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THE MOST GENERAL TRANSFORMATION OF HOMOGENEOUS LINEAR DIFFERENTIAL RETARDED EQUATIONS OF THE FIRST ORDER

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On an interval $I = (a, \infty)$, $a \geqq -\infty$ we shall consider an equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)\,y + \sum_{i=1}^{n} q_i(x)\,y(\xi_i(x)) = 0 \tag{1}$$

with delays

$$\mu_i(x) = x - \xi_i(x), \qquad 1 \leqq i \leqq n,$$

where $p, q_i, \xi_i \in C^0(I)$, $q_k \neq 0$ on I for some k $(1 \leqq k \leqq n)$ and $\xi_i \neq \xi_j$, $\xi_i(x) < x$ on the whole interval I as $i \neq j$, $\xi_i(I) \supseteq I$, $1 \leqq i, j \leqq n$. Suppose that $\xi_i(x) \to \infty$ as $x \to \infty$ for all i.

A continuous function y is said to be a solution of (1) if there exists $b \in I$ such that y satisfies (1) for all $x \in [b, \infty)$. In such a case, we say that y is a solution of (1) on $[b, \infty)$.

Let $b \in I$, $A = [b, \infty)$, $A_i = \{\xi_i(x) : \xi_i(x) < b, x \in A\}$ and $c = \inf \bigcup_i A_i$, $i = 1, 2, \ldots, n$. Then $-\infty \leqq c < b$ and we put $A_b = (-\infty, b]$ if $c = -\infty$. Otherwise let $A_b = [c, b]$.

For a given function $\sigma \in C^0(A_b)$ we say y is a solution of (1) through (b, σ) if y is a solution of (1) on A and $y(s) = \sigma(s)$ for all $s \in A_b$.

If p, q_i, ξ_i, σ $(i = 1, 2, \ldots, n)$ are continuous functions, there exists a solution of (1) through (b, σ) and it is unique (see [1], p. 24).

TRANSFORMATION OF LINEAR DIFFERENTIAL EQUATIONS

Stäckel [2] and Wilczynski [3] proved that the most general point-transformation

$$T : x = f(t, u), \qquad y = g(t, u),$$

converting every linear differential equation of the n-th order of the form

$$(2)\qquad y^{(n)} + a_1(x)\,y^{(n-1)} + \dots + a_n(x)\,y = 0, \qquad \left(y^{(n)} = \frac{d^n y}{dx^n}\right)$$

with continuous coefficients into another equation of the same form, is

(i) $\qquad x = f(t), \quad y = g(t)\,u^{\lambda} \quad$ in the case that $n = 1$,

(ii) $\qquad x = f(t), \quad y = g(t)\,u \quad$ if $n \geqq 2$,

where f and g are arbitrary functions satisfying some additional assumptions, $\lambda \neq 0$ being an arbitrary constant.

Even if the equation (1) is studied by many authors, they did not pay special attention to the question of transformation. Only El'sgol'c [1] considered the transformation (ii) that converts an equation (1) into another of the same form and order. The same transformation is used by Melvin L. Heard [4] and others for a functional differential equation (generally nonhomogeneous).

THE MOST GENERAL TRANSFORMATION OF THE EQUATION (1)

In this paper we derive the most general transformation which convert any linear differential equation (1) into another equation of the same form

$$(3)\qquad \frac{du}{dt} + P(t)\,u + \sum_{i=1}^{n} Q_i(t)\,u(\eta_i(t)) = 0,$$

where $P, Q_i, \eta_i \in C^0(J), \eta_i(J) \supseteq J, \eta_i \neq \eta_j$ as $i \neq j$ and $\operatorname{sgn}(t - \eta_i(t)) = \operatorname{sgn}(t - \eta_j(t)) \neq 0$ on J for all i, j; $Q_m \neq 0$ on the whole interval J for some m, $1 \leqq i, j, m \leqq n$.

We wish that the transformation $T = (f, g)$ be independent of coefficients of (1) and will convert any nontrivial solution of (1) into some nontrivial solution of (3) for the same reason as in [3] (see [3], p. 8).

If y is a solution of (1), there is $b \in I$ such that y is defined on an interval $A \cup A_b$.

A mapping $\Psi : A \cup A_b \to R^2$ defined by

$$\Psi(x) = (x, y(x)), \qquad x \in A \cup A_b,$$

is a one-to-one homeomorphism of $A \cup A_b$ into a graph of the given solution y.

Conversely, to any point $(x_0, y_0) \in R^2$, $x_0 \in A$, there is a solution of (1) such that a graph of y contains the point (x_0, y_0). For example the interval $A = [x_0, \infty)$ and some continuous function τ on A_{x_0} satisfying the condition $\tau(x_0) = y_0$ will do. Due to continuity of p, q_i, ξ_i, τ ($i = 1, 2, \dots, n$) there exists a solution of (1) through (x_0, τ).

We have

Theorem 1. *The most general transformation which converts an equation* (1) *into* (3) *is*

$$x = f(t), \qquad y = g(t)\,u,$$

where $f, g \in C^1(J), f'(t)\,g(t) \neq 0$ for all $t \in J$.

Furthermore, $\xi_i \circ f = f \circ \eta_i$ *on* J *for* $i = 1, 2, \ldots, n$.

Proof: If $G := I \times R \subset R^2$ then G is open and $\Psi(x) = (x, y(x)) \in G$ for any solution y of (1) and each $x \in I$ where y is defined.

Consider a one-to-one homeomorphism Φ taking G into $U \subset R^2$ with properties $\Phi \in C^1(G)$ and Jacobian $|\Phi'(p)| \neq 0$ for all $p \in G$, i.e. Φ is a diffeomorphism. Then U is open and there is $\Phi^{-1} = (f, g)$ such that $|\Phi^{-1\prime}(q)| \neq 0$ for all $q \in U$ (see [5], p. 223).

The mapping Φ^{-1} is a point-transformation. Consequently, for any solution y of (1) and arbitrary fixed $x \in I$ where y is defined there is a unique set of mutually disjoint in U points $(t, u) = \Phi(\Psi(x))$, $(t_i, u_i) = \Phi(\Psi(\xi_i(x)))$, $i = 1, 2, \ldots, n$. Substituting

$$(x, y(x)) = \Phi^{-1}(t, u) = (f(t, u), g(t, u))$$

and

$$(\xi_i(x), y(\xi_i(x))) = \Phi^{-1}(t_i, u_i), \qquad i = 1, 2, \ldots, n,$$

into the equation (1) we get

$$\frac{du}{dt} + \frac{g_t(t, u) + f_t(t, u)\,X}{g_u(t, u) + f_u(t, u)\,X} = 0,$$

where $X = p(f(t, u))\,g(t, u) + \sum_{i=1}^{n} q_i(f(t, u))\,g(t_i, u_i)$ is a function of the coefficients of the equation (1).

The equation (3) gives that the last expression of a previous equation must be a linear combination of X for arbitrary choice of coefficients of (1).

Since

$$|\Phi^{-1\prime}(t, u)| = \begin{vmatrix} f_t & f_u \\ g_t & g_u \end{vmatrix} (t, u) \neq 0,$$

it follows immediatelly $f_u = 0$, $f_t g_u \neq 0$.

Consequently, $x = f(t)$ and $\xi_i(x) = f(t_i)$ by means of $(\xi_i(x), y(\xi_i(x))) = \Phi^{-1}(t_i, u_i)$, $i = 1, 2, \ldots, n$. Therefore, for a given $x \in I$ there is unique $t \in J$.

Define the functions $\eta_i : J \to R$ by virtue of

(6) $$\eta_i(t) = f^{-1}(\xi_i(f(t))) = t_i, \qquad 1 \leq i \leq n.$$

Then $\eta_i \neq \eta_j$ as $i \neq j$, $\eta_i(t) \neq t$ for $\xi_i \neq \xi_j$ as $i \neq j$, $\xi_i(x) \neq x$ and the function f is monotonic on J, $1 \leq i, j \leq n$.

Using (5) and (6) we have immediatelly from (4)

(7) $$\frac{du}{dt} + \frac{g_t(t, u)}{g_u(t, u)} + f'(t)\, p(f(t)) \frac{g(t, u)}{g_u(t, u)} + \sum_{i=1}^{n} f'(t)\, q_i(f(t)) \times \times g(\eta_i(t), u(\eta_i(t))) = 0.$$

The following relations

(8) $$\frac{g(t, u)}{g_u(t, u)} = \alpha(t)\, u,$$

(9) $$\frac{g_t(t, u)}{g_u(t, u)} = \beta(t)\, u,$$

(10) $$\frac{g(\eta_i(t), u(\eta_i(t)))}{g_u(t, u)} = \gamma_i(t)\, u(\eta_i(t)), \qquad i = 1, 2, \ldots, n,$$

must be valid for a suitable functions α, β, γ_i on $J = f^{-1}(I)$ to obtain the equation (3).

For u is a nontrivial solution of (3), and $\alpha(t) \equiv 0$ on some interval $J_1 \subset J$ would imply $g(t, u) \equiv 0$ which would be a contradiction with y to be nontrivial, it is not difficult to show directly from (8) that

(11) $$g(t, u) = \alpha_1(t)\, u^{\alpha_2(t)}, \qquad \alpha_1(t)\, \alpha_2(t) \neq 0$$

on J for suitable functions α_1, α_2.

From (9) and (11) we have

(12) $$\alpha_1' u^{\alpha_2} + u^{\alpha_2} \alpha_2' \alpha_1 \ln | u | = \beta \alpha_1 \alpha_2 u^{\alpha_2 - 1} u$$

for $u \neq 0$. It is clear that only $\alpha_2(t) = \lambda = \text{const}$ complies with (12).

Finally, $\lambda = 1$ for the sake of equations

$$\alpha_1(\eta_i(t))\, u^\lambda(\eta_i(t)) = \lambda \gamma_i(t)\, \alpha_1(t)\, u(\eta_i(t))\, u^{\lambda - 1}(t)$$

$(1 \leqq i \leqq n)$ obtained from (10).

Consequently, $g(t, u) = \alpha_1(t)\, u$, $\alpha_1(t) \neq 0$.

It remains to show the required transformation rewritten as $x = f(t)$, $y = g(t)\, u$ $f'g \neq 0$ on J, $\eta_i(t) = f^{-1}(\xi_i(f(t)))$, $t \in J$, $i = 1, 2, \ldots, n$, converts (1) into (3). In fact we get

$$\frac{du}{dt} + \left(\frac{g'(t)}{g(t)} + p(f(t))\, f'(t) \right) u + \sum_{i=1}^{n} q_i(f(t))\, f'(t)\, u(\eta_i(t)) = 0$$

and the theorem is proved.

Remark 1. Consider the equations

(a) $$\frac{dy}{dx} - \frac{2}{x} y + \frac{2}{x} y\left(\frac{x}{2}\right) = 0 \qquad \text{on} \qquad I = (0, \infty)$$

and

(b) $$\frac{du}{dt} - u + u(t-1) = 0 \quad \text{on some} \quad J \leqq R, J = (b, \infty).$$

There does not exist a transformation converting (a) into (b). Indeed, if $x = f(t)$, $y = g(t)u$ was such a transformation then by Theorem 1 we would have

(c) $$f(t-1) = f(t)/2$$

on J and the transformed equation would be

$$\frac{du}{dt} + \left(\frac{g'(t)}{g(t)} - 2\frac{f'(t)}{f(t)}\right)u + 2\frac{f'(t)\,g(t-1)}{f(t)\,g(t)}u(t-1) = 0,$$

$f(t)g(t) \neq 0$. Due to (5) $g'(t)/g(t) - 2f'(t)/f(t) = -1$ and through integration we would get $g(t) = Kf^2(t)\,e^{-t}$ with an arbitrary constant $K \neq 0$. Similarly

(d) $$2\frac{f'(t)\,g(t-1)}{f(t)\,g(t)} = 1.$$

Furthermore $g(t-1) = Kf^2(t-1)\,ee^{-t} = K\dfrac{f^2(t)}{4}ee^{-t}$. Using this result and integrating (d) we would have $f(t) = C\exp\{2t/e\}$ for $C \neq 0$ an arbitrary constant. This is a contradiction with (c). Hence, such a transformation does not exist.

Remark 2. If the function f is strictly increasing (strictly decreasing), the transformation described in Theorem 1 converts a retarded equation into a retarded (an advanced) equation.

Proof: Let f be strictly increasing. Then $\xi_i(x) = \xi_i(f(t)) = f(\eta_i(t)) < x = f(t)$ for all $x \in I$ implies $\eta_i(t) < t$ for all $t = f^{-1}(x) \in J$, $i = 1, 2, \ldots, n$. An equation (3) is retarded.

If f is strictly decreasing, we prove similarly (3) is an advanced equation, i.e. $\eta_i(t) > t$ on the whole interval J, $1 \leqq i \leqq n$.

Remark 3. For the equation

$$\frac{dy}{dx} + y(x^{-2}) = 0$$

on $I = (1, \infty)$ where $\xi(x) = x^{-2} < x$ on I and our assumption $\xi(I) \supseteq I$ is not satisfied (see also Myškis [6], p. 211). It would be interesting to find the most general transformation for equations the above type characterized by $\xi(I) \cap I = \Phi$.

Example. $\dfrac{dy}{dx} - \dfrac{2}{x}y + \dfrac{2}{x}y\left(\dfrac{x}{2}\right) = 0$ is the retarded equation on the interval $I = (0, \infty)$. There holds $\xi(x) = \dfrac{x}{2} < x$ on I and $\xi(I) = I$. The transformation $x =$

$= 2^t = f(t)$, $y = g(t)u$, $g \in C^1(R = f^{-1}(I))$, $g \neq 0$ on R, $\eta(t) = f^{-1}(\xi(f(t))) =$ $= t - 1 < t$ converts this equation into the retarded equation

$$\frac{du}{dt} + \left(\frac{g'(t)}{g(t)} - \ln 4\right)u + \frac{g(t-1)}{g(t)} \ln 4\, u(t-1) = 0$$

on R.

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