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GENERAL UNIQUENESS THEOREMS FOR ORDINARY DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

Consider an initial value problem

$$x' = f(t, x), \qquad x(t_0) = x_0, \tag{1}$$

where x, f are n-dimensional vectors.

The aim of the present paper is to prove new general uniqueness theorems for the initial value problem (1) and to obtain the well-known uniqueness criteria (especially Lakshmikantham [6], Brauer and Sternberg [3], Brauer [4], Kamke [5], Borůvka [2], by choosing special functions in our theorems.

Notation. Let R and R^n be the real number system and the Euclidean n-space, respectively. Define $R^+ = [0, \infty)$, $R^- = (-\infty, 0]$. By $\| \|$ we denote any convient norm in R^n; $\| \|_e$, and $| |$ denote the Euclidean norm in R^n and in R, respectively. By D^+, D_+ we denote Dini derivatives. For the notation of the inner product in R^n we use the sign . . Let $C[D_1; D_2]$ be the class of all continuous functions $f: D_1 \to D_2$ and let $f(t) = o(g(t))$ as $t \to t_{0+}$ mean $\lim_{t \to t_{0+}} f(t)/g(t) = 0$.

Finally, if $t_0 < t^*$, $b > 0$, we put

$$R_0 = \{(t, x) : t_0 < t \leqq t^*, \| x - x_0 \| \leqq b\},$$

$$\hat{R}_0 = \{(t, x, y) : t_0 < t \leqq t^* + \varepsilon, \| x - x_0 \| \leqq b + \varepsilon, \| y - x_0 \| \leqq b + \varepsilon, \varepsilon > 0\}.$$

Definition. Let $t_0 < t^*$ and $f(t, x) \in C[R_0; R^n]$. We say that a function $x(t)$ is a solution of the initial value problem (1) on $[t_0, t^*]$, if $x(t) \in C[[t_0, t^*]; R^n]$ such that $x(t_0) = x_0$, and $x'(t) = f(t, x(t))$ for $t \in (t_0, t^*)$.

2. MAIN RESULTS

Theorem 1. *Suppose*

(i) $f(t, x) \in C[R_0; R^n]$;

(ii) *there exist a positive function* $B(t) \in C[(t_0, t^*); R^+]$ *and a function* $g(t, u) \in \in C[(t_0, t^*] \times R^+; R]$ *such that for every* $t_1 \in (t_0, t^*)$ *there is* $u(t) \equiv 0$ *the only differentiable function satisfying*

$$u'(t) = g(t, u(t)) \quad \text{for } t \in (t_0, t_1), \tag{2}$$

$$u(t) = o(B(t)) \quad \text{as } t \to t_{0+}; \tag{3}$$

(iii) *there exists a function* $V(t, x, y) \in C[\mathring{R}_0; R^+]$ *such that* $V(t, x, y)$ *is locally Lipschitzian in* x, y *for* $(t, x, y) \in \mathring{R}_0$. *Let any two solutions of* (1) $x(t), y(t)$ *fulfil*

$$V(t, x(t), y(t)) \equiv 0 \Leftrightarrow x(t) \equiv y(t) \text{ on } (t_0, t^*];$$

(iv) *for* $(t, x), (t, y) \in R_0$, $x \neq y$, $t < t^*$ *there is satisfied the condition*

$$D_{+f}V(t, x, y) \leqq g(t, V(t, x, y)), \tag{4}$$

where

$$D_{+f}V(t, x, y) = \liminf_{h \to 0_+} \frac{V(t + h, x + hf(t, x), y + hf(t, y)) - V(t, x, y)}{h}. \tag{5}$$

Then, for any pair of solutions $x(t), y(t)$ *of (1) such that* $x(t) \not\equiv y(t)$ *on the common interval of their existence, there holds the condition*

$$V(t, x(t), y(t)) \neq o(B(t)) \quad \text{as } t \to t_{0+}. \tag{6}$$

Proof. Suppose there are two different solutions $x(t), y(t)$ of the problem (1) on $[t_0, t_1] \subset [t_0, t^*]$ satisfying $V(t, x(t), y(t)) = o(B(t))$ as $t \to t_{0+}$. Define $m(t) = V(t, x(t), y(t))$ for $t \in (t_0, t_1)$. Then there holds

$$D_+m(t) = \liminf_{h \to 0_+} \frac{1}{h}\{m(t + h) - m(t)\} =$$
$$= \liminf_{h \to 0_+} \frac{1}{h}\{V(t + h, x(t + h), y(t + h)) - V(t, x(t), y(t))\} =$$
$$= \liminf_{h \to 0_+} \frac{1}{h}\{V(t + h, x(t) + hf(t, x(t)) + o(h), y(t) + hf(t, y(t)) + o(h)) -$$
$$- V(t, x(t), y(t))\} \leqq D_{+f}V(t, x(t), y(t)) \leqq$$
$$\leqq g(t, V(t, x(t), y(t))) = g(t, m(t))$$

for every $t \in (t_0, t_1)$ such that $m(t) \neq 0$. Define $m(t_0) = 0$. Because of $x(t) \not\equiv y(t)$ on $[t_0, t_1]$ there exists some $c \in (t_0, t_1)$ such that $V(c, x(c), y(c)) = m(c) > 0$.

Let $r(t)$ be the left minimal solution of the problem

$$u' = g(t, u), \qquad u(c) = m(c). \tag{7}$$

It is easy to see that $r(t) \leqq m(t)$ for $t \in (t_2, c)$ where $t_2 = \sup \{t \in [t_0, c) : m(t) = 0\}$. If $t_2 > t_0$ then $r(t) \equiv 0$ on (t_0, t_2). Consequently we obtain $0 \leqq r(t) \leqq m(t)$ on (t_0, c).

Since $m(t) = o(B(t))$ as $t \to t_{0+}$, we get $r(t) = o(B(t))$ as $t \to t_{0+}$. This together with (ii) implies

$$r(t) \equiv 0 \qquad \text{on } (t_0, c),$$

contradicting the assumption $r(c) = m(c) > 0$.

Theorem 2. *Suppose*

(i) $f(t, x) \in C[R_0; R^n]$;

(ii) *there exist functions* $B_i(t) \in C[(t_0, t^*); R^+]$, $i = 1, 2$ *and functions* $g_1(t, u) \in C[(t_0, t^*] \times R^+; R]$, $g_2(t, u) \in C[(t_0, t^*] \times R^-; R]$ *such that for every* $t_1 \in (t_0, t^*)$ *there are* $u_i(t) \equiv 0$ *the only differentiable functions satisfying*

$$(-1)^{i-1} u_i(t) \geqq 0$$

$$u_i'(t) = g_i(t, u_i(t)) \qquad \textit{for } t \in (t_0, t_1), \tag{8}$$

$$u_i(t) = o(B_i(t)) \qquad \textit{as } t \to t_{0+}, \tag{9}$$

where $i = 1, 2$;

(iii) *the hypothesis* (iii) *of Theorem 1 is satisfied except* $V(t, x, y) \in C[\mathring{R}_0; R^+]$ *is replaced by* $V(t, x, y) \in C[\mathring{R}_0; R]$;

(iv) *for* $(t, x), (t, y) \in R_0$, $x \neq y$, $t < t^*$ *there are fulfilled the conditions*

$$D_{+f} V(t, x, y) \leqq g_1(t, V(t, x, y)), \tag{10}$$

$$D_f^+ V(t, x, y) \geqq g_2(t, V(t, x, y)), \tag{11}$$

where

$$D_f^+ V(t, x, y) = \limsup_{h \to 0+} \frac{1}{h} \{V(t + h, x + hf(t, x), y + hf(t, y)) - V(t, x, y)\} \tag{12}$$

and $D_{+f} V(t, x, y)$ *is defined by* (5).

Then, the conclusion of Theorem 1 is valid for $B(t) = \min \{B_1(t), B_2(t)\}$.

Proof. The proof is similar to that of Theorem 1.

Put $m(t) = V(t, x(t), y(t))$, where $x(t), y(t)$ are two different solutions of the problem (1) on $[t_0, t_1] \subset [t_0, t^*]$. Then there exists some $c \in (t_0, t_1)$ so that $m(c) \neq 0$. It follows from the relations (10) and (11) that

$$D_+ m(t) \leqq g_1(t, m(t))$$

and

$$D^+ m(t) \geqq g_2(t, m(t))$$

for every $t \in (t_0, t_1)$ such that $m(t) \neq 0$. Define $m(t_0) = 0$.

First consider the case $m(c) < 0$. Let $r_2(t)$ be the left maximal solution of the problem

$$u' = g_2(t, u), \qquad u(c) = m(c).$$

There holds $r_2(t) \geqq m(t)$ for $t \in (t_2, c)$, where $t_2 = \sup\{t \in [t_0, c) : m(t) = 0\}$. If $t_2 > t_0$ then $r_2(t) \equiv 0$ on (t_0, t_2). Then we obtain $|r_2(t)| \leqq |m(t)|$ on (t_0, c).

Since $m(t) = o(B(t))$ as $t \to t_{0+}$, we have $m(t) = o(B_2(t))$ as $t \to t_{0+}$. Thus $r_2(t) = o(B_2(t))$ as $t \to t_{0+}$. This together with the assumption (ii) implies

$$r_2(t) \equiv 0 \qquad \text{on } (t_0, c),$$

contradicting $r_2(c) < 0$.

The case $m(c) > 0$ contradicts the assumption $r_1(c) > 0$, where $r_1(t)$ is the left minimal solution of the problem

$$u' = g_1(t, u), \qquad u(c) = m(c).$$

This completes the proof.

Remark 1. Assuming in the previous theorems in addition that for any two solutions $x(t), y(t)$ of the problem (1) there holds

$$V(t, x(t), y(t)) = o(B(t)) \qquad \text{as } t \to t_{0+},$$

then the initial value problem (1) has at most one solution.

Corollary 1. *Let $a > 0$, $b > 0$ and $D : t_0 \leqq t \leqq t_0 + a$, $\| x - x_0 \|_e \leqq b$. Assume*

(i) $f(t, x) \in C[D; R^n]$;

(ii) $g(t, u) \in C[(t_0, t_0 + a] \times [0, 2b]; R^+]$ *and* $u(t) \equiv 0$ *is the only function such that*

$$u' = g(t, u) \qquad \text{for } t \in (t_0, t_0 + a],$$
$$\lim_{t \to t_{0+}} \frac{u(t)}{t - t_0} = 0;$$

(iii) *for* $(t, x), (t, y) \in D$, $t \neq t_0$, $x \neq y$ *there holds*

$$\frac{1}{\| x - y \|_e} (f(t, x) - f(t, y)) \cdot (x - y) \leqq g(t, \| x - y \|_e).$$

Then, the initial value problem (1) *has the unique solution.*

Proof. Without loss of generality we may suppose that $g(t, u) \in C[[t_0, t_0 + a] \times R^+; R]$. Set $V(t, x, y) = \| x - y \|_e$, $B(t) = t - t_0$.

For $(t, x), (t, y) \in D$, $t \neq t_0$, $x \neq y$ we get

$$D_{+f} V(t, x, y) = \frac{1}{\| x - y \|_e} (x - y) \cdot (f(t, x) - f(t, y)) \leqq g(t, V(t, x, y)).$$

Since for every solutions $x(t), y(t)$ of (1) the condition

$$\lim_{t \to t_{0+}} \frac{\| x(t) - y(t) \|_e}{t - t_0} = 0$$

is satisfied, in view of Remark 1 we obtain the statement of Corollary 1.

Remark 2. Denot e $J = \{x \in R^n : \| x - x_0 \| \leqq b\}$ and replace the assumption (ii) in Theorem 1 by

(ii') there exist a positive function $B(t) \in C[(t_0, t^*); R^+]$ and a function $g(t, v, u) \in \in C[(t_0, t^*) \times J \times R^+; R]$ such that for every solution $x(t)$ of the problem (1) and for every $t_1 \in (t_0, t^*)$, $u(t) \equiv 0$ is the only function for which

(2') $$u'(t) = g(t, x(t), u(t)) \qquad \text{on } (t_0, t_1),$$

(3') $$u(t) = o(B(t)) \qquad \text{as } t \to t_{0+}.$$

Then, the conclusion of Theorem 1 is valid. As well the proof remains without an essential change.

Remark 3. If $V(t, x, y) = \| x - y \|$ and $f(t, x)$ is continuous only for $t > t_0$, then the choice $B(t) = 1$ in Theorem 1 gives the uniqueness of the problem (1).

Moreover if the function $f(t, x)$ is bounded on D, then it is possible to choose $B(t) = (t - t_0)^k$, where $k \in (-\infty, 1)$.

3. APPLICATIONS

To obtain uniqueness criteria as the corollaries of Theorem 1, resp. Theorem 2, it is sufficient to prove that there hold (*i*) the assumptions of Theorem 1, resp. Theorem 2, and (ii) the relation $V(t, x(t), y(t)) = o(B(t))$ as $t \to t_{0+}$ for any two solutions $x(t)$, $y(t)$ of the problem (1).

The following table shows how to obtain the well-know uniqueness criteria by selecting functions in general theorems. We use the notation quoted in the mentioned literature.

Remark 4. In the original paper by Brauer and Sternberg [3] the uniqueness is asserted unless the assumption $V(t, x(t) - y(t)) = o(t - t_0)$ as $t \to t_{0+}$ for any two solutions $x(t), y(t)$ of (1) would be assumed. The following example shows non-validity of this statement.

Consider the initial value problem $x' = 2\dfrac{x}{t}$, $x(0) = 0$. Put $V(t, x) = \dfrac{|x|}{t^2}$, $g(t, x) = \dfrac{x}{t}$. It is easy to prove the validity of the assumptions of Theorem of Brauer and Sternberg. However, this problem has different solutions $x_1(t) = t^2$, $x_2(t) \equiv 0$.

Remark 5. Theorem of Borůvka [2] is the corollary of Theorem 1 and Remark 2.

Theorem 1.

Author	$B(t)$	$g(t, u)$	$V(t, x, y)$
Kamke (1930) [5]	$t - t_0$	$g(t, u)$	$\|\| x - y \|\|$
Okamura (1942) [9]	1	0	$V(t, x - y)$
Borůvka (1956) [2]	1	$\Phi(t, u, v)$	$\varphi(t, x, y)$
Perov (1958) [10]	1	$a(t) L(u)$	$\|\| x - y \|\|/(t - t_0)^k$
Brauer, Sternberg (1958) [3]	$t - t_0$	$\omega(t, u)$	$V(t, x - y)$
Brauer (1959) [4]	$B(t)$	$\psi_2(t, u)$	$\| x - y \|$
Lakhsmikantham (1962) [6]	$B(t)$	$h(t, u)$	$\| x - y \|$
Moyer (1966) [8]	1	0	$e^{W(t, \|\|x-y\|\|)}$
Witte (1974) [11]	$\int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}s} [\exp \int_{t_0+a}^{s} h(u)\,\mathrm{d}u]\,\mathrm{d}s$	$h(t)\,u$	$\| x - y \|$
Lemmert (1975) [7]	$\exp \int_{t^*}^{t} h(s)\,\mathrm{d}s$	$h(t)\,u$	$\| x - y \|$

Theorem 2.

Author	$B_i(t)$ $i = 1, 2$	$g_1(t, u)$	$g_2(t, u)$	$V(t, x, y)$
Antosiewicz 1962 [1]	1	$\omega_2(t, u)$	$\omega_1(t, u)$	$V(t, x - y)$

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