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JACOBSON'S THEOREM FOR ALTERNATIVE RINGS

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Abstract. We show below that the well known theorem of Jacobson concerning the commutativity of associative rings also holds for the case of alternative rings. Thus, we prove that if for every element x of an alternative ring A there exists an integer $n(x) > 1$ such that $x^{n(x)} = x$ then A is commutative and we also prove that A is associative. In fact we prove that A is a subdirect sum of fields.

We recall that a not necessarily associative ring A is called *alternative* if and only if every subring of A generated by two elements is associative.

In what follows R denotes an alternative ring with more than one element such that

$$x^{n(x)} = x \qquad \textit{for every} \qquad x \in R, \tag{1}$$

where $n(x) > 1$ is an integer depending on x.

Lemma 1. *If R has no zero divisor then R is a division alternative ring of prime characteristic $p > 0$.*

Proof. If R has no zero divisor then the two cancellation laws hold in R. But then if a is a nonzero element of R, from (1) it follows that $a^{n(a)-1}$ is the unit 1 of R and $a^{n(a)-2}$ is the inverse of a. Thus, R is a division ring. However, again from (1) it follows that $(1 + 1)^m = (1 + 1)$ for some integer $m > 1$ which implies that R has a characteristic $p > 0$ which is obviously a prime number.

Next, as in [2] we have:

Lemma 2. *Let R be a division alternative ring with center Z. Let $a \in R$ and $a \notin Z$. Then there exists $x \in R$ such that*

$$xax^{-1} = a^i \neq a \qquad \textit{for some integer } i. \tag{2}$$

Proof. From Lemma 1 it follows that the field P of integers modulo the prime characteristic p of R is the prime field of R. Let $P(a)$ be the division alternative ring obtained by adjoining a to P. From (1) it follows that $P(a)$ is finite and generated by a. Consequently $P(a)$ is associative and therefore $P(a)$ is a finite field. But then the proof of Lemma 1 of [2] applies and (2) is implied by the Corollary of [2].

Lemma 3. *If R is a division alternative ring then R is commutative and associative (i.e., R is a field).*

Proof. We observe that the subring W mentioned in the proof of the Theorem of [2] is generated by two elements a and b. Thus, in our case also W is a finite subfield of R. But then, in view of Lemma 2, the Lemma is proved by applying the proof of the Theorem of [2]. The associativity of R follows from Lemma 6 of [3].

Lemma 4. *R is commutative and associative. In fact R is a subdirect sum of fields.*

Proof. From (1) it follows that R has no nonzero nilpotent element. Thus, from the Theorem of [1] it follows that R is a subdirect sum of alternative rings without zero divisors. But then Lemma 1 and Lemma 3 imply that R is a subdirect sum of fields and therefore R is commutative and associative.

In the above we assumed that R has more than one element. Allowing $\{0\}$ to be considered as a field, Lemma 4 implies:

Theorem. *Let A be an alternative ring in which for every $x \in A$ there exists an integer $n(x) > 1$, depending on x, such that $x^{n(x)} = x$. Then A is commutative and associative. In fact A is a subdirect sum of fields.*

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