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CATEGORIES OF SYSTEMS OF X-RELATIONS

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In [1], M. Armbrust studies the category of congruence systems, i.e. the category whose objects are sets with systems of equivalence relations which are closed with respect to arbitrary intersections and under unions of directed subsystems and which contain the identity relation. The above properties are characteristic for the set of all congruences of any partial algebra. (See [3], [7].) Moreover, some other types of relations compatible with a partial algebra have those properties. (For the quasi-orders see [5].)

This paper investigates the categories $\mathbf{X}$ whose objects are sets with systems of X-relations (e.g. the equivalences, the quasi-orders and the tolerances, respectively, are the special types of X-relations) satisfying the same properties as congruence systems.

It is proved that such arbitrary category $\mathbf{X}$ is bicomplete and concrete and in $\mathbf{X}$ there are characterized the injective and projective objects. Moreover, there is shown a connection between the category of congruence systems and the category of quasi-order systems.

For the notions of the universal algebra and the category theory used in the paper see [3], [4] and [2], [6], respectively.

Let X be a system of relational quasi-identities of the type $\langle 2\rangle$ with the signature $\langle A_1^2\rangle$. Let us suppose that X contains the identity of reflexivity and that each other quasi-identity of X (if it exists) is in the form

$$\forall x_1 \dots \forall x_n(\mathscr{A}_1 \& \dots \& \mathscr{A}_p \Rightarrow \mathscr{A}),$$

where $\mathscr{A}_1, \dots, \mathscr{A}_p, \mathscr{A}$ are primitive formulas and the following conditions are satisfied:

a) for each x_i $(i = 1, \dots, n)$ there exists at least one of $\mathscr{A}_1, \dots, \mathscr{A}_p$ containing x_i,

b) if $p > 1$, then each $\mathscr{A}_k$ $(k = 2, \dots, p)$ contains at least one of the variables containing in $\mathscr{A}_{k-1}$,

c) it holds $\mathscr{A} = A_1^2(x_r, x_q)$ $(r, q \in \{1, \dots, n\})$.

Definition. An *X-relation* on a set A is any relation on A satisfying all quasidentities of X.

Note. It is clear that the equivalences, the quasi-orders (i.e. the reflexive and transitive relations) and the tolerances (i.e. the reflexive and symmetric relations) are X-relations, respectively.

Let $\mathscr{B}$ be a formula of the signature $\langle A_1^2 \rangle$ which variables are contained in $\{x_1, \ldots, x_n\}$, let A be a set, $\varrho \subseteq A \times A$ and $a_1, \ldots, a_n \in A$. Then the symbol $\mathscr{B}^\varrho(a_1, \ldots, a_n)$ means the value of the formula $\mathscr{B}$ in A for the substitution of ϱ instead of A_1^2 and of a_i instead of x_i, for each x_i containing in $\mathscr{B}$.

Lemma 1. a) The *intersection of any system of X-relations on A is also an X-relation on A.*

b) *The identity relation* id_A *is an X-relation on A.*

Proof. a) Let $\Theta_\gamma (\gamma \in \Gamma)$ be X-relations, $\Theta = \bigcap_{\gamma \in \Gamma} \Theta_\gamma$. Then evidently Θ is reflexive. Let $\Gamma \neq \emptyset$ and let $\forall x_1 \ldots \forall x_n (\mathscr{A}_1 \& \ldots \& \mathscr{A}_p \Rightarrow \mathscr{A})$ be a quasi-identity of X. Let $a_1, \ldots, a_n \in A$. Let us suppose that $\mathscr{A}_1^\Theta(a_1, \ldots, a_n) \& \ldots \& \mathscr{A}_p^\Theta(a_1, \ldots, a_n)$ holds. Then $\mathscr{A}_i^\Theta(a_1, \ldots, a_n)$ $(i = 1, \ldots, n)$, therefore also $\mathscr{A}_i^{\Theta_\gamma}(a_1, \ldots, a_n)$ for all $\gamma \in \Gamma$ $(i = 1, \ldots, n)$. Since Θ_γ is an X-relation, $\mathscr{A}^{\Theta_\gamma}(a_1, \ldots, a_n)$ for each $\gamma \in \Gamma$, thus $\mathscr{A}^\Theta(a_1, \ldots, a_n)$, i.e. the implication $\mathscr{A}_1^\Theta(a_1, \ldots, a_n) \& \ldots \& \mathscr{A}_p^\Theta(a_1, \ldots, a_n) \Rightarrow \mathscr{A}^\Theta(a_1, \ldots, a_n)$ is true.

If $\Gamma = \emptyset$, then $\Theta = A \times A$ is an X-relation on A.

b) It is clear that the identity relation is an X-relation on A.

Corollary. *The set* $X_0(A)$ *of all X-relations on A ordered by inclusion forms a complete lattice. The smallest element is* id_A, *the greatest element is* $A \times A$.

Lemma 2. *Let* $\{\Theta_\gamma; \gamma \in \Gamma\}$ *be a directed system of X-relations on A. Then* $\bigcup_{\gamma \in \Gamma} A_\gamma = \bigvee_{\gamma \in \Gamma}^{X_0(A)} \Theta_\gamma$.

Proof. Let $\forall x_1 \ldots \forall x_n (\mathscr{A}_1 \& \ldots \& \mathscr{A}_p \Rightarrow \mathscr{A})$ be a quasi-identity of X, $\mathscr{A} = A_1^2(x_k, x_j)$. Let us denote $\Psi = \bigcup_{\gamma \in \Gamma} \Theta_\gamma$. Let $a_1, \ldots, a_n \in A$, $(a_1^{(1)}, a_2^{(1)}), \ldots, (a_1^{(p)}, a_2^{(p)}) \in \Psi$, $a_1^{(1)}, a_2^{(1)}, \ldots, a_1^{(p)}, a_2^{(p)} \in \{a_1, \ldots, a_n\}$. Then there exist $\gamma_1, \ldots, \gamma_p \in \Gamma$ such that $(a_1^{(1)}, a_2^{(1)}) \in \Theta_{\gamma_1}, \ldots, (a_1^{(p)}, a_2^{(p)}) \in \Theta_{\gamma_p}$. Then by the assumption there exists $\Theta \in \{\Theta_\gamma; \gamma \in \Gamma\}$ such that $\Theta_{\gamma_1}, \ldots, \Theta_{\gamma_p} \subseteq \Theta$. But this means $(a_k, a_j) \in \Theta$, therefore $(a_k, a_j) \in \Psi$.

Let $\mathfrak{A} = (A, F)$ be a partial algebra of the type τ, where $A \neq \emptyset$ is the support of $\mathfrak{A}$ and $F \neq \emptyset$ is a set of partial operations on A. If $f \in F$, then n_f denotes the arity of f and $\mathrm{D}(f, A)$ denotes the definition domain of f. Let Θ be an X-relation on A. Then Θ is called an $\mathscr{X}$*-relation* on $\mathfrak{A}$ if for each $f \in F$ it holds:

(X) *If* $(a_1, \ldots, a_{n_f}), (b_1, \ldots, b_{n_f}) \in \mathrm{D}(f, A)$ *and* $(a_i, b_i) \in \Theta$, $i = 1, \ldots, n_f$, *then* $(a_1 \ldots a_{n_f} f, b_1 \ldots b_{n_f} f) \in \Theta$.

Lemma 3. *The intersection of any system of* $\mathscr{X}$*-relations on* $\mathfrak{A}$ *is an* $\mathscr{X}$*-relation on* $\mathfrak{A}$.

Corollary. *The set* $\mathscr{X}(\mathfrak{A})$ *of all* $\mathscr{X}$*-relations on* $\mathfrak{A}$ *ordered by inclusion is a complete* *lattice which is a closed $\wedge$-subsemilattice of the complete lattice $X_0(A)$. The smallest element in $\mathscr{X}(\mathfrak{A})$ is id_A, the greatest element is $A \times A$.*

Lemma 4. *Let $\{\Theta_\gamma;\ \gamma \in \Gamma\}$ be a directed system of $\mathscr{X}$-relations on a partial algebra $\mathfrak{A} = (A, F)$. Then $\bigvee_{\gamma\in\Gamma}^{\mathscr{X}(\mathfrak{A})} \Theta_\gamma = \bigcup_{\gamma\in\Gamma} \Theta_\gamma$.*

Proof. By Lemma 2, $\bigcup_{\gamma\in\Gamma} \Theta_\gamma$ is an X-relation on A. Let $f \in F$, $(x_1, \ldots, x_{n_f})$, $(y_1, \ldots, y_{n_f}) \in \mathrm{D}(f, A)$, $(x_i, y_i) \in \bigcup_{\gamma\in\Gamma} \Theta_\gamma$ $(i = 1, \ldots, n_f)$. Then there exist $\gamma_1, \ldots, \gamma_{n_f} \in \Gamma$ such that $(x_i, y_i) \in \Theta_{\gamma_i}$ $(i = 1, \ldots, n_f)$. Thus there exists $\Theta \in \Gamma$ such that $\Theta_i \subseteq \Theta$ $(i = 1, \ldots, n_f)$. Therefore $(x_1 \ldots x_{n_f} f, y_1 \ldots y_{n_f} f) \in \bigcup_{\gamma\in\Gamma} \Theta_\gamma$.

Lemma 5. *The set $X_0(A)$ of all X-relations on a set A ordered by inclusion is an algebraic lattice.*

Proof. Follows from the algebraicity of the lattice $R_0(A)$ of all reflexive relations on A and from [7, Folgerung 4.7].

Let $\mathfrak{A} = (A, F)$ be a partial algebra, $\varrho \subseteq A \times A$. Let us denote the smallest element of $\mathscr{X}(\mathfrak{A})$ containing ϱ by Θ_ϱ. Then it is clear that the mapping $\lambda : X_0(A) \to X_0(A)$ defined by $\varrho\lambda = \Theta_\varrho$ for each $\varrho \in X_0(A)$ is a closure operator on $X_0(A)$.

Theorem 6. *If $\mathfrak{A} = (A, F)$ is a partial algebra, then the lattice $\mathscr{X}(\mathfrak{A})$ is algebraic.*

Proof. By Lemma 5 and by [7, Lemma 4.7], the closure operator $\lambda : X_0(A) \to$ $\to X_0(A)$ is algebraic. Thus by [7, Lemma 4.2], $\mathscr{X}(\mathfrak{A}) = (X_0(A))\,\lambda$ is an algebraic lattice.

Definition. An *x-system* is an ordered pair $(A, \mathscr{X})$, where A is a set and $\mathscr{X} \subseteq X_0(A)$ a system which is closed with respect to arbitrary intersections and under unions of directed subsystems and which contains the identity relation id_A.

Let $(A, \mathscr{X})$, $(B, \mathscr{Y})$ be x-systems, $\varphi : A \to B$. Then we say that φ is an *x-morphism* from $(A, \mathscr{X})$ to $(B, \mathscr{Y})$ if $\varrho\varphi^{-1} \in \mathscr{X}$ for each $\varrho \in \mathscr{Y}$. Now, we denote by $\mathbf{X}$ the category whose class of objects is exactly the class of all x-systems and whose morphisms are precisely the x-morphisms between these x-systems. Evidently, always $(A, X_0(A)) \in \mathbf{X}$.

Theorem 7. *Let $\mathfrak{A} = (A, F)$ be a partial algebra, $\mathscr{X} = \mathscr{X}(\mathfrak{A})$ the set of all $\mathscr{X}$-relations on $\mathfrak{A}$ and let $(B, \mathscr{Y})$ be an x-system. Then a mapping $\varphi : A \to B$ is a morphism from $(A, \mathscr{X})$ to $(B, \mathscr{Y})$ in $\mathbf{X}$ if and only if for each $f \in F$ there exists an n_f-ary partial operation g on B such that $(x_1, \ldots, x_{n_f}) \in \mathrm{D}(f, A)$ implies $(x_1\varphi, \ldots, x_{n_f}\varphi) \in \mathrm{D}(g, B)$ and $x_1 \ldots x_{n_f} f\varphi = x_1\varphi \ldots x_{n_f}\varphi g$.*

Proof. a) Let $\varphi : (A, \mathscr{X}) \to (B, \mathscr{Y}) \in \mathrm{Mor}\, \mathbf{X}$, $f \in F$. Let us define an n_f-ary partial operation g on B as follows: $(y_1, \ldots, y_{n_f}) \in \mathrm{D}(g, B)$ if and only if there exists $(x_1, \ldots, x_{n_f}) \in \mathrm{D}(f, A)$ such that $x_1\varphi = y_1, \ldots, x_{n_f}\varphi = y_{n_f}$ and $y_1 \ldots y_{n_f} g =$ $= x_1 \ldots x_{n_f} f\varphi$. Let $\varrho \in \mathscr{Y}$. Then there exists $\tau \in \mathscr{X}$ such that $\tau = \varrho\varphi^{-1}$. Suppose that $(y_1, \ldots, y_{n_f})$, $(z_1, \ldots, z_{n_f}) \in \mathrm{D}(g, B)$ and that $(y_1, z_1), \ldots, (y_{n_f}, z_{n_f}) \in \varrho$. Let $x_1, \ldots,$ $x_{n_f}, u_1, \ldots, u_{n_f} \in A$ be such that $x_1\varphi = y_1, \ldots, x_{n_f}\varphi = y_{n_f}$, $u_1\varphi = z_1, \ldots, u_{n_f}\varphi =$

$= z_{n_f}$, $(x_1, \ldots, x_{n_f})$, $(u_1, \ldots, u_{n_f}) \in \mathrm{D}(f, A)$. Then $(x_1, u_1), \ldots, (x_{n_f}, u_{n_f}) \in \tau$ and thus $x_1 \ldots x_{n_f} f \tau u_1 \ldots u_{n_f} f$. But hence $x_1 \ldots x_{n_f} f \varphi \varrho u_1 \ldots u_{n_f} f \varphi$, therefore $y_1 \ldots y_{n_f} g \varrho z_1 \ldots z_{n_f} g$.

b) Let $\varphi : A \to B$ be such mapping that for each $f \in F$ there exists an n_f-ary partial operation g on B compatible with all $\varrho \in \mathscr{Y}$ and satisfying the conditions from the assumption of Theorem. Let $\varrho \in \mathscr{Y}$, $f \in F$, $(x_1, \ldots, x_{n_f})$, $(y_1, \ldots, y_{n_f}) \in \mathrm{D}(f, A)$, $(x_1, y_1), \ldots, (x_{n_f}, y_{n_f}) \in \varrho\varphi^{-1}$. Then, by the assumption, $(x_1\varphi, y_1\varphi), \ldots, (x_{n_f}\varphi, y_{n_f}\varphi) \in \varrho$, hence $(x_1\varphi \ldots x_{n_f}\varphi g, y_1\varphi \ldots y_{n_f}\varphi g) \in \varrho$. But this implies $(x_1 \ldots x_{n_f} f, y_1 \ldots x_{n_f} f) \in \varrho\varphi^{-1}$. Therefore $\varrho\varphi^{-1}$ is compatible with all partial operations of F, i.e. $\varrho\varphi^{-1} \in \mathscr{X}$.

Theorem 8. *Let* $\varphi : (A, \mathscr{X}) \to (B, \mathscr{Y}) \in \mathrm{Mor}\,\mathbf{X}$. *Then*

a) *φ is a monomorphism iff it is injective;*

b) *φ is an epimorphism iff it is surjective.*

Proof. a) Let $\varphi : (A, \mathscr{X}) \to (B, \mathscr{Y}) \in \mathrm{Mor}\,\mathbf{X}$, $a_1, a_2 \in A$, $a_1 \neq a_2$, $a_1\varphi = a_2\varphi$. Let $(C, \mathscr{Z}) \in \mathrm{Ob}\,\mathbf{X}$ be such that $C = \{c\}$, $\mathscr{Z} = \{C \times C\}$. Let us suppose that ψ_1, $\psi_2 : C \to A$ are such mappings that $c\psi_1 = a_1$, $c\psi_2 = a_2$. If $\varrho \in \mathscr{X}$, then from the reflexivity it follows $\varrho\psi_1^{-1} = \varrho\psi_2^{-1} = \{C \times C\} \in \mathscr{Z}$. This means that ψ_1, ψ_2 are morphisms from $(C, \mathscr{Z})$ to $(A, \mathscr{X})$, therefore φ is not a monomorphism in $\mathbf{X}$.

b) Let $\varphi : (A, \mathscr{X}) \to (B, \mathscr{Y}) \in \mathrm{Mor}\,\mathbf{X}$ and let $B \backslash A\varphi \neq \emptyset$. Let $b \in B \backslash A\varphi$, $d \notin B$, $D = B \cup \{d\}$, $\mathscr{U} = \{D \times D, \mathrm{id}_D\}$. Let us suppose that $\chi_1, \chi_2 : B \to D$ are defined by $B\chi_1 = 1_{B,D}$, $\chi_2 \mid (B \backslash \{b\}) = \chi_1 \mid (B \backslash \{b\})$, $b\chi_2 = d$. Then $(D \times D)\,\chi_1^{-1} = (D \times D)\,\chi_2^{-1} = B \times B \in \mathscr{Y}$, $\mathrm{id}_D\chi_1^{-1} = \mathrm{id}_D\chi_2^{-1} = \mathrm{id}_B \in \mathscr{Y}$, hence χ_1, χ_2 are morphisms from $(B, \mathscr{Y})$ to $(D, \mathscr{U})$. Since $\varphi\chi_1 = \varphi\chi_2$, φ is not an epimorphism in $\mathbf{X}$.

Remark 1. a) The category $\mathbf{X}$ is not balanced.

b) The isomorphisms in $\mathbf{X}$ are exactly all bijective morphisms such that their inverse mappings are morphisms, too.

Remark 2. Let us consider any subobject of an object $(A, \mathscr{X})$ in $\mathbf{X}$. Then it is possible to choose as a representative of this subobject such a monomorphism $(A', \mathscr{X}') \to (A, \mathscr{X})$ that $(A', \mathscr{X}') \in \mathrm{Ob}\,\mathbf{X}$, $A' \subseteq A$, $\mathscr{X}' \supseteq \{\varrho \cap A' \times A';\ \varrho \in \mathscr{X}\}$.

Remark 3. Let us consider any quotient object of an object $(A, \mathscr{X})$ of $\mathbf{X}$. Let $\varphi : (A, \mathscr{X}) \to (B, \mathscr{Y})$ be a representative of this quotient object. Then evidently the mapping $\varphi : A \to B$ is surjective. If we denote the natural mapping of A onto the quotient set $A/\varphi\varphi^{-1}$ by ν_φ, then there exists a bijection $f_\varphi : B \to A/\varphi\varphi^{-1}$ such that $\varphi f_\varphi = \nu_\varphi$. Denote by $\mathscr{X}/\varphi\varphi^{-1}$ the set of all relations on $A/\varphi\varphi^{-1}$ induced by such relations $\sigma \in \mathscr{X}$ for which there exist such $\varrho \in \mathscr{Y}$ that $\sigma = \varrho\varphi^{-1}$.

It holds $(a_1, a_2) \in \mathrm{id}_B\varphi^{-1}$ if and only if $a_1\varphi = a_2\varphi$ $(a_1, a_2 \in A)$. Hence $\mathrm{id}_{A/\varphi\varphi^{-1}}$ is induced by $\mathrm{id}_B\varphi^{-1}$.

Let $\Theta_\gamma \in \mathscr{X}/\varphi\varphi^{-1}$ $(\gamma \in \Gamma)$, let Θ_γ be induced by $\sigma_\gamma \in \mathscr{X}$ and let $\sigma_\gamma = \varrho_\gamma\varphi^{-1}$ $(\varrho_\gamma \in \mathscr{Y}$, $\gamma \in \Gamma)$. Then $\bigcap_{\gamma \in \Gamma} \sigma_\gamma = (\bigcap_{\gamma \in \Gamma} \varrho_\gamma)\,\varphi^{-1}$. Let $a_1, a_2 \in A$. Then $(a_1\varphi\varphi^{-1}, a_2\varphi\varphi^{-1}) \in \bigcap_{\gamma \in \Gamma} \Theta_\gamma$ if and only if $(a_1\varphi\varphi^{-1}, a_2\varphi\varphi^{-1}) \in \Theta_\gamma$ for each $\gamma \in \Gamma$. This is true if and only if for each $\gamma \in \Gamma$ there exist $x_1^\gamma \in a_1\varphi\varphi^{-1}$, $x_2^\gamma \in a_2\varphi\varphi^{-1}$ such that $(x_1^\gamma, x_2^\gamma) \in \sigma_\gamma$. By the assumption, the last fact is equivalent to $(a_1\varphi, a_2\varphi) \in \varrho_\gamma$ for each $\gamma \in \Gamma$, and this holds if and only if $(a_1\varphi, a_2\varphi) \in \bigcap_{\gamma \in \Gamma} \varrho_\gamma$. This is true if and only if $(a_1, a_2) \in \bigcap_{\gamma \in \Gamma} \sigma_\gamma$. Therefore $(a_1\varphi\varphi^{-1}, a_2\varphi\varphi^{-1})$ belongs to the relation $(\bigcap_{\gamma \in \Gamma} \sigma_\gamma)/\varphi\varphi^{-1}$ induced on $A/\varphi\varphi^{-1}$ by the relation $\bigcap_{\gamma \in \Gamma} \sigma_\gamma$.

Conversely, let $(a_1\varphi\varphi^{-1}, a_2\varphi\varphi^{-1}) \in (\bigcap_{\gamma \in \Gamma} \sigma_\gamma)/\varphi\varphi^{-1}$. This is satisfied if and only if there exist $x_1 \in a_1\varphi\varphi^{-1}, x_2 \in a_2\varphi\varphi^{-1}$ such that $(x_1, x_2) \in \bigcap_{\gamma \in \Gamma} \sigma_\gamma$. Therefore $(a_1\varphi\varphi^{-1}, a_2\varphi\varphi^{-1}) \in \bigcap_{\gamma \in \Gamma} \Theta_\gamma$. Hence $\bigcap_{\gamma \in \Gamma} \Theta_\gamma = (\bigcap_{\gamma \in \Gamma} \sigma_\gamma)/\varphi\varphi^{-1}$.

Similarly we can prove that if $(\Theta_\gamma;\ \gamma \in \Gamma)$ is a directed subsystem in $\mathscr{X}/\varphi\varphi^{-1}$ and if Θ_γ is induced by $\sigma_\gamma \in \mathscr{X}$ $(\gamma \in \Gamma)$, then $\bigcup_{\gamma \in \Gamma} \Theta_\gamma$ is induced by $\bigcup_{\gamma \in \Gamma} \sigma_\gamma$.

Therefore $(A/\varphi\varphi^{-1}, \mathscr{X}/\varphi\varphi^{-1}) \in \mathrm{Ob}\,\mathbf{X}$.

Let us show that f_φ is an isomorphism from $(B, \mathscr{Y})$ onto $(A/\varphi\varphi^{-1}, \mathscr{X}/\varphi\varphi^{-1})$. Let $\bar{\varrho} \in \mathscr{X}/\varphi\varphi^{-1}$ be induced by $\varrho\varphi^{-1}$, where $\varrho \in \mathscr{Y}$. Let $(\bar{a}_1, \bar{a}_2) \in \bar{\varrho}$. Then there exist $(x_1, x_2) \in \varrho\varphi^{-1}, x_1 \in \bar{a}_1, x_2 \in \bar{a}_2$, and hence $(x_1\varphi, x_2\varphi) \in \varrho$. Therefore also $(a_1\varphi, a_2\varphi) \in \varrho$, thus $(\bar{a}_1 f_\varphi^{-1}, \bar{a}_2 f_\varphi^{-1}) \in \varrho$. If $(b_1, b_2) \in \varrho$, then $(b_1\varphi^{-1}, b_2\varphi^{-1}) \in \bar{\varrho}$. Hence $\varrho f_\varphi^{-1} = \varrho \in \mathscr{Y}$, i.e. f_φ is a morphism from $(B, \mathscr{Y})$ to $(A/\varphi\varphi^{-1}, \mathscr{X}/\varphi\varphi^{-1})$.

Let $\varrho \in \mathscr{Y}$, $(b_1, b_2) \in \varrho$. Then $(b_1\varphi^{-1}, b_2\varphi^{-1}) \in \bar{\varrho}$. If $y_1, y_2 \in B$, $(y_1\varphi^{-1}, y_2\varphi^{-1}) \in \bar{\varrho}$, then there exist $z_1 \in y_1\varphi^{-1}$, $z_2 \in y_2\varphi^{-1}$ such that $(z_1, z_2) \in \varrho\varphi^{-1}$, thus $(z_1\varphi, z_2\varphi) \in \varrho$, and so $(y_1, y_2) \in \varrho$. Therefore f_φ^{-1} is a morphism from $(A/\varphi\varphi^{-1}, \mathscr{X}/\varphi\varphi^{-1})$ to $(B, \mathscr{Y})$.

This means that for any quotient object of an object $(A, \mathscr{X})$ in $\mathbf{X}$ it is possible to choose as a representative of this quotient object such an epimorphism $\varphi : (A, \mathscr{X}) \to (\bar{A}, \bar{\mathscr{X}})$ that $\bar{A} = A/\sim$ for a certain equivalence relation $\sim$ on A and that $\bar{\mathscr{X}}$ is a subset of the set of all X-relations on $\bar{A}$ induced by the elements of $\mathscr{X}$ containing $\sim$.

Specially, let us suppose that X contains the transitivity quasi-identity. Let $(A, \mathscr{X}) \in \mathrm{Ob}\,\mathbf{X}$ and let $\sim$ be an arbitrary equivalence on A. Let us denote by $\mathscr{X}/\sim$ the system of all X-relations on the quotient set $A/\sim$ induced by such $\varrho \in \mathscr{X}$ that $\varrho \supseteq \sim$. If $v_\sim : A \to A/\sim$ is the natural mapping, then we denote $\bar{\varrho} = \varrho v_\sim$ for any such ϱ. Let $(\bar{a}_1, \bar{a}_2) \in \bar{\varrho}$. Then there exist $x_1 \in \bar{a}_1$, $x_2 \in \bar{a}_2$, $(x_1, x_2) \in \varrho$. Thus (a_1, x_1), (x_1, x_2), $(x_2, a_2) \in \varrho$, therefore also $(a_1, a_2) \in \varrho$. This implies $\bar{\varrho} v_\sim^{-1} = \varrho \in \mathscr{X}$, and so $v_\sim : (A, \mathscr{X}) \to (A/\sim, \mathscr{X}/\sim) \in \mathrm{Mor}\,\mathbf{X}$.

Theorem 9. *If $A = \{a\}$, then $(A, \{A \times A\})$ is a free object over $\mathbf{X}$ with the free generating set $\{a\}$.*

Proof. Let $(B, \mathscr{Y}) \in \mathrm{Ob}\,\mathbf{X}$, $\varphi : \{a\} \to B$. If $\varrho \in \mathscr{Y}$, then, by the reflexivity of ϱ, it follows $\varrho\varphi^{-1} = A \times A$, hence $\varphi : (A, \{A \times A\}) \to (B, \mathscr{Y}) \in \mathrm{Mor}\,\mathbf{X}$ and this morphism is the unique extension of the mapping φ.

Corollary. $\mathbf{X}$ *is a concrete category.*

Theorem 10. *An object* $(A, \mathscr{X}) \in \mathrm{Ob}\,\mathbf{X}$ *is injective in* $\mathbf{X}$ *if and only if* A *is a one-element set (and* $\mathscr{X} = \{A \times A\}$*).*

Proof. Let $(A, \mathscr{X}), (B, \mathscr{Y}), (B', \mathscr{Y}') \in \mathrm{Ob}\,\mathbf{X}$, $\varphi : (B', \mathscr{Y}') \to (B, \mathscr{Y})$, $\psi : (B', \mathscr{Y}') \to$ $\to (A, \mathscr{X}) \in \mathrm{Mor}\,\mathbf{X}$, and let φ be a monomorphism. By Remark 2 it is possible to assume that $B' \subseteq B$, $\mathscr{Y}' \supseteqq \{\varrho \cap B' \times B';\ \varrho \in \mathscr{Y}\}$ and $\varphi = 1_{B',B}$.

Let us assume that $(A, \mathscr{X})$ is injective in $\mathbf{X}$. Let $A = B = B'$, $\mathscr{X} = \mathscr{Y}'$, $\mathscr{Y}' \supset \mathscr{Y}$ and let $\psi = 1_A$. Then $\chi = 1_A$ is the unique mapping of B to A for which $\varphi\chi = \psi$. But, by the assumption, there exists $\varrho \in \mathscr{X}$ such that $\varrho\psi^{-1} \notin \mathscr{Y}$, i.e. χ is not a morphism from $(B, \mathscr{Y})$ to $(A, \mathscr{X})$. Hence, if $(A, \mathscr{X})$ is injective, then $\mathscr{X} = X_0(A)$.

Let card $A \geqq 2$. Let us suppose that $B = B' = \{b_1, b_2\}$, $\mathscr{Y} = \{B \times B,\ \mathrm{id}_B\}$, $\mathscr{Y}' = X_0(B)$. Let $a_1, a_2 \in A$, $a_1 \neq a_2$ and let $\psi : B' \to A$ be a mapping such that $b_1\psi = a_1$, $b_2\psi = a_2$. Since a coimage of an X-relation is in any mapping also an X-relation, ψ is a morphism from $(B', \mathscr{Y}')$ to $(A, \mathscr{X})$. Furthermore $1_{B',B}$ is a monomorphism from $(B', \mathscr{Y}')$ to $(B, \mathscr{Y})$ and $\chi = \psi$ is the unique mapping of B to A for which $1_{B',B}\chi = \psi$.

Let us show that $\chi \notin \mathrm{Mor}\,\mathbf{X}$. Let $\varrho = \mathrm{id}_A \cup \{a_1, a_2\}$. Then $\varrho \in X_0(A)$. But $\varrho\chi^{-1} =$ $= \mathrm{id}_B \cup \{(b_1, b_2)\} \notin \mathscr{Y}$.

Finally, it is clear that $(A, \mathscr{X})$ is injective for $A = \{a\}$.

Theorem 11. *An object* $(A, \mathscr{X})$ *is projective in* $\mathbf{X}$ *if and only if* $\mathscr{X} = X_0(A)$.

Proof. a) Let us suppose that $(A, \mathscr{X}) \in \mathrm{Ob}\,\mathbf{X}$ is projective. Let $A = B$ and $\sim\ = \mathrm{id}_B$. Let $\bar{\mathscr{Y}} = \mathscr{X}$ and $\bar{\mathscr{Y}} \subset \mathscr{Y}$. Suppose that $\varphi = \psi = 1_A$. Then $\varphi : (A, \mathscr{X}) \to$ $\to (\bar{B}, \bar{\mathscr{Y}})$, $\psi : (B, \mathscr{Y}) \to (\bar{B}, \bar{\mathscr{Y}}) \in \mathrm{Mor}\,\mathbf{X}$ and ψ is an epimorphism. Furthermore $\chi = 1_A$ is the unique mapping of A to B such that $\chi\psi = \varphi$. By the assumption there exists $\varrho \in \mathscr{Y}$ for which $\varrho\chi^{-1} \notin \mathscr{X}$, therefore χ is not a morphism from $(A, \mathscr{X})$ to $(B, \mathscr{Y})$. Hence it must be $\mathscr{X} = X_0(A)$.

b) Let $(A, \mathscr{X}), (B, \mathscr{Y}), (\bar{B}, \bar{\mathscr{Y}}) \in \mathrm{Ob}\,\mathbf{X}$, $\varphi : (A, \mathscr{X}) \to (\bar{B}, \bar{\mathscr{Y}})$, $\psi : (B, \mathscr{Y}) \to (\bar{B}, \bar{\mathscr{Y}}) \in$ $\in \mathrm{Mor}\,\mathbf{X}$ and let ψ be an epimorphism. Let us suppose that $\bar{B} = B/\sim$, where $\sim$ is an equivalence on B, and that ψ is the natural mapping. Let us denote by χ a mapping of A to B such that $a\chi \in a\varphi\varphi^{-1}$ for each $a \in A$. Then $\chi\psi = \varphi$ and χ is a morphism from $(A, \mathscr{X})$ to $(B, \mathscr{Y})$, because a coimage of an X-relation is an X-relation for each mapping.

Remark. a) If $(A, \mathscr{X}) \in \mathrm{Ob}\,\mathbf{X}$, $A' \subseteq A$, $\mathscr{X}' \supseteqq \{\varrho \cap A' \times A';\ \varrho \in \mathscr{X}\}$, then a necessary condition for $1_{A',A} : (A', \mathscr{X}') \to (A, \mathscr{X})$ being a section is $\mathscr{X}' = \{\varrho \cap A' \times A';\ \varrho \in \mathscr{X}\}$.

b) If $\mathscr{X}' = \{\varrho \cap A' \times A';\ \varrho \in \mathscr{X}\}$, $A'' = A \backslash A'$, then $1_{A',A} : (A', \mathscr{X}') \to (A, \mathscr{X})$ is a section if and only if there exists a mapping $\chi : A'' \to A'$ such that for each a_1, $a_2 \in A''$ and for each $\varrho \in \mathscr{X}$ it is $(a_1, a_2) \in \varrho \cap A'' \times A''$ if and only if $(a_1\chi, a_2\chi) \in$ $\in \varrho \cap A' \times A'$.

Theorem 12. $\mathbf{X}$ *is a complete category.*

Proof. a) We prove the existence of the kernels in $\mathbf{X}$. Let $\varphi, \psi : (A, \mathscr{X}) \to (B, \mathscr{Y}) \in \in \operatorname{Mor} \mathbf{X}$. Let us suppose that $A' \neq \emptyset$ and let us consider the embedding $1_{A',A} : (A', \mathscr{X}') \to (A, \mathscr{X})$, where $\mathscr{X}' = \{\varrho \cap A' \times A';\ \varrho \in \mathscr{X}\}$. Let $\chi : (C, \mathscr{Z}) \to \to (A, \mathscr{X}) \in \operatorname{Mor} \mathbf{X}$ and let $\chi\varphi = \chi\psi$. Then $C\chi = A'$ and hence for the mapping $\mu : C \to A'$ such that $c\chi = c\mu$ for each $c \in C$ there is $\mu 1_{A',A} = \chi$. Let $\varrho' \in \mathscr{X}'$. Then there exists $\varrho \in \mathscr{X}$ such that $\varrho' = \varrho \cap A' \times A'$. But this means that $\varrho'\mu^{-1} = \varrho\chi^{-1} \in \mathscr{Z}$. Hence $\mu : (C, \mathscr{Z}) \to (A', \mathscr{X}') \in \operatorname{Mor} \mathbf{X}$, and therefore $1_{A',A}$ is a kernel of φ and ψ.

Let $A' = \emptyset$. Then for $\emptyset : A' \to A$ it holds $\emptyset : (A', \mathscr{X}') \to (A, \mathscr{X}) \in \operatorname{Mor} \mathbf{X}$ and $\emptyset$ is a kernel of φ and ψ.

b) We shall show that there exist the products in $\mathbf{X}$. Let $(A_\gamma, \mathscr{X}_\gamma) \in \operatorname{Ob} \mathbf{X}$, $\gamma \in \Gamma$. Let us denote $A = \prod_{\gamma \in \Gamma} A_\gamma$. If $\bar{\varrho} = (\dots, \varrho_\gamma, \dots) \in \prod_{\gamma \in \Gamma} \mathscr{X}_\gamma$ $(\varrho_\gamma \in \mathscr{X}_\gamma)$, we denote by ϱ a relation on A such that for each $x = (\dots, x_\gamma, \dots)$, $y = (\dots, y_\gamma, \dots) \in A$ $(x_\gamma, y_\gamma \in A_\gamma$, $\gamma \in \Gamma)$ it holds $(x, y) \in \varrho$ if and only if $(x_\gamma, y_\gamma) \in \varrho_\gamma$ for each $\gamma \in \Gamma$. Evidently, ϱ is an X-relation on A. The fact that ϱ arises from $\bar{\varrho}$ by the above manner, we shall denote by $\varrho \leftrightarrow \bar{\varrho}$. We denote by $\mathscr{X}$ the set of all relations on A arising from the elements of $\prod_{\gamma \in \Gamma} \mathscr{X}_\gamma$. If $\varrho^{(\delta)} \in \mathscr{X}$, $\delta \in \Delta$, then we shall write $\varrho = \bigcap_{\delta \in \Delta} \varrho^{(\delta)}$. Let $\varrho^{(\delta)} \leftrightarrow (\dots, \varrho_\gamma^{(\delta)}, \dots)$, $\delta \in \Delta$. Let us suppose that $x = (\dots, x_\gamma, \dots)$, $y = (\dots, y_\gamma, \dots) \in A$ and that $(x, y) \in \varrho$. But this is in the case if and only if for each $\gamma \in \Gamma$ it is $(x_\gamma, y_\gamma) \in \bigcap_{\delta \in \Delta} \varrho_\gamma^{(\delta)}$. By the assumption, $\mathscr{X}_\gamma$ is closed under intersections, hence $\bigcap_{\delta \in \Delta} \varrho_\gamma^{(\delta)} \in \mathscr{X}_\gamma$. Therefore it holds $\varrho \leftrightarrow (\dots, \bigcap_{\delta \in \Delta} \varrho_\gamma^{(\delta)}, \dots)$, and so $\varrho \in \mathscr{X}$. Hence $\mathscr{X}$ is also closed with respect to intersections.

Let, in addition, the system $(\varrho^{(\delta)}; \delta \in \Delta)$ be directed. We shall write $\tau = \bigcup_{\delta \in \Delta} \varrho^{(\delta)}$. Then for $x = (\dots, x_\gamma, \dots)$, $y = (\dots, y_\gamma, \dots)$ it is $(x, y) \in \tau$ if and only if $(x_\gamma, y_\gamma) \in \in \bigcup_{\gamma \in \Gamma} \varrho_\gamma^{(\delta)}$ for each $\gamma \in \Gamma$. Indeed, for each $\gamma \in \Gamma$ it holds that the system $(\varrho_\gamma^{(\delta)}; \delta \in \Delta)$ is directed, hence $\bigcup_{\gamma \in \Gamma} \varrho_\gamma^{(\delta)} \in \mathscr{X}_\gamma$. Therefore $\tau \leftrightarrow (\dots, \bigcup_{\delta \in \Delta} \varrho_\gamma^{(\delta)}, \dots)$, i.e. $\tau \in \mathscr{X}$.

Finally, it is evident that $\mathrm{id}_A \leftrightarrow (\dots, \mathrm{id}_{A_\gamma}, \dots)$, hence $\mathrm{id}_A \in \mathscr{X}$.

Therefore $(A, \mathscr{X}) \in \operatorname{Ob} \mathbf{X}$.

Now, let $\varphi_\gamma : A \to A_\gamma$ be the projection for each $\gamma \in \Gamma$. Let $\gamma_0 \in \Gamma$, $\tau_{\gamma_0} \in X_{\gamma_0}$. It holds that $\tau_{\gamma_0}\varphi_{\gamma_0}^{-1}$ contains exactly all ordered pairs $((\dots, x_{\gamma_0}, \dots), (\dots, y_{\gamma_0}, \dots))$, where $(x_{\gamma_0}, y_{\gamma_0}) \in \tau_{\gamma_0}$. But this means that $\tau_{\gamma_0}\varphi_{\gamma_0}^{-1} \in \mathscr{X}$, because $\tau_{\gamma_0}\varphi_{\gamma_0}^{-1} \leftrightarrow (\dots, \varrho_\gamma, \dots)$, where $\varrho_\gamma = A_\gamma \times A_\gamma$ for $\gamma \neq \gamma_0$, and $\varrho\gamma_0 = \tau_{\gamma_0}$. Thus φ_γ is a morphism from $(A, \mathscr{X})$ to $(A_\gamma, \mathscr{X}_\gamma)$ for each $\gamma \in \Gamma$.

Let $\psi_\gamma : (B, \mathscr{Y}) \to (A_\gamma, \mathscr{X}_\gamma) \in \operatorname{Mor} \mathbf{X}$ for each $\gamma \in \Gamma$. Let us show that the product ψ of the mappings ψ_γ $(\gamma \in \Gamma)$ is a morphism from $(B, \mathscr{Y})$ to $(A, \mathscr{X})$. Let $\varrho \in \mathscr{X}$, $\varrho \leftrightarrow \leftrightarrow (\dots, \varrho_\gamma, \dots)$, $(b_1, b_2) \in \varrho\psi^{-1}$. Then $(b_1\psi, b_2\psi) \in \varrho$, i.e. $(b_1\psi_\gamma, b_2\psi_\gamma) \in \mathscr{X}_\gamma$ for each $\gamma \in \Gamma$. Thus $(b_1, b_2) \in \varrho_\gamma\psi_\gamma^{-1}$ for each $\gamma \in \Gamma$, hence $(b_1, b_2) \in \bigcap_{\gamma \in \Gamma} \varrho_\gamma\psi_\gamma^{-1}$. Since $\varrho_\gamma\psi_\gamma^{-1} \in \mathscr{Y}$

for each $\gamma \in \Gamma$, $\bigcap_{\gamma \in \Gamma} \varrho_\gamma \psi_\gamma^{-1} \in \mathscr{Y}$. Let $(c_1, c_2) \in \bigcap_{\gamma \in \Gamma} \varrho_\gamma \psi_\gamma^{-1}$. Then $(c_1\psi_\gamma, c_2\psi_\gamma) \in \varrho_\gamma$ for each $\gamma \in \Gamma$, therefore $((\ldots, c_1\psi_\gamma, \ldots), (\ldots, c_2\psi_\gamma, \ldots)) \in \varrho$, hence $(c_1, c_2) \in \varrho\psi^{-1}$. Thus $\varrho\psi^{-1} = \bigcap_{\gamma \in \Gamma} \varrho_\gamma \psi_\gamma^{-1} \in \mathscr{Y}$, and so $\psi \in \operatorname{Mor} \mathbf{X}$. But this means that $((A, \mathscr{X}), \varphi_\gamma; \gamma \in \Gamma)$ is the product of the objects $(A_\gamma, \mathscr{X}_\gamma)$, $\gamma \in \Gamma$.

Theorem 13. *$\mathbf{X}$ is a cocomplete category.*

Proof. a) We shall denote the existence of the cokernels in $\mathbf{X}$. Let $\varphi, \psi : (A, \mathscr{X}) \to (B, \mathscr{Y}) \in \operatorname{Mor} \mathbf{X}$ and let $\sim$ be the smallest equivalence on B containing all ordered pairs $(a\varphi, a\psi)$, where $a \in A$. Then the natural morphism $v_\sim : (B, \mathscr{Y}) \to (B/\sim, \mathscr{Y}/\sim)$ is a coequalizer of φ and ψ.

Let $\chi : (B, \mathscr{Y}) \to (C, \mathscr{Z}) \in \operatorname{Mor} \mathbf{X}$ be a coequalizer of φ and ψ. Let us consider the equivalence relation $\chi\chi^{-1}$ on B. Evidently $\sim \subseteq \chi\chi^{-1}$. Let us denote by $\chi' : B/\sim \to B/\chi\chi^{-1}$ the natural mapping. Let $\tau : B/\chi\chi^{-1} \to C$ be a mapping such that $b\chi\chi^{-1}\tau = b\chi$ for each $b \in B$. Then $\chi = v_\sim \chi' \tau$. Moreover, it is evident that $\chi'\tau \in \operatorname{Mor} \mathbf{X}$. Hence $v_\sim$ is the cokernel of φ and ψ.

b) We shall show that there exist the coproducts in $\mathbf{X}$. Let $(A_\gamma, \mathscr{X}_\gamma) \in \operatorname{Ob} \mathbf{X}$, $\gamma \in \Gamma$, and let $A = \dot{\bigcup}_{\gamma \in \Gamma} A_\gamma$. (The symbol $\dot{\cup}$ means always the disjoint union.) Let $\Sigma \subseteq \Gamma$, $\overline{\mathscr{X}_\Sigma} = \prod_{\gamma \in \Gamma} \mathscr{X}_\gamma$. If $\bar{\varrho} = (\ldots, \varrho_\sigma, \ldots) \in \overline{\mathscr{X}_\Sigma}$, $\varrho_\sigma \in \mathscr{X}_\sigma$, then we denote by ϱ such relation on A that

$$\varrho = \dot{\bigcup_{\sigma \in \Sigma}} \varrho_\sigma \;\dot{\cup} \dot{\bigcup_{\gamma, \varepsilon \in \Gamma \setminus \Sigma}} A_\gamma \times A_\varepsilon .$$

The fact, that ϱ arises from $\bar{\varrho}$ by this manner, will be denoted by $\varrho \approx \bar{\varrho}$. Now, let $\mathscr{X}_\Sigma$ be the set of all such relations ϱ on A. Further, we shall denote $\mathscr{X} = \dot{\bigcup}_{\Sigma \subseteq \Gamma} \mathscr{X}_\Sigma$.

It holds $\mathscr{X} \subseteq X_0(A)$. Indeed, let $\Sigma \subseteq \Gamma$, $\bar{\varrho} = (\ldots, \varrho_\sigma, \ldots) \in \overline{\mathscr{X}_\Sigma}$, $\varrho_\sigma \in \mathscr{X}_\sigma$, $\sigma \in \Sigma$, $\varrho \approx \bar{\varrho}$. Then for $a \in A_\sigma$, $\sigma \in \Sigma$, it is $(a, a) \in \varrho_\sigma$. For $a \in A_\gamma$, $\gamma \in \Gamma \setminus \Sigma$, it is $(a, a) \in A_\gamma \times A_\gamma$. Therefore ϱ is reflexive. Further, let $\forall x_1 \ldots \forall x_n(\mathscr{A}_1 \& \ldots \& \mathscr{A}_p \Rightarrow \mathscr{A})$ be a quasi-identity from X. Let us suppose that $a_1, \ldots, a_n \in A$ and that it holds $\mathscr{A}_1^\varrho(a_1, \ldots, a_n), \ldots, \mathscr{A}_p^\varrho(a_1, \ldots, a_n)$. Then the following cases are possible:

1. $a_1, \ldots, a_n \in A_\sigma$, $\sigma \in \Sigma$;
2. $a_1 \in A_{\alpha_1}, \ldots, a_n \in A_{\alpha_n}$, $\alpha_1, \ldots, \alpha_n \in \Gamma \setminus \Sigma$.

In the first case it is $\mathscr{A}_1^{\varrho_\sigma}(a_1, \ldots, a_n), \ldots, A_p^{\varrho_\sigma}(a_1, \ldots, a_n)$, hence also $\mathscr{A}^{\varrho_\sigma}(a_1, \ldots, a_n)$. And since $\varrho_\sigma \subseteq \varrho$, it holds $\mathscr{A}^\varrho(a_1, \ldots, a_n)$.

Let us suppose that $\mathscr{A} = A_1^2(x_r, x_p)$, $r, q \in \{1, \ldots, n\}$. Then in the second case, $(a_r, a_q) \in A_{\alpha_r} \times A_{\alpha_q} \subseteq \varrho$, thus $\mathscr{A}^\varrho(a_1, \ldots, a_n)$.

Therefore ϱ satisfies all quasi-identities from X.

Let us suppose that $\varrho^{(\delta)} \in \mathscr{X}$, $\delta \in \Delta$. We shall write $\varrho = \bigcap_{\delta \in \Delta} \varrho^{(\delta)}$. Let $\varrho^{(\delta)} \approx (\ldots, \varrho_\sigma^{(\delta)}, \ldots) \in \overline{\mathscr{X}_{\Sigma(\delta)}}$, $\varrho_\sigma^{(\delta)} \in \mathscr{X}_\sigma$, $\sigma \in \Sigma(\delta) \subseteq \Gamma$. We shall show that $\varrho \in \mathscr{X}_\Sigma$, where $\Sigma = \bigcup_{\sigma \in \Sigma} \Sigma(\delta)$, and that $\varrho_\sigma = \bigcap_{\sigma \in \Sigma(\delta)} \varrho_\sigma^{(\delta)}$ for each $\sigma \in \Sigma$.

It is clear that if $\gamma_1, \gamma_2 \in \Gamma$, then the component of ϱ in $A_{\gamma_1} \times A_{\gamma_2}$ is equal to the intersection of the components of all relations $\varrho^{(\delta)}(\delta \in \Delta)$ in $A_{\gamma_1} \times A_{\gamma_2}$. If $\sigma \in \Sigma$, then this intersection on A_σ is $\varrho_\sigma = \bigcap_{\sigma \in \Sigma(\delta)} \varrho_\sigma^{(\delta)}$, and hence ($\mathscr{X}_\sigma$ is closed with respect to intersections) $\varrho_\sigma \in \mathscr{X}_\sigma$. Let $\sigma \in \Sigma$, $\gamma \in \Gamma$, $\sigma \neq \gamma$. Then at least for one $\delta_0 \in \Delta$ it holds that the component $\varrho^{(\delta_0)}$ in $A_\sigma \times A_\gamma$ (in $A_\gamma \times A_\sigma$) is void. Hence also the intersection of the components of all ϱ_δ $(\delta \in \Delta)$ in $A_\sigma \times A_\gamma$ (in $A_\gamma \times A_\sigma$) is void.

Finally, let $\gamma_1, \gamma_2 \in \Gamma \backslash \Sigma$. Then for each relation $\varrho^{(\delta)}$ $(\delta \in \Delta)$ it holds that its component in $A_{\gamma_1} \times A_{\gamma_2}$ is equal to $A_{\gamma_1} \times A_{\gamma_2}$, and this is equal to the intersection of all such components.

Therefore, $\mathscr{X}$ is closed under intersections. Let, in addition, the system $(\varrho^{(\delta)}, \delta \in \Delta)$ be directed. Let us write $\tau = \bigcup_{\delta \in \Delta} \varrho^{(\delta)}$. Evidently, for each $\gamma_1, \gamma_2 \in \Gamma$, the component of τ in $A_{\gamma_1} \times A_{\gamma_2}$ is equal to the union of the components of all relations $\varrho^{(\delta)}$ $(\delta \in \Delta)$ in $A_{\gamma_1} \times A_{\gamma_2}$. Let us write $\Sigma' = \bigcap_{\delta \in \Delta} \Sigma(\delta)$. Let $\gamma \in \Gamma$. If $\gamma \in \Sigma'$, then for each $\delta \in \Delta$, the component $\varrho_\gamma^{(\delta)}$ of a relation $\varrho^{(\delta)}$ on A_γ is an element of $\mathscr{X}_\gamma$. Moreover, from the directedness of the system $(\varrho^{(\delta)}, \delta \in \Delta)$ there follows also the directedness of the system $(\varrho_\gamma^{(\delta)}, \delta \in \Delta)$, hence $\bigcup_{\delta \in \Delta} \varrho_\gamma^{(\delta)} \in \mathscr{X}_\gamma$. If $\gamma_1, \gamma_2 \in \Gamma \backslash \Sigma'$, then there exists at least one $\delta_0 \in \Delta$ such that the component of $\varrho^{(\delta_0)}$ in $A_{\gamma_1} \times A_{\gamma_2}$ is equal to $A_{\gamma_1} \times A_{\gamma_2}$, therefore also the component of τ in $A_{\gamma_1} \times A_{\gamma_2}$ is equal to $A_{\gamma_1} \times A_{\gamma_2}$. Let $\gamma_1 \in \Sigma'$, $\gamma_2 \in \Gamma$. Then the component of each $\varrho^{(\delta)}$ $(\delta \in \Delta)$ in $A_{\gamma_1} \times A_{\gamma_2}$ (in $A_{\gamma_2} \times A_{\gamma_1}$) is void, hence also the component of τ in $A_{\gamma_1} \times A_{\gamma_2}$ (in $A_{\gamma_2} \times A_{\gamma_1}$) is void.

Therefore $\tau \in \mathscr{X}_{\Sigma'}$, and $\tau_\sigma = \bigcup_{\sigma \in \Sigma(\delta)} \varrho_\sigma^{(\delta)} (\sigma \in \Sigma')$. Finally, it is evident that $\mathrm{id}_A \in \mathscr{X}_\Gamma$, hence $\mathrm{id}_A \in \mathscr{X}$.

Then we have known that $(A, \mathscr{X}) \in \mathrm{Ob}\, \mathbf{X}$.

Now, we denote by φ_γ the embedding of A_γ in A $(\gamma \in \Gamma)$. Let $\varrho \in \mathscr{X}_\Sigma$, $\Sigma \subseteq \Gamma$, $\varrho \approx (\ldots, \varrho_\sigma, \ldots)$, $\sigma \in \Sigma$, $\varrho_\sigma \in \mathscr{X}_\sigma$. If $\gamma \in \Gamma \backslash \Sigma$, then $\varrho\varphi_\gamma^{-1} = A_\gamma \times A_\gamma \in \mathscr{X}_\gamma$. If $\gamma \in \Sigma$, then $\varrho\varphi_\gamma^{-1} = \varrho_\gamma \in \mathscr{X}_\gamma$. Therefore φ_γ is a morphism from $(A_\gamma, \mathscr{X}_\gamma)$ to $(A, \mathscr{X})$ for each $\gamma \in \Gamma$. Let us suppose that $\psi_\gamma : (A_\gamma, \mathscr{X}_\gamma) \to (B, \mathscr{X}) \in \mathrm{Mor}\, \mathbf{X}$ for each $\gamma \in \Gamma$. Let $\psi : A \to B$ be the unique mapping such that $\varphi_\gamma\psi = \psi_\gamma$ for each $\gamma \in \Gamma$. If $\tau \in \mathscr{Y}$, then $\tau\psi_\gamma^{-1} \in \mathscr{X}_\gamma$ for each $\gamma \in \Gamma$, therefore $\tau\psi^{-1} \approx (\ldots, \tau\psi_\gamma^{-1}, \ldots) \in \overline{\mathscr{X}_\Gamma}$. But this means that $\tau\psi^{-1} \in \mathscr{X}$, hence τ is a morphism from $(A, \mathscr{X})$ to $(B, \mathscr{Y})$. Therefore $((A, \mathscr{X}), \varphi_\gamma; \gamma \in \Gamma)$ is the coproduct of $(A_\gamma, \mathscr{X}_\gamma)$, $\gamma \in \Gamma$.

If the system X contains exactly the identity of reflexivity and the quasi-identity of transitivity (the identity of reflexivity and the quasi-identities of symmetry and transitivity), then we shall denote the corresponding category of all x-systems and all x-morphisms by $\mathbf{Q}$ (by $\mathbf{C}$). It holds

Theorem 14. *The category $\mathbf{C}$ is a reflective, full subcategory of the category $\mathbf{Q}$. The corresponding reflector preserves and reflects monomorphisms.*

Proof. Let $(A, \mathscr{Q}) \in \mathrm{Ob}\, \mathbf{Q}$. Let us denote by $\hat{\mathscr{Q}}$ the system of all equivalences on A

belonging to $\mathcal{Q}$. Let us suppose that $\varepsilon_\beta \in \hat{\mathcal{Q}}$, $\beta \in J$. Then $\bigcap_{\beta \in J} \varepsilon_\beta \in \mathcal{Q}$ and $\bigcap_{\beta \in J} \varepsilon_\beta$ is an equivalence on A, therefore $\bigcap_{\beta \in J} \varepsilon_\beta \in \hat{\mathcal{Q}}$. Let $\varepsilon_\beta \in \hat{\mathcal{Q}}$, $\beta \in J$, and let $\bigcup_{\beta \in J} \varepsilon_\beta \in \mathcal{Q}$. Then $\bigcup_{\beta \in J} \varepsilon_\beta$ is evidently symmetric, hence $\bigcup_{\beta \in J} \varepsilon_\beta \in \hat{\mathcal{Q}}$. This implies that if $\{\varepsilon_\beta;\ \beta \in J,\ \varepsilon_\beta \in \hat{\mathcal{Q}}\}$ is a directed system, then $\bigcup_{\beta \in J} \varepsilon_\beta \in \hat{\mathcal{Q}}$. And since $\mathrm{id}_A \in \hat{\mathcal{Q}}$, $(A, \hat{\mathcal{Q}}) \in \mathrm{Ob}\ \mathbf{C}$.

We shall show that $\mathbf{C}$ is a reflective subcategory of $\mathbf{Q}$. Let us define a function which assigns to each $(A, \mathcal{Q}) \in \mathrm{Ob}\ \mathbf{Q}$ the object $R((A, \mathcal{Q})) = (A, \hat{\mathcal{Q}}) \in \mathrm{Ob}\ \mathbf{C}$. Further, let us define a function which assigns to each $(A, \mathcal{Q}) \in \mathrm{Ob}\ \mathbf{Q}$ the morphism $\Phi_R((A, \mathcal{Q})) = 1_A : (A, \mathcal{Q}) \to (A, \hat{\mathcal{Q}}) \in \mathrm{Mor}\ \mathbf{Q}$. Let $(A, \mathcal{Q}) \in \mathrm{Ob}\ \mathbf{Q}$, $(B, \mathcal{F}) \in \mathrm{Ob}\ \mathbf{C}$, $\varphi : (A, \mathcal{Q}) \to (B, \mathcal{F}) \in \mathrm{Mor}\ \mathbf{Q}$. Let us suppose that $\varrho \in \mathcal{F}$, $a_1, a_2 \in A$, $(a_1, a_2) \in \varrho\varphi^{-1}$. Then $(a_1\varphi, a_2\varphi) \in \varrho$, thus $(a_2\varphi, a_1\varphi) \in \varrho$. Hence $(a_2, a_1) \in \varrho\varphi^{-1}$, i.e. $\varrho\varphi^{-1} \in \hat{\mathcal{Q}}$. But this means that $\mathcal{F}\varphi^{-1} \subseteq \hat{\mathcal{Q}}$, therefore φ is also a morphism from $(A, \hat{\mathcal{Q}})$ to $(B, \mathcal{F})$ in $\mathbf{C}$. And since $1_A \,.\, \varphi = \varphi$, the diagram

$$\begin{tikzcd} (A, \mathcal{Q}) \arrow[r, "\varphi"] \arrow[d, "\Phi_R((A, \mathcal{Q}))"'] & (B, \mathcal{F}) \\ R((A, \mathcal{Q})) \arrow[ur, "\varphi"'] & \end{tikzcd}$$

commutes.

Let $\psi : (A, \hat{\mathcal{Q}}) \to (B, \mathcal{F})$ be a morphism in $\mathbf{C}$ for which $\Phi_R((A, \mathcal{Q})) \,.\, \psi = \varphi$. But then $1_A \,.\, \psi = \varphi$, i.e. $\psi = \varphi$.

The corresponding reflector $R : \mathbf{Q} \to \mathbf{C}$ assigns to each morphism $\varphi : (A, \mathcal{Q}) \to (A_1, \mathcal{Q}_1) \in \mathrm{Mor}\ \mathbf{Q}$ the morphism $R(\varphi) : (A, \hat{\mathcal{Q}}) \to (A_1, \hat{\mathcal{Q}}_1) \in \mathrm{Mor}\ \mathbf{C}$ which as a mapping of the set A to the set A_1 is equal to φ.

The statement concerning the monomorphisms is now trivial.

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