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Archivum Mathematicum, Vol. 16 (1980), No. 4, 185--187

Persistent URL: http://dml.cz/dmlcz/107072

ARCH. MATH. 4, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XVI: 185—187, 1980

A WELL ORDERING OF THE CARTESIAN PRODUCT OF TWO ORDINALS

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(Received April 2, 1979)

ABSTRACT. *For ordinals u and v a well ordering of $u \times v$ is given which is simpler than the canonical well ordering of $u \times v$ and yet has the desirable properties of the latter.*

We consider the ordinals given in their usual definition [4, p. 14] whereby every ordinal is well ordered by $\in$ (the elementhood symbol) and is the set of all the ordinals less than it.

Also, as usual, an ordinal is called a *cardinal* if it is not equipollent to any of its elements [4, p. 24]. Thus, denoting the unique cardinal equipollent to the ordinal (or for that matter equipollent to a well ordered set) u by $\bar{\bar{u}}$, for every cardinal k we have:

$$(1) \qquad v \in k \quad \textit{implies} \quad \bar{\bar{v}} < k.$$

Let r be an ordinal and $u \in r$. We denote by $I(u)$ the set of all ordinals less than u and we call $I(u)$ the *initial segment of r determined by u*. Clearly, $I(u) = u$. Moreover, for every cardinal k, we have:

$$(2) \qquad k < \bar{\bar{r}} \quad \textit{implies} \quad k = \overline{\overline{I(u)}} = \bar{\bar{u}} \quad \textit{for some} \quad u \in r.$$

Let u and v be ordinals. The Cartesian product $u \times v$ can be well ordered (based on the abovementioned well ordering of u and v) in various ways. The two most frequently used ways are the *lexicographic well ordering* [4, p. 18] and the *canonical well ordering* [4, p. 20] which is particularly suitable for proving $m = m^2$ for infinite cardinals m.

Based on the usual notion of the ordinal sum $a + b$ of ordinals a and b, we define below another way of well ordering $u \times v$. The present well ordering of $u \times v$ has the desirable properties of the canonical well ordering and yet is simpler than the latter inasmuch as it has two defining clauses (whereas the canonical well ordering is defined via three clauses).

Definition. *Let u and v be ordinals. For every pair of elements (a, b) and (p, q) of the Cartesian product $u \times v$, we let:*

(3) $$(a, b) < (p, q) \quad \textit{if and only if} \quad a + b < p + q \quad \textit{or} \quad a + b = p + q \textit{ and } a < p.$$

It is trivial to verify that $<$ given by (3) well orders $u \times v$ (in fact, along the -1 slope diagonals).

From (1), it also readily follows that:

(4) $$\overline{\overline{I((a, b))}} \leqq \overline{\overline{a + b}} \,.\, \overline{\overline{a + b}} \qquad \textit{(the desirable property mentioned above).}$$

As an application of the above Definition, we prove the Theorem below where no use of the axiom of Choice is made since all the sets involved are well ordered.

Theorem. *Let m be an infinite cardinal. Then*

(5) $$m = \overline{\overline{m \times m}} = m^2.$$

Proof. Let us assume to the contrary, and let k be the smallest infinite cardinal such that:

(6) $$k < \overline{\overline{k \times k}}.$$

But then, for every $a \in k$ and $b \in k$, we have:

(7) $$\overline{\overline{a + b}} < k.$$

Indeed, if a and b are finite then (7) is trivial. Otherwise, let $v = \max\{a, b\}$. Obviously, $v \in k$ and hence $\bar{\bar{v}} < k$ by (1). Thus, by our assumption, $\bar{\bar{v}} = \overline{\overline{v \times v}}$. But then, $\overline{\overline{a + b}} \leqq \overline{\overline{v + v}} \leqq \overline{\overline{v \times v}} = \bar{\bar{v}} < k$, establishing (7).

Also, by (7), in view of our assumption, we have:

(8) $$\overline{\overline{a + b}} \,.\, \overline{\overline{a + b}} = \overline{\overline{a + b}}.$$

Next, let us consider the well ordering of $k \times k$ given by (3). From (6) and (2) it follows that k is equipollent to an initial segment of $k \times k$, say $I((a, b))$ determined by (a, b). But then, in view of (4) and (8), we have:

$$k = \overline{\overline{I((a, b))}} \leqq \overline{\overline{a + b}} \,.\, \overline{\overline{a + b}} = \overline{\overline{a + b}}$$

contradicting (7). Hence our assumption is false and the Theorem is proved.

Remark 1. In mathematical literature, for every infinite cardinal m, the equality $m = m^2$ is proved in various ways. For instance, in [2, p. 219] it is proved based on the notion of the *natural sums* introduced by [3, p. 593]. In [1, p. 186] it is proved based on Zorn's lemma. In [4, p. 25] and [5, p. 281] it is proved based on the notion of the canonical well ordering, and, in the above, based on the notion of well ordering given by (3).

Remark 2. As far as a simplest proof of $m = m^2$ for infinite cardinal m is concerned, we give the following proof (making a basic use of (2) and the fact that every infinite cardinal k is obviously a limit ordinal) where Σ refers to pairwise disjoint summands.

Let us assume to the contrary, and let k be the smallest infinite cardinal such that $k < \overline{\overline{k \times k}}$. But then, $k < \overline{\overline{k \times k}} = \overline{\overline{\bigcup_{a \in k} a \times a}} \leqq \sum_{a \in k} \bar{\bar{a}}^2 = \sum_{a \in k} \bar{\bar{a}}$, which by (2) implies $k \leqq \bar{\bar{v}}^2$ for some $\bar{\bar{v}} < k$. However, $k \leqq \bar{\bar{v}}^2$ by our assumption implies $k \leqq \bar{\bar{v}}$, contradicting $\bar{\bar{v}} < k$. Thus, our assumption is false and $m = m^2$ is proved, as desired.

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