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# ON THE ASYMPTOTIC BEHAVIOUR OF THE EQUATION $\frac{dz}{dt} = f(t, z)$ WITH A COMPLEX-VALUED FUNCTION $f$

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## 1. INTRODUCTION

This paper deals with the asymptotic properties of the solutions of an equation

$$\dot{z} = f(t, z), \qquad \dot{} = \frac{d}{dt}, \tag{1.1}$$

where $f$ is a continuous complex-valued function of a real variable $t$ and a complex variable $z$. Some results concerning the asymptotic behaviour of the solutions of (1.1) are obtained in [2]. The principial tool used in this paper is the technique of Liapunov-like functions.

The approach of the present paper is based on the same method. It is convenient to write the equation (1.1) in the form

$$\dot{z} = G(t, z)\,[h(z) + g(t, z)], \tag{1.2}$$

where $G$ is a real-valued function and $g, h$ are complex-valued functions. We shall assume that the function $h$ is holomorphic and that the right-hand side of (1.2) is in a suitable meaning "close" to this function.

The organization of the paper is as follows: In Section 2 we give our fundamental results concerning the asymptotic behaviour of the solutions of (1.2). In Section 3 we attempt to generalize some results of [3], [4] applying the results of Section 2 to the equation

$$\dot{z} = q(t, z) - p(t)\, z^2.$$

The proof of Theorem 2.3 is based on the well-known Ważewski principle. For the reader's convenience we shall quote in the Appendix some fundamental notions and basic results of this theory; for more details we refer, for example, to [1].

Throughout the paper we use the following notation:

| | |
|---|---|
| $\boldsymbol{C}$ | – Set of all complex numbers |
| $\boldsymbol{N}$ | – Set of all positive integers |
| $\mathrm{Re}\, b$ | – Real part of a complex number $b$ |
| $\mathrm{Im}\, b$ | – Imaginary part of a complex number $b$ |
| $\bar{b}$ | – Conjugate of $b$ |
| $\lvert b \rvert$ | – Absolute value of $b$ |
| $\mathrm{Bd}\, \Gamma$ | – Boundary of a set $\Gamma \subset \boldsymbol{C}$ |
| $\mathrm{Cl}\, \Gamma$ | – Closure of a set $\Gamma \subset \boldsymbol{C}$ |
| $\mathrm{Int}\, \Gamma$ | – Interior of a Jordan curve $z = z(t)$, $t \in [\alpha, \beta]$ whose points $z$ form a set $\Gamma$; $\Gamma$ will be called the *geometric image* of the Jordan curve $z = z(t)$, $t \in [\alpha, \beta]$ |
| $I$ | – Interval $[t_0, \infty)$ |
| $\Omega$ | – Simply connected region in $\boldsymbol{C}$ such that $0 \in \Omega$ |
| $C[\alpha, \infty)$ | – Class of all continuous real-valued functions defined on the interval $[\alpha, \infty)$ |
| $C(\Gamma)$ | – Class of all continuous real-valued functions defined on the set $\Gamma$ |
| $\tilde{C}(\Gamma)$ | – Class of all continuous complex-valued functions defined on the set $\Gamma$ |
| $\mathscr{H}(\Gamma)$ | – Class of all complex-valued functions defined and holomorphic in the region $\Gamma \subset \boldsymbol{C}$ |
| $\mathrm{D}_f U(t, z)$ | – Trajectory derivative of a function $U(t, z)$ for the equation $\dot{z} = f(t, z)$; this derivative is defined by the relation |

$$\mathrm{D}_f U(t, z) = \frac{\partial U(t, z)}{\partial t} + \frac{\partial U(t, z)}{\partial \,\mathrm{Re}\, z} \mathrm{Re}\, f(t, z) + \frac{\partial U(t, z)}{\partial \,\mathrm{Im}\, z} \mathrm{Im}\, f(t, z).$$

Suppose that $h(z) \in \mathscr{H}(\Omega)$ is a function such that $h'(0) \neq 0$ and $h(z) = 0 \Leftrightarrow z = 0$. Following [2] we define

$$r(z) = \begin{cases} \dfrac{zh'(0) - h(z)}{zh(z)} & \text{for } z \in \Omega,\ z \neq 0, \\ -\dfrac{h''(0)}{2h'(0)} & \text{for } z = 0, \end{cases}$$

$$w(z) = z \exp\left[\int_0^z r(z^*)\, \mathrm{d}z^*\right]$$

and

$$W(z) = \lvert w(z) \rvert.$$

All of these functions are well-defined on $\Omega$. Let $\Xi$ be the system of all simply connected regions $\Gamma \subset \Omega$ with the property $0 \in \Omega$. For any $\Gamma \in \Xi$ put

$$\lambda_0^\Gamma = \liminf_{M \to \infty} \inf_{z \in \Gamma_M} W(z),$$

where

$$\Gamma_M = \{z \in \Gamma : \inf_{z^* \in \mathrm{Bd}\,\Gamma} |z - z^*| < M^{-1}\} \cup \{z \in \Gamma : |z| > M\}.$$

Denote

$$\lambda_0 = \sup_{\Gamma \in \Xi} \lambda_0^\Gamma.$$

Clearly $0 < \lambda_0 \leqq \infty$.

For $0 < \lambda < \lambda_0$ define the sets $\hat{K}(\lambda) \subset \Omega$ in the following way: choose $\Gamma \in \Xi$ so that $\lambda_0^\Gamma > \lambda$ and put

$$\hat{K}(\lambda) = \{z \in \Gamma : W(z) = \lambda\}.$$

According to [2] this definition is correct, and, denoting

$$\hat{K}(0) = \{0\},$$

$$K(\lambda) = \bigcup_{0 \leqq \mu < \lambda} \hat{K}(\mu) \qquad \text{for } 0 < \lambda \leqq \lambda_0,$$

$$K(\lambda_1, \lambda_2) = \bigcup_{\lambda_1 < \mu < \lambda_2} \hat{K}(\mu) \qquad \text{for } 0 \leqq \lambda_1 < \lambda_2 \leqq \lambda_0,$$

we have the following statement:

**Theorem 1.1.** *$K = K(\lambda_0)$ is a simply connected region and $\lambda_0^K = \lambda_0$. Every set $\hat{K}(\lambda)$, where $0 < \lambda < \lambda_0$, is the geometric image of a certain Jordan curve, and,*

$$\hat{K}(\lambda) = \{z \in K(\lambda_0) : W(z) = \lambda\},$$

$$\mathrm{Int}\,\hat{K}(\lambda) = \{z \in K(\lambda_0) : W(z) < \lambda\}.$$

*Moreover,*

$$K(\lambda) = \mathrm{Int}\,\hat{K}(\lambda) \qquad \textit{for } 0 < \lambda < \lambda_0,$$

$$K(\lambda_1, \lambda_2) = K(\lambda_2) - \mathrm{Cl}\,K(\lambda_1) \qquad \textit{for } 0 < \lambda_1 < \lambda_2 \leqq \lambda_0,$$

*and*

$$K(0, \lambda) = K(\lambda) - \{0\} \qquad \textit{for } 0 < \lambda \leqq \lambda_0.$$

## 2. MAIN RESULTS

Consider the equation

$$\dot{z} = G(t, z)\,[h(z) + g(t, z)], \tag{2.1}$$

where $G(t, z)\,[h(z) + g(t, z)] \in \tilde{C}(I \times \Omega)$, $G \in C(I \times (\Omega - \{0\}))$, $g \in \tilde{C}(I \times (\Omega - \{0\}))$, $h \in \mathscr{H}(\Omega)$. Assume that $h'(0) \neq 0$ and $h(z) = 0 \Leftrightarrow z = 0$. Let $W(z)$, $\lambda_0$, $\hat{K}(\lambda)$, $K(\lambda)$, $K(\lambda_1, \lambda_2)$ be defined as before. The number $\lambda_0$ and the numbers $\vartheta \leqq \lambda_0$ ($\vartheta_n \leqq \lambda_0$) in the present section may take the value $\infty$.

**Theorem 2.1.** *Assume* $0 < \gamma < \lambda_0$. *Suppose that*

$$G(t, z) > 0 \tag{2.2}$$

*and*

$$\operatorname{Re}\left[g(t, z)\frac{h'(0)}{h(z)}\right] < -\operatorname{Re} h'(0) \tag{2.3}$$

*hold for* $t \geqq t_0$, $z \in \hat{K}(\gamma)$.

*If a solution* $z(t)$ *of* (2.1) *satisfies*

$$z(t_1) \in \mathrm{Cl}\, K(\gamma), \tag{2.4}$$

*where* $t_1 \geqq t_0$, *then* $z(t) \in K(\gamma)$ *for* $t > t_1$.

Proof. Let $z = z(t)$ be any solution of (2.1) satisfying (2.4). Put $\mathscr{M} = \{t \geqq t_1 : z(t) \in K(0, \lambda_0)\}$. For any $t \in \mathscr{M}$ we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} W^2(z) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[w(z)\overline{w(z)}\right] = \\ &= 2\operatorname{Re}\left[w'(z)\overline{w(z)}\,\dot{z}\right] = \\ &= 2\operatorname{Re}\left\{w(z)\overline{w(z)}\left[z^{-1} + r(z)\right]\dot{z}\right\} = \\ &= 2\, W^2(z)\operatorname{Re}\left[h'(0)\, h^{-1}(z)\,\dot{z}\right],\end{aligned}$$

where $z = z(t)$. Hence

$$\begin{aligned}\dot{W}(z) &= W(z)\operatorname{Re}\left[h'(0)\, h^{-1}(z)\,\dot{z}\right] = \\ &= G(t, z)\, W(z)\operatorname{Re}\left\{h'(0)\, h^{-1}(z)\left[h(z) + g(t, z)\right]\right\} = \\ &= G(t, z)\, W(z)\operatorname{Re}\left\{h'(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\}\end{aligned}$$

for $t \in \mathscr{M}$. If there is a $t_2 \geqq t_1$ such that $z(t_2) \in \hat{K}(\gamma)$, then (2.2) and (2.3) imply

$$\dot{W}(z(t_2)) < 0. \tag{2.5}$$

Suppose that there exists a $t^* > t_1$ for which $z(t^*) \notin K(\gamma)$. Define $t_3 = \inf\{t^* > t_1 : z(t^*) \notin K(\gamma)\}$. In view of (2.5) we have $t_3 > t_1$. Furthermore $z(t_3) \in \hat{K}(\gamma)$, and $z(t) \in K(\gamma)$ holds for $t \in (t_1, t_3)$. But on account of (2.5) we know that there is a $t_4 \in (t_1, t_3)$ such that $W(z(t_4)) > \gamma$. Thus our supposition is impossible and $z(t) \in K(\gamma)$ for $t > t_1$.

The proof of the following theorem is analogous to that of Theorem 2.1.

**Theorem 2.2.** *Assume* $0 < \gamma < \lambda_0$. *Suppose that* (2.2) *and*

$$-\operatorname{Re}\left[g(t, z)\frac{h'(0)}{h(z)}\right] < \operatorname{Re} h'(0) \tag{2.6}$$

*hold for* $t \geqq t_0$, $z \in \hat{K}(\gamma)$.

*If a solution* $z(t)$ *of* (2.1) *satisfies*

$$z(t_1) \notin K(\gamma),$$

*where* $t_1 \geqq t_0$, *then*

$$z(t) \notin \mathrm{Cl}\, K(\gamma)$$

*for all* $t > t_1$ *for which* $z(t)$ *is defined.*

It is clear that if the hypotheses of Theorem 2.1 are fulfilled, then (2.1) possesses a bounded solution. The following theorem establishes the existence of a bounded solution of (2.1) on the assumptions of Theorem 2.2.

**Theorem 2.3.** *Let the assumptions of Theorem* 2.2 *be satisfied. Then for any* $t_1 > t_0$ *there exists a solution* $z(t)$ *of* (2.1) *satisfying*

$$z(t) \in K(\gamma) \tag{2.7}$$

*for* $t \geqq t_1$.

Proof. Choose $t_1 > t_0$. Put

$$U(t, z) = W^2(z) - \gamma^2,$$

$$V(t, z) = \frac{1}{2}(t_0 + t_1) - t,$$

$$\Omega^0 = \left\{(t, z) : z \in K(\lambda_0),\ W(z) < \gamma,\ t > \frac{1}{2}(t_0 + t_1)\right\},$$

$$\mathscr{U} = \left\{(t, z) : z \in K(\lambda_0),\ W(z) = \gamma,\ t \geqq \frac{1}{2}(t_0 + t_1)\right\},$$

$$\mathscr{V} = \left\{(t, z) : z \in K(\lambda_0),\ W(z) \leqq \gamma,\ t = \frac{1}{2}(t_0 + t_1)\right\}.$$

Denoting $f(t, z) = G(t, z)\,[h(z) + g(t, z)]$, we have

$$\mathrm{D}_f U(t, z) = 2\,\mathrm{Re}\,[w'(z)\,\overline{w(z)}\,f(t, z)] =$$
$$= 2G(t, z)\,W^2(z)\,\mathrm{Re}\,\{h'(0)\,h^{-1}(z)\,[h(z) + g(t, z)]\} =$$
$$= 2\gamma^2 G(t, z)\left\{\mathrm{Re}\,h'(0) + \mathrm{Re}\left[\frac{h'(0)}{h(z)}\,g(t, z)\right]\right\} > 0$$

for $(t, z) \in \mathscr{U}$. Further,

$$\mathrm{D}_f V(t, z) = -1 < 0 \qquad \text{for } (t, z) \in \mathscr{V}.$$

Thus $\Omega^0$ is a $(U, V)$-subset with respect to (2.1). Using the first part of the Ważewski theorem (see Appendix) we infer that the set of all egress points of $\Omega^0$ is

$$\Omega_e^0 = \left\{(t, z) : z \in K(\lambda_0),\ W(z) = \gamma,\ t > \frac{1}{2}(t_0 + t_1)\right\}.$$

Put

$$\Xi = \{(t_1, z) : z \in K(\lambda_0),\ W(z) \leqq \gamma\}.$$

The set

$$\Xi \cap \Omega_e^0 = \{(t_1, z) : z \in K(\lambda_0),\ W(z) = \gamma\}$$

is a retract of $\Omega_e^0$, as it can be seen by choosing the retraction $(t, z) \mapsto (t_1, z)$. Next we shall show that $\Xi \cap \Omega_e^0$ is not a retract of $\Xi$. Suppose on the contrary that there is a retraction $p_1 : \Xi \to \Xi \cap \Omega_e^0$. Because of the Riemann theorem we can find a conformal mapping of $K(\gamma)$ onto $\{z : |z| < 1\}$. Since Bd $K(\gamma) = \hat{K}(\gamma)$ is the geometric image of a Jordan curve, there exists a homeomorphism $p_2$ of Cl $K(\gamma)$ onto $\{z : |z| \leqq \leqq 1\}$ which is an extension of this mapping. Let $p_3 : \mathrm{Cl}\, K(\gamma) \to \Xi$ be defined by $z \mapsto (t_1, z)$. The composite mapping $v(z) = p_2(p_3^{-1}(p_1(p_3(p_2^{-1}(z)))))$ is a retraction of $\{z : |z| \leqq 1\}$ onto $\{z : |z| = 1\}$. Clearly, $-v$ is a continuous map of $\{z : |z| \leqq 1\}$ into itself without fixed points, which is impossible by the fixed point theorem of Brouwer. Therefore $\Xi \cap \Omega_e^0$ is not a retract of $\Xi$. Using the Ważewski theorem we infer that there exists a solution $z(t)$ of (2.1) such that (2.7) holds for $t \geqq t_1$.

Now, we recall one result of [2], Theorem 2.5:

**Theorem 2.4.** *Assume* $\delta > 0$, $\vartheta_n \leqq \lambda_0$, $s_n \geqq t_0$ *for* $n \in N$. *Suppose there are functions* $E_n(t) \in C[t_0, \infty)$ *such that:*

(i) *for* $n \in N$ *there are fulfilled the conditions*

$$\int_{t_0}^{\infty} E_n(s)\, \mathrm{d}s = -\infty,$$

$$\sup_{s_n \leqq s \leqq t < \infty} \int_s^t E_n(\xi)\, \mathrm{d}\xi = \varkappa_n < \infty,$$

$$\delta e^{\varkappa_n} < \vartheta_n;$$

(ii) *the inequality*

$$-G(t, z)\, \mathrm{Re}\left\{h'(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq E_n(t)$$

*holds for* $t \geqq s_n$, $z \in K(\delta, \vartheta_n)$, $n \in N$.

*Denote*

$$\vartheta = \sup_{n \in N} [\vartheta_n e^{-\varkappa_n}].$$

*If a solution* $z(t)$ *of* (2.1) *satisfies*

$$z(t_1) \in K(\delta e^{\varkappa_1}, \lambda_0),$$

*where* $t_1 \geqq s_1$, *then to any* $\varepsilon, 0 < \varepsilon < \vartheta$, *there exists a* $T = T(\varepsilon, t_1) > 0$ *independent of* $z(t)$ *such that*

$$z(t) \notin \mathrm{Cl}\, K(\varepsilon)$$

*for all* $t \geqq t_1 + T$ *for which* $z(t)$ *is defined.*

Using Theorems 2.3 and 2.4, we can prove the following

**Theorem 2.5.** *Let* $\beta_n < 1$, $0 \leqq \delta_n < \vartheta_n \leqq \lambda_0$, $s_n \geqq t_0$ *hold for* $n \in N$. *Assume* $\operatorname{Re} h'(0) > 0$,

$$\lim_{n\to\infty} \delta_n = \delta < \vartheta = \lim_{n\to\infty} \vartheta_n.$$

*Suppose that*

(i) *there are nonnegative functions* $D_n(t) \in C[t_0, \infty)$ *such that*

$$\int_{t_0}^{\infty} D_n(t)\, dt = \infty \tag{2.8}$$

*and*

$$G(t, z) \geqq D_n(t) \tag{2.9}$$

*for* $t \geqq s_n$, $z \in K(\delta_n, \vartheta_n)$, $n \in N$;

(ii) *the inequality*

$$-\operatorname{Re}\left[g(t, z)\frac{h'(0)}{h(z)}\right] \leqq \beta_n \operatorname{Re} h'(0) \tag{2.10}$$

*holds for* $t \geqq s_n$, $z \in K(\delta_n, \vartheta_n)$, $n \in N$;

(iii) *there is a* $\gamma$, $\delta < \gamma < \vartheta$ *such that*

$$G(t, z) > 0 \tag{2.2}$$

*for* $t \geqq t_0$, $z \in \hat{K}(\gamma)$.

*Then there exists a solution* $z(t)$ *of* (2.1) *with the property that to any* $\varepsilon$, $\delta < \varepsilon < \lambda_0$, *a* $t_1 = t_1(\varepsilon) > t_0$ *can be found such that*

$$z(t) \in K(\varepsilon)$$

*for* $t \geqq t_1$.

Proof. Without loss of generality it may be assumed that $\delta_n > 0$ for $n \in N$. Pick $N \in N$ such that $\delta_N < \gamma < \vartheta_N$. For $t \geqq s_N$, $z \in \hat{K}(\gamma)$ we have

$$-\operatorname{Re}\left[g(t, z)\frac{h'(0)}{h(z)}\right] \leqq \beta_N \operatorname{Re} h'(0) < \operatorname{Re} h'(0).$$

By Theorem 2.3 there exists a solution $z(t)$ of (2.1) satisfying

$$z(t) \in K(\gamma) \tag{2.7}$$

for $t \geqq s_N + 1$.

Putting $E_n(t) = (\beta_n - 1)\, D_n(t) \operatorname{Re} h'(0)$, we obtain

$$-G(t, z) \operatorname{Re}\left\{h'(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq E_n(t)$$

for $t \geqq s_n$, $z \in K(\delta_n, \vartheta_n)$, $n \in N$. Choose $\varepsilon$, $\delta < \varepsilon < \gamma$. Let $n$ be a positive integer such that $\delta_n < \varepsilon < \gamma < \vartheta_n$. Denote $t_1 = t_1(\varepsilon) = \max[s_N + 1, s_n]$. We claim $z(t) \in K(\varepsilon)$

for $t \geqq t_1$. Suppose for the sake of argument that there is a $t_2 \geqq t_1$ for which $z(t_2) \in K(\varepsilon, \vartheta_n)$. Using Theorem 2.4 we infer that there exists a $t_3 \geqq t_2$ such that $z(t_3) \notin K(\gamma)$. Since it contradics (2.7), it follows that $z(t) \in K(\varepsilon)$ for $t \geqq t_1$.

## 3. APPLICATION TO THE EQUATION $\dot{z} = q(t, z) - p(t) z^2$

In this section we propose to establish certain results concerning the asymptotic behaviour of the equation

$$\dot{z} = q(t, z) - p(t) z^2, \tag{3.1}$$

where $p \in \tilde{C}(I)$, $q \in \tilde{C}(I \times C)$. Some results of this type are given in [2]. The special case of (3.1) is studied in [3], [4], where M. Ráb has obtained results describing the asymptotic properties of the Riccati differential equation

$$\dot{z} = q(t) - p(t) z^2$$

with complex-valued coefficients $p$, $q$.

If $a, b \in C$, $\operatorname{Re}[(a - b) p(t)] > 0$, then (3.1) can be written in the form

$$\begin{aligned} \dot{z} = \frac{\operatorname{Re}[(a-b)p(t)]}{|a-b|^2} \Bigg[ (\bar{b} - \bar{a})(z-a)(z-b) + \frac{|a-b|^2 q(t,z)}{\operatorname{Re}[(a-b)p(t)]} - \\ - \frac{|a-b|^2 p(t)}{\operatorname{Re}[(a-b)p(t)]} z^2 + (\bar{a} - \bar{b})(z-a)(z-b) \Bigg]. \end{aligned} \tag{3.2}$$

Denote $c = a - b$. Substituting $z_1 = z - b$, we get

$$\dot{z}_1 = G(t, z_1)[h(z_1) + g(t, z_1)], \tag{3.2$_1$}$$

where

$$G(t, z_1) = \frac{\operatorname{Re}[cp(t)]}{|c|^2}, \qquad h(z_1) = -\bar{c} z_1(z_1 - c),$$

$$g(t, z_1) = \frac{|c|^2 q(t, z_1 + b)}{\operatorname{Re}[cp(t)]} - \frac{|c|^2 p(t)}{\operatorname{Re}[cp(t)]}(z_1 + b)^2 + \bar{c} z_1(z_1 - c).$$

Put

$$\Omega = \{z_1 : 2 \operatorname{Re}[\bar{c} z_1] < |c|^2\}$$

and consider the equation (3.2$_1$) on the set $I \times \Omega$. We observe that $W(z_1) = |c| \, |z_1| \, |z_1 - c|^{-1}$, $\lambda_0 = |c|$ and $K(\lambda_0) = \Omega$. Moreover, we have

$$\hat{K}(\lambda) = \{z_1 \in \Omega : |c| \, |z_1| = \lambda |z_1 - c|\}$$

for $0 \leqq \lambda < \lambda_0$. Notice that

$$(3.3) \qquad \left| z_1 - \frac{c}{2} \right| > \frac{1}{2} |c| \frac{|c| - \lambda}{|c| + \lambda}$$

for $z_1 \in K(\lambda)$, where $0 < \lambda \leqq \lambda_0$.

Suppose that there is an $H(t) \in C[t_0, \infty)$ such that

$$| q(t, z_1 + b) + abp(t) - (a + b) p(t) (z_1 + b) | \leqq H(t)$$

for $t \geqq t_0$, $z_1 \in \Omega$.

1° Assume that

$$(3.4) \qquad \operatorname{Re}[cp(t)] > 0 \qquad \text{for } t \geqq t_0$$

and

$$(3.5) \qquad \sup_{t \geqq t_0} \frac{H(t)}{\operatorname{Re}[cp(t)]} < \frac{1}{4} |c|.$$

If $\delta \leqq |c|$ is defined by

$$(3.6) \qquad \sup_{t \geqq t_0} \frac{H(t)}{\operatorname{Re}[cp(t)]} = \frac{\delta |c|^2}{2(|c|^2 + \delta^2)},$$

then $0 \leqq \delta < |c| = \lambda_0$. Notice that the function

$$\varphi(s) = \frac{s}{|c|^2 + s^2}$$

is increasing in $[0, |c|)$. Thus we have

$$-\operatorname{Re}\left[ g(t, z_1) \frac{h'(0)}{h(z_1)} \right] =$$

$$= \frac{|c|^2}{\operatorname{Re}[cp(t)]} \operatorname{Re}\left\{ [q(t, z_1 + b) + abp(t) - (a + b) p(t) (z_1 + b)] \frac{c}{z_1(z_1 - c)} \right\} \leqq$$

$$\leqq \frac{H(t)}{\operatorname{Re}[cp(t)]} \frac{|c|^3}{|z_1| \, |z_1 - c|} \leqq \frac{\delta |c|^2}{2(|c|^2 + \delta^2)} \frac{|c|^3}{|z_1| \, |z_1 - c|} \leqq$$

$$\leqq \frac{W(z_1)}{2[|c|^2 + W^2(z_1)]} \frac{|c|^5}{|z_1| \, |z_1 - c|} \leqq$$

$$\leqq \frac{1}{2} |c|^5 \frac{|c| \, |z_1|}{|z_1 - c|} \left[ |c|^2 + \frac{|c|^2 \, |z_1|^2}{|z_1 - c|^2} \right]^{-1} \frac{1}{|z_1| \, |z_1 - c|} \leqq$$

$$\leqq \frac{1}{2} |c|^4 \frac{1}{|z_1 - c|^2 + |z_1|^2} \leqq \frac{1}{4} |c|^4 \left[ \left| z_1 - \frac{c}{2} \right|^2 + \left| \frac{c}{2} \right|^2 \right]^{-1}$$

for $t \geqq t_0$ and $z_1 \in K(\delta, \vartheta)$, where $\delta < \vartheta < \lambda_0$. Hence using (3.3), we get

$$-\operatorname{Re}\left[g(t, z_1)\frac{h'(0)}{h(z_1)}\right] \leqq \frac{1}{4}|c|^4\left[\frac{1}{4}|c|^2\left(\frac{|c|-\vartheta}{|c|+\vartheta}\right)^2 + \left|\frac{c}{2}\right|^2\right]^{-1} \leqq$$
$$\leqq |c|^2 \frac{(|c|+\vartheta)^2}{2(|c|^2+\vartheta^2)} < |c|^2 = \operatorname{Re} h'(0).$$

Using Theorem 2.3 we obtain the following statement: *To any $\gamma$, $\delta < \gamma < |c|$, and to any $T > t_0$, there is a solution $z_1(t)$ of* (3.2$_1$) *such that*

$$|c|\,|z_1(t)| < \gamma\,|z_1(t) - c|$$

*for $t \geqq T$.*

2° Suppose (3.4),

(3.7) $$\int_{t_0}^{\infty} \operatorname{Re}[cp(t)]\,dt = \infty$$

and

(3.8) $$\lim_{t\to\infty} \frac{H(t)}{\operatorname{Re}[cp(t)]} = 0.$$

Put

$$\delta_n = \frac{|c|}{n+1}, \qquad n \in \boldsymbol{N}.$$

For $n \in \boldsymbol{N}$ choose $s_n \geqq t_0$ so that

$$\sup_{t \geqq s_n} \frac{H(t)}{\operatorname{Re}[cp(t)]} \leqq \frac{(n+1)|c|}{2[(n+1)^2+1]} \qquad \left(= \frac{\delta_n |c|^2}{2(|c|^2+\delta_n^2)}\right).$$

Then for $t \geqq s_n$, $z_1 \in K(\delta_n, \vartheta)$, $n \in \boldsymbol{N}$, where $\frac{|c|}{2} < \vartheta < |c| = \lambda_0$, the inequality

$$-\operatorname{Re}\left[g(t, z_1)\frac{h'(0)}{h(z_1)}\right] \leqq |c|^2 \frac{(|c|+\vartheta)^2}{2(|c|^2+\vartheta^2)}$$

holds again. Applying Theorem 2.5 with $\vartheta_n = \vartheta$ and

$$D_n(t) = \frac{\operatorname{Re}[cp(t)]}{|c|^2}, \qquad \beta_n = \frac{1}{2}(|c|+\vartheta)^2(|c|^2+\vartheta^2)^{-1},$$

we get: *There exists a solution $z_1(t)$ of* (3.2$_1$) *such that*

$$\lim_{t\to\infty} z_1(t) = 0.$$

By using 1° we obtain the following generalization of Theorem 2 of [3]:

**Theorem 3.1.** *Assume that there are $a, b \in \boldsymbol{C}$ and $H(t) \in C[t_0, \infty)$ such that*

(3.9) $$| q(t, z) + abp(t) - (a + b) p(t) z | \leqq H(t) \qquad \textit{for } t \geqq t_0, z \in C,$$

(3.10) $$\operatorname{Re}[(a - b) p(t)] > 0 \qquad \textit{for } t \geqq t_0$$

*and*

(3.11) $$\sup_{t \geqq t_0} \frac{H(t)}{\operatorname{Re}[(a - b) p(t)]} < \frac{1}{4} | a - b |.$$

*Define $\delta \in [0, 1)$ by the relation*

(3.12) $$\sup_{t \geqq t_0} \frac{H(t)}{\operatorname{Re}[(a - b) p(t)]} = \frac{\delta | a - b |}{2(1 + \delta^2)}.$$

*Let $\gamma$ be any real number satisfying $\delta < \gamma < 1$. Then to every $T > t_0$ there is a solution $z(t)$ of (3.1) such that*

$$| z(t) - b | < \gamma | z(t) - a |$$

*for all $t \geqq T$.*

Combining Theorem 3.2 of [2] with 2°, we can generalize Theorem 3 of [3]:

**Theorem 3.2.** *Suppose there are $a, b \in C$ and $H(t) \in C[t_0, \infty)$ such that there hold* (3.9), (3.10),

(3.13) $$\int_{t_0}^{\infty} \operatorname{Re}[(a - b) p(t)] \, dt = \infty$$

*and*

(3.14) $$\lim_{t \to \infty} \frac{H(t)}{\operatorname{Re}[(a - b) p(t)]} = 0.$$

*Then there exists at least one solution $z_0(t)$ of (3.1) for which*

$$\lim_{t \to \infty} z_0(t) = b.$$

*Let $T \geqq t_0$ be such that*

$$\sup_{t \geqq T} \frac{H(t)}{\operatorname{Re}[(a - b) p(t)]} < \frac{1}{4} | a - b |.$$

*Then any solution $z(t)$ of (3.1) satisfying* $\operatorname{Re}[(\bar{a} - \bar{b})(2z(t_1) - a - b)] \geqq 0$, *where $t_1 \geqq T$, is defined for all $t \geqq t_1$ and*

$$\lim_{t \to \infty} z(t) = a.$$

## 4. APPENDIX

Here we recall, for the reader's convenience, some fundamental notions and basic results of the theory of Ważewski; for more details we refer, for example, to [1, pp. 278–283]. In what follows we assume $f \in \tilde{C}(I \times \Omega)$.

Let $\Gamma_1$ be a topological space, $\Gamma_2 \subset \Gamma_1$. A continuous mapping $\psi$ of $\Gamma_1$ onto $\Gamma_2$ is called a *retraction* of $\Gamma_1$ onto $\Gamma_2$, if the restriction of $\psi$ to $\Gamma_2$ is the identity mapping. The set $\Gamma_2$ is said to be a *retract* of $\Gamma_1$, if there exists a retraction of $\Gamma_1$ onto $\Gamma_2$.

An open subset $\Omega^\circ$ of $I \times \Omega$ is called a *(U, V)-subset* with respect to

$$\dot{z} = f(t, z), \tag{4.1}$$

if there exists a number of real-valued functions $U_1(t, z), \ldots, U_n(t, z)$; $V_1(t, z), \ldots, V_m(t, z)$ defined on $I \times \Omega$ which are of the class $C^1$ (with respect to $t$, $\operatorname{Re} z$, $\operatorname{Im} z$) such that

$$\Omega^0 = \{(t, z) : U_j(t, z) < 0 \text{ and } V_k(t, z) < 0 \text{ for all } j, k\}$$

and

$$\begin{aligned} D_f U_\alpha(t, z) > 0 \quad &\text{for} \quad (t, z) \in \mathscr{U}_\alpha, \\ D_f V_\beta(t, z) < 0 \quad &\text{for} \quad (t, z) \in \mathscr{V}_\beta, \end{aligned}$$

where

$$\begin{aligned} \mathscr{U}_\alpha &= \{(t, z) : U_\alpha(t, z) = 0 \quad \text{and} \quad U_j(t, z) \leqq 0, V_k(t, z) \leqq 0 \quad \text{for all} \quad j, k\}, \\ \mathscr{V}_\beta &= \{(t, z) : V_\beta(t, z) = 0 \quad \text{and} \quad U_j(t, z) \leqq 0, V_k(t, z) \leqq 0 \quad \text{for all} \quad j, k\}. \end{aligned}$$

**Ważewski theorem.** (i) *Let $\Omega^0$ be a $(U, V)$-subset with respect to* (4.1). *Denote by $\Omega^0_e$ the set of egress points of $\Omega^0$, and by $\Omega^0_{se}$ the set of strict egress points of $\Omega^0$. Then*

$$\Omega^0_e = \Omega^0_{se} = \bigcup_{j=1}^{n} \mathscr{U}_j - \bigcup_{k=1}^{m} \mathscr{V}_k.$$

(ii) *Let $\Omega^0$ be a $(U, V)$-subset with respect to* (4.1) *and let $\Xi \subset \Omega^0 \cup \Omega^0_e$ be a non-empty compact set satisfying the condition that $\Xi \cap \Omega^0_e$ is not a retract of $\Xi$ but is a retract of $\Omega^0_e$. Then there exists at least one point $(t_1, z_1) \in \Xi \cap \Omega^0$ such that the graph of a solution $z(t)$ of* (4.1), *$z(t_1) = z_1$ is contained in $\Omega^0$ on its right maximal interval of existence.*

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