field. Then*

$$\mathrm{card}\ \Delta^-(K) = l^k,$$

where

$$k = \Sigma \mathrm{h}(T)\ (T \in \mathscr{T}, d \mid T).$$

1.9. Definition. Let $K = [l, d]$ be a field. The *index of irregularity of K* is the number

$$\mathrm{i}(K) = \mathrm{i}([l, d]) = \mathrm{card}\left\{1 \leqq a \leqq \frac{l-3}{2} : d \mid l - 2a, l \mid B_{2a}\right\}.$$

Let $\mathrm{r}^-(K)$ denote the rank of the group $\Delta^-(K)$, hence

$$\mathrm{r}^-(K) = \mathrm{card}\ \mathscr{I}^-(K).$$

Obviously, *for a real field K we have*

$$\mathrm{i}(K) = \mathrm{r}^-(K) = 0.$$

For an imaginary field K we get the best contemporary result (1.11) on the relation between $\mathrm{i}(K)$ and $\mathrm{r}^-(K)$ from *Ribet's Theorem*.

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[2]) For $K \cap \mathbf{R} = \mathbf{Q}$ the symbol $\Delta(K \cap \mathbf{R})$ is not defined. Then the g.c.d. of $\frac{l-1}{d}$ and $\frac{l-1}{2}$ is 1 ($K = [l, d]$), thus $d = \frac{l-1}{2}$ is odd and $\mathscr{I}^+(K) = \emptyset$.

[3]) If $\mathscr{T} = \emptyset$, then by $\Sigma \mathrm{h}(T)\ (T \in \mathscr{T})$ we understand the integer 0.

1.10. Theorem. (*Ribet* [5]). *Let* $1 \leqq a \leqq \frac{l-3}{2}$, $l \mid B_{2a}$. *Then there exists* $i \in \mathscr{I}^{-}(\mathbb{Q}(\zeta))$ *such that* $T_i = l - 2a$.

From this Theorem we obtain easily

1.11. Theorem. *Let K be a field. Then*

$$r^{-}(K) \geqq i(K).$$

1.12. Theorem. (*Pollaczek* [4], *Satz VI*). *Let*

$$z(T) = \text{card}\,\{i \in \mathscr{I}(\mathbb{Q}(\zeta)) : T = T_i\}$$

for $2 \leqq T \leqq l - 2$. *Then for T odd we have*

$$z(l - T) \leqq z(T) \leqq z(l - T) + 1.$$

2. THE TABLE OF IRREGULAR IMAGINARY FIELDS

2.1. Definition. An imaginary field K is said to be *irregular* if the group $\Delta^{-}(K)$ is non-trivial.

2.2. Proposition. *An imaginary field K is irregular if and only if* $i(K) > 0$.

Proof. Let $K = [l, d]$ be an imaginary field.

I. If the field K is irregular, then according to 1.8 there exists $T \in \mathscr{T}$, $d \mid T$. Put $2a = T + 1$. Then $1 \leqq a \leqq \frac{l-3}{2}$, $d \mid l - 2a$ and $l \mid B_{2a}$, thus $i(K) > 0$.

II. If $i(K) > 0$, then there exists $1 \leqq a \leqq \frac{l-3}{2}$ such that $d \mid l - 2a$ and $l \mid B_{2a}$. Put $T = 2a - 1$. Then $T \in \mathscr{T}$, $d \mid T$ and $h(T) > 0$. Hence according to 1.8 the field K is irregular.

2.3. Remark. *Adachi* ([1], *Theorem A* (*ii*)) introduces this Proposition 2.2 but with another proof (without using *Carlitz* result 1.8).

From the following table we can read easily all the irregular imaginary fields $K = [l, d]$ and their index of irregularity for $l < 125\,000$ and $d \neq 1$. The case $d = 1$ is involved in *Wagstaff's* paper [7]. The first column gives the prime l, the third column the integer d and the middle one the integer $2a$ such that $1 \leqq a \leqq \frac{l-3}{2}$, $d \mid l - 2a$ and $l \mid B_{2a}$. The given integer d has the property that $d' \mid l - 2a$ for an integer $1 < d' < l - 1$, $d \mid d'$, $d \neq d'$. The symbol $*$ at the prime in the first column points out larger appearance of this prime in the table.

From this table it follows that there are 1745 irregular imaginary fields $K = [l, d]$ for $l < 125\,000$ and $d \neq 1$. From them there are 1648 fields of index of irregularity 1, 90 of $i(K) = 2$, 6 of $i(K) = 3$ and only one of index of irregularity 4. This field is equal to $K = [43\,189, 3]$ and for the fields $K = [l, d]$ with $i(K) = 3$ we have $d = 3$ and $l = 37\,057$, 56 131, 71 191, 108 877, 109 789, and 109 843.

We got this table from the *Wagstaff's table* [7] using the computer *"Odra 1013"* at the *Technical University of Brno.*

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Table 1

l	2a	d	l	2a	d
67	58	3	3853	748	3.3
307	88	3	3881	1686	5
379	100	3.3	3967	106	3
461	196	5	*4003	82	3
463	130	3	*4003	142	3
491	336	5	4027	2332	3
523	400	3	4261	2068	3
541	86	5	4339	214	3
577	52	3	4451	2896	5
607	592	3	4523	456	7
631	226	3.3.5	4561	436	3.5
673	502	3	4591	3596	5
683	32	31	4639	3226	3
757	514	3.3.3	4657	2416	3
811	544	3	4663	4278	7
877	868	3	4813	2620	3
971	166	5	4861	4678	3
1153	802	3.3	4903	3106	3
1201	676	3.5.5	4909	1462	3
1237	874	3	5081	3016	5
1291	206	5	5101	190	3
*1297	202	3	5179	4732	3
*1297	220	3	5231	3466	5
1301	176	5.5	5413	1702	3
1327	466	3	5441	4726	5
1381	266	5	5501	666	5
1559	862	41	5527	5206	3
1669	388	3	5531	3438	7
1753	712	3	5557	3196	3
1777	1192	3	5791	1258	3
1871	1794	11	5821	1150	3
1933	1058	7	5839	2308	3
1951	1656	5	5923	4240	3.3
2017	1204	3	5953	3274	3
2137	1624	3	6217	4186	3
2267	2234	11	6247	1492	3
2383	2278	3	6287	5034	7
2411	2126	5	6343	750	7
2441	366	5	6451	3236	5
2663	1244	11	6491	346	5
2791	2554	3	6521	236	5
2861	352	13	6529	1564	3
3011	1496	5	6571	1744	3
3049	700	3	6577	1312	3
3083	1450	23	6733	1690	3
3181	3142	3	6763	4144	3
3433	1300	3	6793	2686	3
3469	1174	3.17	6971	2010	41
3511	1416	5	7057	4972	3
3529	3490	3	7127	6798	7
3581	1466	5	7351	1466	5
*3637	2526	101	7547	5644	11
*3637	3202	3	7591	2620	3
3697	1884	7	7687	1246	3
3821	3296	5	7901	4286	5
3851	216	5	7927	6448	3

(*Table 1 — continue*)

l	2*a*	*d*
8011	622	3.3
8059	874	3
8101	5968	3.3.3
8161	2758	3
8191	7680	7
8209	8056	3.3
8221	4900	3
8231	4806	5
8443	1894	3
8467	2368	3
8779	1072	3.7
8839	5584	3
8923	5614	3
8929	4126	3
9011	4294	53
9059	8100	7
9277	2422	3
9349	28	3
9431	2766	5
9433	178	3
9463	3760	3
9511	1132	3
9677	7094	41
9811	1366	3.5
9871	2980	3
9883	9622	3.3
9907	5968	3.13
*9949	4810	3
*9949	9112	3
10009	3952	3.3
10243	8134	3
10429	3652	3
10453	4378	3
10531	2172	13
10597	6478	3
10663	9430	3
10729	7528	3
10831	1136	5
10867	1390	3
11027	4620	149
11047	6568	3
11059	7886	19
11149	10114	3
11437	4960	3
11503	1078	3
11701	3346	3.5
11743	9580	3.103
11789	6868	7
*12073	2458	3
*12073	6874	3
12143	6462	13
12343	11506	3
12451	6726	5.5
*12613	502	3
*12613	9400	3
12697	10052	23.23

l	2*a*	*d*
12703	2782	3
12781	2716	3.5
12821	8886	5
12907	11842	3
12979	4960	3.3
13033	6718	3
13063	3796	3
13217	2640	7
13249	12724	3
13267	11848	3.11
13297	1438	3
*13411	5974	3
*13411	7450	3
13441	3016	3.5
13513	3430	3
13567	2990	7
13693	6560	7
13721	218	7
13759	8386	3
14153	4148	29
14323	10198	3.11
14401	13372	3
14407	6688	3
14449	7996	3
*14533	2884	3
*14533	3998	7
*14533	8896	3
14551	7330	3
14561	2304	7
14737	4498	3
*14767	238	3
*14767	8494	3
14831	13256	5
14843	8406	41
14851	12520	3.3
14891	11256	5
15313	7316	11
15541	2916	5
15601	11818	3.13
15619	10180	3
15667	9904	3
15739	14260	3
*15787	9316	3.3
*15787	11884	3
*15823	13552	3
*15823	15748	3
16069	5470	3
16519	15688	3
16573	4432	3
16843	16840	3
16879	16780	3
16901	9986	5
17107	14722	3
17191	16930	3.3
17209	15880	3
17231	7916	5

(*Table 1 — continue*)

l	2*a*	*d*	*l*	2*a*	*d*
17341	14896	3.5	22381	17668	3
17419	16228	3	22541	11908	7.7
17467	6184	3	22573	3896	19
17681	11826	5	22639	20120	11
17863	8068	3	22807	22060	3.3
18199	16522	3	22963	21046	3
18289	3938	127	23059	19708	3
18439	2602	3	23201	11066	5
18583	6724	3	23227	10634	7.7
18899	6920	11	23623	4252	3
19051	11696	5	23633	22828	7
19069	13030	3	*23719	14680	3
19211	4106	5	*23719	17434	3
19213	12772	3	23773	17584	3
19373	14066	29	24049	12904	3
19381	4812	17	24091	14276	5
*19531	1376	5	24151	21582	7
*19531	7820	7	24181	2486	5
19699	16822	3.7	24373	10990	3.3
19843	6922	3	24379	2874	17
19891	14842	3.3.17	24571	5050	3.3.3
19979	9472	7	24631	1510	3
19991	16996	5	24691	13564	3
20011	8452	3	24821	12136	5
20161	19316	5	24919	14620	3
20177	7242	13	25013	23492	13.13
20201	6056	5	25117	15544	3
20269	10018	3.3	25153	3406	3
20341	4120	3	*25357	13474	3
20407	3052	3	*25357	17446	3
20411	12052	13	25391	18146	5
*20521	3402	19	25439	19272	7
*20521	6280	3	25523	9822	7
20533	11734	3	26111	3690	7
20551	9106	3.5	26113	298	3
20749	5578	3.13	26171	23706	5
20983	5422	3.13	26251	23336	5
21013	15420	17	26267	8396	23
21061	17946	5	26431	3166	3.5
21067	2278	3	26479	24280	3
21193	610	3	26737	2116	3
21211	1204	3	26801	4056	5
21319	2872	3.11	26953	7906	3
21391	7462	3	26981	18564	19
21649	12134	11	27017	22122	11
*21661	3426	5	27067	21880	3.13
*21661	7738	3	27103	2314	3
21817	18856	3.3	27277	9778	3
21871	17306	5	*27361	1540	3.3.19
21961	14494	3	*27361	10900	3.3
*22051	10086	5	27551	8216	5
*22051	12748	3.7	27581	21296	5
22063	1138	3	27751	2182	3
22291	20008	3	27793	27616	3
22369	20068	3	27823	24214	3

l	2a	d
27967	12832	3
28001	26736	5
28151	27076	5.5
28201	9958	3
28351	23826	5.5
*28393	1804	3
*28393	6178	3
28429	7792	3
28463	3960	107
28549	17740	3.3
28579	17878	3
28627	18820	3
28631	20316	5
*28657	8116	3
*28657	20614	3
28663	16610	17
28771	25916	5
*28789	2020	3
*28789	23782	3
28837	5980	3
28843	1924	3
29017	18718	3
29137	2980	3
29207	28510	17
29327	27072	11
29437	4936	3
*29527	12790	3.7
*29527	22990	3
29803	14434	3
29863	5182	3
29917	10192	3
29947	9916	3
29989	20532	7.7
*30071	23226	5
*30071	28796	5
30169	1972	3.3
30223	6712	3
30553	1822	3
30727	18382	3
30757	15514	3
30817	148	3
30829	22162	3
*31051	25402	3
*31051	27712	3.3
31069	8908	3
31181	26846	5
31183	4822	3
31387	19492	3
31481	3496	5
31627	28680	7
*31687	19642	3
*31687	26698	3
31721	29336	5
31729	5848	3
31741	8434	3
31771	4942	3.3

l	2a	d
32083	2032	3
32119	11284	3
32143	916	3
32191	12922	3
32251	7406	5
32297	28876	11
32321	25716	5
32327	11404	7
*32341	15996	5.7
*32341	20294	7
32369	29360	17
32377	11872	3
32411	606	5
32779	10048	3
32831	6700	7
32839	27508	3
32869	982	3.3
32887	11386	3.3
32957	6140	7
33013	26286	7
33083	19322	139
33181	30586	3.5
33331	16156	3.5
33343	24100	3
33403	17704	3
33427	33226	3
33457	30058	3
33487	28240	3
33493	8248	3
33589	9304	3
33641	30156	5
33757	29958	29
34033	24172	3
34141	20416	3.5
34421	28236	5
*34471	2560	3
*34471	23392	3.3
34483	18052	3
34511	17166	5
*34543	15058	3.3
*34543	24232	3
34651	29698	3
34687	28636	3
34693	33790	3.7
34841	1316	5
35051	29816	5
35227	14920	3
35353	31972	3
35407	12892	3
*35533	15220	3.3
*35533	21502	3.3
35671	2368	3
35729	25070	11
35839	11662	3
35923	22714	3
36007	30982	3

l	2*a*	*d*
27967	12832	3
28001	26736	5
28151	27076	5.5
28201	9958	3
28351	23826	5.5
*28393	1804	3
*28393	6178	3
28429	7792	3
28463	3960	107
28549	17740	3.3
28579	17878	3
28627	18820	3
28631	20316	5
*28657	8116	3
*28657	20614	3
28663	16610	17
28771	25916	5
*28789	2020	3
*28789	23782	3
28837	5980	3
28843	1924	3
29017	18718	3
29137	2980	3
29207	28510	17
29327	27072	11
29437	4936	3
*29527	12790	3.7
*29527	22990	3
29803	14434	3
29863	5182	3
29917	10192	3
29947	9916	3
29989	20532	7.7
*30071	23226	5
*30071	28796	5
30169	1972	3.3
30223	6712	3
30553	1822	3
30727	18382	3
30757	15514	3
30817	148	3
30829	22162	3
*31051	25402	3
*31051	27712	3.3
31069	8908	3
31181	26846	5
31183	4822	3
31387	19492	3
31481	3496	5
31627	28680	7
*31687	19642	3
*31687	26698	3
31721	29336	5
31729	5848	3
31741	8434	3
31771	4942	3.3
32083	2032	3
32119	11284	3
32143	916	3
32191	12922	3
32251	7406	5
32297	28876	11
32321	25716	5
32327	11404	7
*32341	15996	5.7
*32341	20294	7
32369	29360	17
32377	11872	3
32411	606	5
32779	10048	3
32831	6700	7
32839	27508	3
32869	982	3.3
32887	11386	3.3
32957	6140	7
33013	26286	7
33083	19322	139
33181	30586	3.5
33331	16156	3.5
33343	24100	3
33403	17704	3
33427	33226	3
33457	30058	3
33487	28240	3
33493	8248	3
33589	9304	3
33641	30156	5
33757	29958	29
34033	24172	3
34141	20416	3.5
34421	28236	5
*34471	2560	3
*34471	23392	3.3
34483	18052	3
34511	17166	5
*34543	15058	3.3
*34543	24232	3
34651	29698	3
34687	28636	3
34693	33790	3.7
34841	1316	5
35051	29816	5
35227	14920	3
35353	31972	3
35407	12892	3
*35533	15220	3.3
*35533	21502	3.3
35671	2368	3
35729	25070	11
35839	11662	3
35923	22714	3
36007	30982	3

(*Table 1 — continue*)

l	2*a*	*d*
36013	13942	3
36097	14092	3
36107	10900	7
36229	15496	3
36457	5830	3
36479	23050	13
36493	1972	3
36583	23506	3
36607	31366	3
36691	7312	3
36697	2626	3
36761	23586	5
36877	5160	7
36913	25648	3
*37057	30754	3
*37057	31240	3
*37057	34576	3
37087	5542	3
37159	23816	11
*37189	4528	3.3
*37189	13540	3
37363	9328	3
37369	7774	3
37447	29302	3
37483	12898	3
37493	17298	7
37571	26816	5
37591	10876	3.5
37633	20322	7
37811	29046	5
37831	9836	5
37957	10654	3
*38053	1186	3
*38053	11376	7
*38053	25702	3
38113	10144	3
38197	23488	3
38287	16222	3
38299	33520	3
38371	14710	3
38431	22058	7
38671	1522	3
*38677	9520	3
*38677	9964	3
38767	23882	13
38821	19746	5
38891	27426	5
38971	276	5
39079	24574	3
39097	30958	3
39191	30346	5
39301	5866	3.5
39521	4866	5
39541	17470	3
39607	4110	7
39631	32276	5
39679	10978	3
39709	3202	3
39733	23128	3
*40093	22920	13
*40093	27148	3
*40093	34846	3
40177	6736	3
*40351	13828	3
*40351	38830	3
40357	33748	3
40543	27682	3
40591	28642	3
40597	17356	3
*40801	24604	3
*40801	31476	5.5
40867	16918	3
41017	5218	3
41143	31216	3
41203	8772	7
41227	23812	3
41231	16036	5
41233	13528	3
*41389	1978	3
*41389	4192	3
41467	23110	3
*41521	23886	5
*41521	37706	5
41609	18306	7
41617	13954	3
41641	21806	5
41719	10780	3
*41737	11740	3
*41737	25786	3
*41911	7166	5
*41911	16222	3
*41911	23038	3
41953	15220	3.19
42337	28012	3
42349	41260	3
42379	10978	3
42391	11616	5
42457	622	3
42697	3934	3.3
42701	35302	7
42751	1958	19
42859	28588	3
*42961	9868	3
*42961	13174	3
42979	29632	3
43063	39478	3
*43189	9454	3
*43189	14464	3
*43189	26380	3
*43189	35578	3.59
43261	5370	7
43577	6098	13

(*Table 1 — continue*)

l	2a	d
43609	9394	3
43777	37564	3.19
43793	40138	17
43867	26158	3
43891	26126	5.11.19
43963	33868	3
43991	8146	5
44059	43180	3
44129	24340	7
*44201	19046	5.13
*44201	40274	17
44257	37834	3
44269	14428	3.7
44371	35786	5.17
44587	14656	3
44617	15856	3
44621	40056	5
*44701	26494	3
*44701	44556	5
*45139	16972	3
*45139	27484	3
45179	8240	7
45427	21400	3
45497	31230	11
45631	11956	3.5
45697	35098	3
45841	33136	3.5
45893	30844	149
45971	19066	5
46093	3430	3
46171	36694	3.3.3.3.3
46187	31466	7
*46351	7626	5.5
*46351	40684	3
46447	42454	3
46549	24520	3
46681	8548	3
46819	9592	3
46933	7534	3
*47041	9712	3
*47041	32316	5
*47041	39148	3
47059	9758	11
47149	29998	3
47353	35230	3
47581	1480	3
47741	9828	31
47743	4702	3
47881	17824	3
47951	16806	5
47981	1636	5
48271	11884	3
48409	36568	3
*48539	23612	7
*48539	40720	7
48679	43990	3
48817	48178	3.3
*48991	39032	23
*48991	40096	3.5
49057	18572	7
49201	11758	3
49333	22846	3
49451	38158	23
49537	39286	3.3
*49597	17338	3
*49597	17422	3
49681	29812	3
49801	16526	5.5
49843	24130	3.3
49991	2266	5
50047	16348	3
50077	19270	3.3
*50101	4546	3.5
*50101	10948	3
50231	22216	5
50359	44014	3
50497	13486	3
50503	14902	3
50539	9166	3
50723	20238	7
51071	40936	5
51193	26920	3.3.3
51241	31446	5
51307	41002	3
51341	906	5
*51517	4126	3
*51517	48328	3
*51721	24466	3.5
*51721	35338	3
51817	25858	3.17
51973	5974	3
52021	49396	3.5
52177	37834	3
52181	30426	5
52237	45772	3
52249	45172	3.7
52289	42770	19
52321	27936	5
52561	596	5
52571	26006	5.7
52627	38020	3
*53101	7504	3
*53101	25426	3.3.5.5
53197	16182	11
53453	42436	23
53617	24988	3
53681	31904	61
53791	32044	3.11
53857	49754	11
54013	8450	7
54133	31198	3
54217	39148	3

(*Table 1 — continue*)

l	2*a*	*d*
54409	47482	3
54517	6244	3
54631	4288	3
54709	14584	3
54829	41212	3.3
54877	45460	3
55109	28774	23
55217	3792	17
55331	7386	5
55603	33484	3
55639	13678	3
55987	2626	3.7
56099	22688	7
*56131	9898	3
*56131	22648	3
*56131	52312	3
56149	28294	3
56167	3004	3.11
56263	21958	3
56299	16720	3
56311	46174	3
56527	5752	3
56531	15646	5
56569	51364	3
56629	30430	3.3
56809	49564	3.3
56923	16420	3
56989	40888	3.3
57191	20784	7
57241	3156	5
57493	48694	3
57649	47584	3
57709	9286	3
57793	14818	3
*57829	19876	3
*57829	26716	3
58027	5926	3
58057	14212	3
58147	46982	11
58151	41616	5
58153	54106	3
*58231	76	3.5
*58231	41590	3.3
58417	8524	3
*58441	9476	5
*58441	16588	3
58549	4918	3
58573	20908	3.3
58693	44958	67
58711	20344	3
*58741	52	3
*58741	37480	3
58771	44044	3
58787	34120	17
*58831	3892	3
*58831	20608	3
58897	56716	3
58901	32016	5.19
59021	16346	5
59023	1726	3
59119	22576	3
59159	21946	11
59167	45724	3
59341	16108	3
59443	16216	3
59611	4132	3
59743	39652	3
60169	13036	3
60343	15820	3
60457	28128	11
60589	52384	3
60811	17596	3.5
61091	53656	5
61141	46924	3
61231	53278	3
*61261	9716	5
*61261	48636	5
61483	36190	3
61603	45730	3
61837	38446	3
62129	1750	11
62143	45874	3
62171	30776	5
62191	51964	3
62323	56832	17
62743	26824	3
*62827	20392	3
*62827	49828	3
62989	52804	3
63211	15786	5.7
63241	16806	5
*63577	3472	3
*63577	34102	3.3
*63589	16668	7
*63589	25462	3
63617	23990	7
63691	34948	3.11
63799	60286	3
63997	49912	3
64279	3256	3
64489	7522	3
64513	56526	7
64601	45646	5.17
64747	20374	3
64891	7348	3
65119	29998	3
65203	38182	3
65293	26950	3
65323	574	3.191
65419	28798	3
65479	61622	7
65551	25582	3

l	2*a*	*d*
65599	6040	3
65617	52210	3
65651	45406	5
65677	45694	3
65809	46444	3
65929	4156	3
65983	44134	3
66067	48676	3.11
66161	17976	5
66221	20456	5
66271	1946	5
*66403	1716	7
*66403	19954	3.3
66529	53068	3.7
66541	2614	3
66571	8128	3.7
66643	19692	29
*66721	49300	3
*66721	57172	3
*66739	19054	3
*66739	31382	7
66947	24850	11
67021	10390	3
67453	14180	11
*67763	59994	17
*67763	63462	17
67777	916	3
67867	52744	3
67891	36472	3
67927	58978	3
67957	1600	3
68053	52714	3
68059	34114	3
68161	44776	3.5
68239	58564	3.3
68351	3266	5
68581	65338	3
*68659	1168	3
*68659	5086	3
*68683	4660	3
*68683	8788	3
68767	10510	3
69031	66670	3
69073	16462	3
*69151	6670	3
*69151	53536	3.5
*69259	5530	3.97
*69259	65066	7
69337	68248	3.3
69403	9416	269
69493	37786	3
69931	57286	3.3.5
69991	59186	5
70051	44496	5
70099	45028	3
70183	5098	3

l	2a	*d*
70393	60082	3.7
*70489	32932	3.3
*70489	35272	3.3
70573	38386	3
70627	63046	3
70687	31692	11
71011	19646	5
71023	33388	3
71161	33936	5
71167	8050	3
*71191	33226	3.5
*71191	43972	3
*71191	57136	3.5
71233	31438	3.7
71237	6282	11
71257	20428	3
71287	20026	3
71317	22402	3.3
71341	50666	5
71359	37276	3.7
71479	8200	3.3
71569	50828	7
71711	64296	5
71899	60882	23
71941	67330	3
72103	22900	3
72211	16916	5
72223	57976	3
72271	43766	5
72277	45220	3
72493	3724	3
72551	1496	5
72559	52744	3
*72613	10948	3
*72613	65608	3
72643	63862	3
*72817	20422	3
*72817	62596	3
73309	31834	3
73363	29596	3
73483	38374	3
73877	68344	11
73907	66494	7
73939	15550	3
74167	16522	3
74323	48106	3.3
74377	22828	3
74509	11770	3
74527	3268	3
74611	33310	3.3
74707	17434	3
74779	43330	3.11
74797	14368	3
74941	45976	3.5
*75133	38980	3.3
*75133	53572	3

l	*2a*	*d*
75337	68414	43
75401	21308	13
75431	54526	5
*75571	35476	3.5.11
*75571	66154	3
75583	29432	19
75629	15112	73
75641	13546	5
75941	48896	5
*75967	42736	3.11
*75967	53656	3
76207	27676	3
76213	62844	29
76289	72266	149
76387	39876	29
76441	43380	7
76481	53156	5
76651	54258	7
76771	75096	5
76831	33246	5
76963	25096	3
77029	18862	3
77041	53704	3.3
77191	45056	5
77239	25922	7
77291	43756	5
77317	20584	3
77369	58350	19
77509	54958	3
77617	65318	7.7
77647	69244	3
77761	52402	3
78079	15114	7
78259	27274	3
*78301	10056	5
*78301	48814	3
78427	76606	3
78649	49594	3
78721	77336	5
78781	21828	13
78787	77842	3.3.3
78889	23662	3
79031	9710	7
79187	70282	137
79279	62026	3
79423	64450	3.7.31
79537	53260	3
79561	4948	3.17
79613	49102	13
79777	8902	3.3
79843	65404	3
79867	5506	3
80209	68854	3
80221	68546	5
80317	5758	3
80347	52046	7
80627	24558	13
80749	45532	3.3
80803	11452	3
80831	1866	5
80989	54256	3
81001	8230	3
81031	32554	3
81223	59890	3
81343	3118	3
81371	47996	5
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*81463	72484	3
81649	78842	7
81817	6916	3
81847	21352	3
81883	31714	3
81973	30046	3
82021	19366	3.5
82141	32346	5
82153	29362	3
*82189	8986	3
*82189	43576	3
82219	80380	3
82301	7286	5
82339	48118	3
82483	17314	3
82531	34480	3.3
82609	11008	3
82963	38356	3
83059	50056	3
83077	4732	3
83117	51580	11
83401	8602	3
83437	78922	3
83471	20706	5
83653	57760	3
83689	24370	3
83773	19514	13
83777	35400	7
83833	37060	3
83843	12926	11
83911	25706	5
84067	44794	3
84121	29606	5
84131	27946	5
84349	82024	3
84391	52294	3
84421	51136	3.5.7
84431	45676	5
84449	37818	13
84457	37384	3.17
84481	70500	11
84589	83240	19
84691	23998	3
84793	78154	3
*85021	2756	5

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85081	25126	3.5
85201	31924	3
85333	26602	3
85451	69586	5
85667	42106	7
*85669	16226	11.59
*85669	55288	3
85819	3250	3
85931	59706	5
86011	26496	5
86017	37144	3
86029	81994	3
86171	5866	5
86257	48730	3
86291	79206	5
*86293	8508	47
*86293	71710	3
*86293	73756	3.3
86491	5740	3
86501	27766	5
86767	38662	3
86843	41084	7
*86861	5196	5
*86861	42476	5
86929	16864	3
86981	51706	5
*87313	11236	3.107
*87313	48676	3
87403	74404	3.7
87421	47872	3
87491	78906	5
87541	34804	3
87547	41164	3
87557	960	7
87679	34516	3.3
87767	65416	7
87877	66808	3.3
87911	51286	5
88513	76852	3
88607	12468	7
88609	39974	71
88661	45748	13
88741	15430	3
88793	77430	11
88811	73086	5
88817	48224	7
88969	57490	3
*89071	366	5
*89071	13646	5
89317	34750	3.3.3
89443	11968	3
89491	11320	3
89527	26824	3
89783	39260	11
89833	37628	197

l	2a	d
90127	17380	3.3
90203	5006	7
90247	24772	3
90481	59200	3
90679	57844	3
90709	12184	3
90841	44836	3.5
90847	3382	3.7.7
90997	46342	3
91141	84600	31
91159	20230	3
91183	77020	3
91243	50872	3
91291	2524	3
91297	55366	3
91387	16702	3
91393	10924	3
91801	80164	3.3.3
91837	75796	3
91873	31438	3.3
*91909	12946	3
*91909	40030	3
91969	22210	3
91997	10356	109
92179	55912	3
92203	81772	3
92251	44218	3.3
92347	38218	3
92353	71368	3
92387	22982	7
92431	66296	5
92551	89166	5
*92557	8734	3
*92557	41752	3.3
92581	76616	5
92681	5692	7
92707	9760	3
*92821	9612	7
*92821	77546	5.13
*93103	15670	3
*93103	89268	59
93133	31060	3.3
93169	28984	3
93187	20188	3.3
93251	73236	5
93283	25072	3
93329	5758	19
93463	34816	3
93523	47136	11
93761	4406	5
93851	67296	5
*93901	2842	3
*93901	37120	3
93937	84808	3
94063	61900	3
94153	38284	3

(*Table 1 — continue*)

l	2a	d
94201	74140	3
94291	36338	7
*94351	38440	3
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94541	1266	5
94561	84244	3
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*94573	93778	3
*94693	11636	13
*94693	54754	3
*94693	80650	3
94999	59824	3
95131	9250	3
95143	14914	3
95317	67094	13.13
95467	75742	3
95581	61616	5
95857	5950	3
95873	5104	7
95929	20302	3
*95971	4766	5
*95971	78906	5
96149	74036	13
96181	87996	5
96337	10036	3.3.223
96451	49078	3
96757	74908	3
97039	74476	3.3
97081	8092	3
97231	24364	3
97387	92446	3
97441	4296	5
*97459	29422	3
*97459	86962	3
97501	60712	3
97579	88648	3.13
97687	16216	3
97711	49978	3
*97813	44416	3.3
*97813	52660	3.3
97843	4966	3
97849	86950	3.3
97967	44804	11
98081	81916	5
98101	68136	5
98347	29560	3
98389	96808	3
98563	82360	3
98713	75928	3
98737	17986	3.11
99133	45130	3
99139	47650	3
99181	31466	5.29
99191	62926	5
99259	63850	3
99371	20126	5

l	2a	d
99401	58386	5
99409	56938	3
99439	82900	3
99529	35392	3
99709	77554	3.7
99793	67586	7
99871	11032	3
*100069	24466	3
*100069	75486	31
100129	64226	7
100153	90556	3
100517	97410	13
100669	81622	3
*100987	11626	3
*100987	65872	3
101063	5006	13
101149	40204	3
101173	27988	3
101287	30166	3
101323	82336	3
*101341	40736	5
*101341	99748	3.3
101627	69826	7.7
101719	42766	3
101771	926	5
101869	64450	3
*102031	48022	3
*102031	91156	3.5
102061	4196	5
102077	9582	13
102103	69694	3.13
102121	84376	3.5
102181	43774	3
102251	4506	5
*102301	74876	5.5
*102301	84130	3
102451	97630	3
102461	11234	47
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