Archivum Mathematicum

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Archivum Mathematicum, Vol. 16 (1980), No. 3, 153--159

Persistent URL: `http://dml.cz/dmlcz/107067`

ARCH. MATH. 3, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XVI: 153—160, 1980

A NOTE ON HIGHER MONOTONICITY PROPERTIES OF CERTAIN STURM LIOUVILLE FUNCTIONS

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(Received December 12, 1978)

1. DEFINITIONS AND NOTATIONS

A function $\varphi(x)$ is said to be n-times monotonic (or monotonic of order n) on an interval I if

(1.1) $$(-1)^i \varphi^{(i)}(x) \geqq 0 \qquad i = 0, 1, 2, \ldots, n; \; x \in I.$$

For such a function we write $\varphi(x) \in \mathcal{M}_n(I)$ or $\varphi(x) \in \mathcal{M}_n(a, b)$ in case that I is an open interval (a, b). In case the strict inequality holds throughout (1.1) we write $\varphi(x) \in \in \mathcal{M}_n^*(I)$ or $\varphi(x) \in \mathcal{M}_n^*(a, b)$. We say that $\varphi(x)$ is completely monotonic on I if (1.1) holds for $n = \infty$.

A sequence $\{\mu_k\}_{k=1}^{\infty}$, denoted simply by $\{\mu_k\}$, is said to be n-times monotonic if

(1.2) $$(-1)^i \Delta^i \mu_k \geqq 0 \qquad i = 0, 1, 2, \ldots, n; \; k = 0, 1, 2, \ldots$$

Here $\Delta\mu_k = \mu_{k+1} - \mu_k$; $\Delta^2\mu_k = \Delta(\Delta\mu_k)$ etc. For such a sequence we write $\{\mu_k\} \in \mathcal{M}_n$. In case that strict inequality holds throughout (1.2) we write $\{\mu_k\} \in \mathcal{M}_n^*$. $\{\mu_k\}$ is called completely monotonic if (1.2) holds for $n = \infty$.

As usual, we write $[a, b)$ to denote the interval $\{x \mid a \leqq x < b\}$. $\varphi(x) \in C_n(I)$ means that $\varphi(x)$ has continuous derivatives of the n-th order.

$$D_x[\varphi(x)] \text{ denotes the first derivative } \frac{d\varphi(x)}{dx}.$$

2. NEW BASIC RESULTS

Consider an equation

(2.1) $$[g(x) y']' + f(x) y = 0 \qquad g(x) > 0,$$

with $f(x)$ and $g(x)$ continuous, $g(x) > 0$ for $a < x < \infty$. The change of variable

(2.2) $$\xi = \int_a^x \frac{du}{g(u)\psi^2(u)} \qquad \psi(x) > 0, \qquad \psi(x) \in C_2(a, \infty),$$

where the integral is assumed convergent, transforms (2.1) into

$$\frac{d^2\eta}{d\xi^2} + \varphi(\xi)\,\eta = 0, \tag{2.3}$$

where $\eta(\xi) = \frac{y(x)}{\psi(x)}$ and $\varphi(\xi) = [(g(x)\,\psi'(x))' + f(x)\,\psi(x)]\,\psi^3(x)\,g(x)$ (see [5] p. 597).

The first theorem is a generalization of ([2] Theorem 3.1).

Theorem 2.1. *Let* $y(x)$, $z(x)$ *be solutions of* (2.1) *on* (a, ∞) *where*

$$0 < \lim_{x\to\infty} [(g(x)\,\psi'(x))' + f(x)\,\psi(x)]\,\psi^3(x)\,g(x) \leqq \infty$$

for some function $\psi(x) > 0$, $\psi(x) \in C_2(a, \infty)$ *and suppose that* $z(x)$ *has consecutive zeros at* $x_1, x_2,$ *on* $[a, \infty)$. *Suppose also that* $g(x)\,\psi^2(x)$, $D_x\{[(g\psi')' + f\psi]\,\psi^3 g\}$ *and* $W(x)$ *are positive and belong to* $\mathcal{M}_n(a, \infty)$ *for some* $n \geqq 0$. *Then, for fixed* $\lambda > -1$

$$\left\{\int_{x_k}^{x_{k+1}} W(x)\,\frac{1}{g(x)\,\psi^2(x)}\left|\frac{y(x)}{\psi(x)}\right|^{\lambda} dx\right\} \in \mathcal{M}_n^*. \tag{2.4}$$

Remark 1. Under the hypotheses of Theorem 2.1, if $g(x)\,\psi^{2+\lambda}(x) \in \mathcal{M}_n(a, \infty)$, we can write

$$\{\int_{x_k}^{x_{k+1}} W(x)\,|\,y(x)\,|^{\lambda}\,dx\} \in \mathcal{M}_n^* \tag{2.5}$$

because (2.4) is still valid when $W(x)$ is replaced by $W(x)\,g(x)\,\psi^{2+\lambda}(x)$, since this last function belongs to $\mathcal{M}_n(a, \infty)$.

Proof: For $n \geqq 1$, $g(x)\,\psi^2(x)$ is non-increasing. Hence, the mapping (2.2) takes the interval (a, ∞) into the ξ-interval $(0, \infty)$. By hypothesis, $0 < \varphi(\infty) \leqq \infty$, since $\varphi(\xi) = [(g\psi')' + f\psi]\,\psi^3 g$. This shows (in case $n \geqq 1$) that $z(x)$ does indeed have an infinite sequence of zeros on $[a, \infty)$. Using the change of variable (2.2) we get

$$\int_{x_k}^{x_{k+1}} W(x)\,\frac{1}{g(x)\,\psi^2(x)}\left|\frac{y(x)}{\psi(x)}\right|^{\lambda} dx = \int_{\xi_k}^{\xi_{k+1}} W[x(\xi)]\,|\,\eta(\xi)\,|^{\lambda}\,d\xi,$$

where $\xi_1, \xi_2, \ldots$ are the zeros of $\zeta(\xi)$ corresponding, respectively, to the zeros $x_1, x_2, \ldots$ of $z(x)$. $\left(\text{Here } \zeta(\xi) = \frac{z(x)}{\psi(x)}\right)$. In case $n \geqq 2$ and $x_1 > a$, the present theorem will follow from ([1] Theorem 3.3) as applied to the equation (2.3) provided we show that

$$\varphi'(\xi) > 0, \qquad \varphi'(\xi) \in \mathcal{M}_n(0, \infty). \tag{2.6}$$

and that

$$W[x(\xi)] > 0, \qquad W[x(\xi)] \in \mathcal{M}_n(0, \infty). \tag{2.7}$$

Now, $\varphi'(\xi) = D_x[((g\psi')' + f\psi)\psi^3 g]\, x'(\xi) = g\psi^2 D_x[((g\psi')' + f\psi)\psi^3 g] > 0$. **But,** $g\psi^2 \in \mathcal{M}_n(a, \infty)$ so that a slight modification of ([1] Lemma 2.2) [in which $p'(x) \leqq 0$ replaces $p'(x) < 0$ and $\geqq$ replaces $>$ in (2.7)] implies that $x'(\xi) \in \mathcal{M}_n(0, \infty)$. **Hence,** in view of ([1] Lemma 2.1) our hypotheses on $W(x)$ show that $W[x(\xi)] \in \mathcal{M}_n(0, \infty)$ and (2.7) holds. Since $D_x[\varphi(\xi)]$, considered as a function of x, belongs to $\mathcal{M}_n(0, \infty)$ and $x'(\xi) \in M_n(0, \infty)$, ([1] Lemma 2.1) shows that $D_\xi[\varphi(\xi)] \in \mathcal{M}_n(0, \infty)$. Hence, (2.6) holds and the proof of Theorem 2.1 is complete, in case $n \geqq 2$ and $x_1 > a$.

In case $n = 0$, $n = 1$ and $x_1 = a$ the proof is done in the same way as that of ([2] Theorem 3.1).

In case $x_1 = a$ the function $W(x)$ must be chosen in such a way that the integrals occuring in the statement of the theorem exist.

Remark 2. ([2] Theorem 3.1) can be easily obtained from Theorem 2.1 (in the present paper) if we choose $\psi(x) = 1$.

Example 1. Bessel function $y = C_\nu(x)$ satisfies the differential equation

$$(xy')' + (x^2 - \nu^2)\frac{1}{x}y = 0, \qquad x \in (0, \infty). \tag{2.8}$$

This equation does not fulfil hypotheses of ([2] Theorem 3.1).

Choose $\psi(x) = 1/\sqrt{x}$ and find out whether the hypotheses of Theorem 2.1 are satisfied:

$$g(x) = x(> 0); \qquad g\psi^2 = 1 \in \mathcal{M}_\infty(0, \infty); \qquad \varphi(\xi) = 1 - \frac{\nu^2 - 1/4}{x^2}.$$

Hence

$$\varphi(\infty) = 1, \qquad D_x[\varphi(\xi)] = 2\frac{\nu^2 - 1/4}{x^3} \in \mathcal{M}_\infty(0, \infty) \qquad \text{if } |\nu| \geqq 1/2.$$

Result. (2.8) fulfils hypotheses of Theorem 2.1 for $|\nu| \geqq 1/2$ and $n = \infty$.

Example 2. Consider a differential equation

$$(xy')' + \left(cx^4 - \frac{k}{x} - \frac{1}{x^{\alpha-2}}\right)y = 0, \qquad c > 0, k > 0, \alpha > -2. \tag{2.9}$$

This equation does not fulfil hypotheses of ([2] Theorem 3.1) as well.

Choose $\psi(x) = 1/x^a$, $a \geqq 0$. By easy calculus we find out that

$$\varphi(\xi) = \frac{a^2 - k}{x^{4a}} + \frac{c}{x^{4a-5}} - \frac{1}{x^{\alpha-3+4a}}.$$

If $4a - 5 = 0$, hence $a = 5/4$ and $\psi(x) = x^{-5/4}$. Then

$$\varphi(\xi) = \frac{25/16 - k}{x^5} + c - \frac{1}{x^{\alpha+2}} \Rightarrow \varphi(\infty) = c(> 0)$$

and
$$\varphi'(\xi) = \frac{5(k - 25/16)}{x^6} + (\alpha + 2)\frac{1}{x^{\alpha+3}}.$$

If $k \geqq 25/16$ then $\varphi'(\xi) \in \mathcal{M}_\infty(0, \infty)$.

Further we have
$$g\psi^2 = \frac{1}{\sqrt{x^3}} \in \mathcal{M}_\infty(0, \infty)$$
and finally
$$g\psi^{2+\lambda} = \frac{1}{x^{3/2+\lambda_0}}.$$

If $3/2 + \lambda/4 \geqq 0 \Rightarrow \lambda \geqq -6$, then $g\psi^{2+\lambda} \in \mathcal{M}_\infty(0, \infty)$.

Result. (2.9) fulfils the hypotheses of Theorem 2.1 and Remark 1 if $k \geqq 25/16$.

The following theorem is a generalization of ([2] Theorem 3.2).

Theorem 2.2. *Let $y(x)$, $z(x)$ be solutions of (2.1) on (a, ∞) where $f(x) > 0$, $\psi > 0$, $D_x[((g\psi')' + f\psi)\,\psi^3 g] > 0$ and where $g(x)\,\psi^2(x)$ and $D_x[((g\psi')' + f\psi)\,\psi^3 g]$ belong to $\mathcal{M}_n(a, \infty)$ for some $n \geqq 2$. Let $\left(\frac{z(x)}{\psi(x)}\right)'$ have consecutive zeros $x'_1, x'_2, \ldots$ on $[a, \infty)$.*
Let $W(x) > 0$, $W(x) \in H_{n-2}(a, \infty)$.
Then, for fixed $\lambda > -1$,
$$\left\{\int_{x'_k}^{x'_{k+1}} W(x)\,\frac{1}{g(x)\,\psi^2(x)}\left|\frac{(y'\psi - y\psi')\sqrt{g}}{\sqrt{((g\psi') + f\psi)\,\psi^3}}\right|^\lambda \mathrm{d}x\right\} \in \mathcal{M}^*_{n-2}.$$

Proof. The change of variable (2.2) yields, as in the proof of Theorem 2.1
$$\left\{\int_{x'_k}^{x'_{k+1}} W(x)\,\frac{1}{g(x)\,\psi^2(x)}\left|\frac{(y'\psi - y\psi')\sqrt{g}}{\sqrt{((g\psi')' + f\psi)\,\psi^3}}\right|^\lambda \mathrm{d}x = \int_{\xi'_k}^{\xi'_{k+1}} W[x(\xi)]\left|\frac{\eta'(\xi)}{\sqrt{\varphi(\xi)}}\right|^\lambda \mathrm{d}\xi,$$
where $\xi_1, \xi_2, \ldots$ are the zeros of $\zeta(\xi) = (z'\psi - z\psi')\,g$ corresponding to the zeros $x'_1, x'_2, \ldots$ of $\left(\frac{z(x)}{\psi(x)}\right)'$. Now, $\varphi'(\xi) = D_x[((g\psi')' + f\psi)\,\psi^3 g]\,g\psi^2 > 0$ for $0 < \xi < \infty$ and, as in the proof of Theorem 2.1, (2.6) and (2.7) hold with n replaced by $n - 2$. The theorem follows on applying ({1} Theorem 3.4) to solutions of the equation (2.3). (We require an extended form of ([1] Theorem 3.4) in which a possible end-point zero and the cases $n = 2, 3$ are included. This can be established in much the same way as was the extension of ([1] Theorem 3.3)).

The existence of an infinite sequence of zeros is a consequence of the hypothesis on $f(x)$ and $g(x)$.

Remark 3. ([2] Theorem 3.2) can be obtained from Theorem 2.2 if we choose $\psi(x) \equiv 1$.

3. THE CASE $\lambda \to -1$

([1] Lemma 8.1): Suppose that $u(x)$ is defined over the closed interval $[a, b]$, that it vanishes only for $x = x_k$ and changes sign at $x = x_k$, $a < x_k < b$, that $u'(x$ and $u'(x)\,q(x)$ are Lebesgue integrable over $[a, b]$, that $q'(x)$ is Lebesgue integrable over $[x_k - \delta, x_k + \delta]$ for some $\delta > 0$. Then

$$\lim_{\mu \to 0+} \int_a^b \mu q(x)\, u'(x)\, |u(x)|^{\mu - 1}\, dx = 2q(x_k) \operatorname{sgn} u(b). \tag{3.1}$$

([1] Lemma 8.1) leads directly to the construction of new higher monotonic sequences. However, it is convenient to modify slightly our earlier notation. Throughout this section we shall take $M_k(W, \lambda)$ to be

$$M_k(W, \lambda) = \int_{\zeta_k}^{\zeta_{k+1}} W(x) \frac{1}{g(x)\, \psi^2(x)} \left| \frac{y(x)}{\psi(x)} \right|^{\lambda} dx \qquad \lambda > -1,\ k = 1, 2, \ldots \tag{3.2}$$

where $\zeta_1, \zeta_2, \ldots$ are consecutive zeros (in the open interval I) of a solution $z(x)$ of (2.1) linearly independent of $y(x)$. The consecutive zeros of $y(x)$ in I are $x_1, x_2, \ldots$ with $\zeta_1 < x_1 < \zeta_2$.

Lemma 3.1. Let $M_k(W, \lambda)$ be defined by (3.2) and let $W'(x)$ be integrable on (a, b). Then

$$\lim_{\lambda \to -1+} (1 + \lambda)\, M_k(W, \lambda) = 2 \frac{W(x_k)}{g(x_k)\, |\psi(x_k)\, y'(x_k) - \psi'(x_k)\, y(x_k)|} = 2 \frac{W(x_k)}{|g(x_k)\, \psi(x_k)\, y'(x_k)|}. \tag{3.3}$$

Proof. In (3.1), put $\mu = 1 + \lambda$, $q(x) = \dfrac{W(x)}{g(x)\, \psi^2(x) \left(\dfrac{y(x)}{\psi(x)} \right)'} = \dfrac{W(x)}{g(x)\, (y'\psi - y\psi')}$,

$a = \zeta_k$, $b = \zeta_{k+1}$. To establish the existence of $\delta > 0$ such that $q'(x)$ is integrable over $[x_k - \delta, x_k + \delta]$, it is sufficient to note the existence of $\delta > 0$ such that $g(y'\psi - y\psi') \neq 0$ in this closed interval. This is obvious, because $g(y'\psi - y\psi')$ is continuous in $[\zeta_k, \zeta_{k+1}] \subset I$, $g(x) > 0$, $\psi(x) > 0$ on I, $y(x_k) = 0$, so that $g(x_k)\,[y'(x_k)\, \psi(x_k) - y(x_k)\, \psi'(x_k)] = g(x_k)\, y'(x_k)\, \psi(x_k) \neq 0$, since the derivative of a non-trivial solution of (2.1) cannot vanish at interior zeros of solution.

The difference operator being a finite linear combination, Lemma 3.1 implies the following result.

Theorem 3.1. *If* $(-1)^n \Delta^n M_k(W, \lambda) \geqq 0$, $n = 0, 1, 2, \ldots, N$; $k = 1, 2, \ldots$, *where* M_k *is defined by* (3.2), *with* $W'(x)$ *integrable and* $W(x) \geqq 0$, *then*

$$(-1)^n \Delta^n \left\{ \left| \frac{W(x_k)}{g(x_k)\, \psi(x_k)\, y'(x_k)} \right| \right\} \geqq 0 \qquad n = 1, 2, \ldots, N;\ k = 1, 2, \ldots \tag{3.4}$$

If the factor $(-1)^n$ is deleted from the hypothesis, then (3.4) holds with the same deletion. In particular, the hypothesis holds [and with it (3.4)] e.g. if the hypotheses of Theorem 2.1 are satisfied.

Remark 4. It should be noted that strengthening the hypothesis by replacing "$\geqq 0$" by "> 0" does not appear to permit, in general, a corresponding strengthening of the conclusion (3.4), due to the limit process. However, this improvement can be made for the case of complete monotonicity as follows.

Theorem 3.2. If the differential equation (2.1) is oscillatory on (a, ∞), $D_x[((g\psi')' + f\psi)\psi^3 g]$ is continuous and non-negative, $W(x) > 0$, $W'(x) \leqq 0$, $0 < x < \infty$ and if $(-1)^n \Delta^n M_k \geqq 0$, $(k, n = 1, 2, \ldots)$, then

$$(3.5) \qquad (-1)^n \Delta^n \left\{ \left| \frac{W(x_k)}{g(x_k)\,\psi(x_k)\,y'(x_k)} \right| \right\} > 0 \qquad n, k = 1, 2, \ldots$$

unless $((g\psi')' + f\psi)\psi^3 g$ is constant.

In particular, if the hypotheses of Theorem 2.1 are satisfied, then (3.5) holds, provided $((g\psi')' + f\psi)\psi^3 g$ is not constant.

Proof. To prove this theorem, it suffices to show that its hypotheses, a strengthening of those of Theorem 3.1, imply that equality can never occur in (3.4) when $N = \infty$. Now we use a modify form (for our case) of result of [3] which says: if there should exist a single pair of values of n and k for which equality occurs in (3.4), when $N = \infty$ then

$$\left| \frac{W(x_k)}{g(x_k)\,\psi(x_k)\,y'(x_k)} \right| = \left| \frac{W(x_{k+1})}{g(x_{k+1})\,\psi(x_{k+1})\,y'(x_{k+1})} \right|$$

for all $k = 2, 3, \ldots$ Clearly, $g(x) > 0$, $\psi > 0$ for all $x \in I$, $y'(x_k)$ and $y'(x_{k+1})$ are of opposite signs $(k = 1, 2, \ldots)$ while $W(x) > 0$, so that the above equality is reduced to

$$(3.6) \qquad \frac{W(x_k)}{g(x_k)\,\psi(x_k)\,y'(x_k)} = \frac{W(x_{k+2})}{g(x_{k+2})\,\psi(x_{k+2})\,y'(x_{k+2})}, \qquad k = 2, 3, \ldots$$

It remains to show that the equality (3.6) implies, in the light of our assumptions, that $[(g\psi')' + f\psi]\psi^3 g$ is constant. This follows from a formula of Wiman ([6] p. 125) which states, in our notation,

$$[g(x_{k+2})\,\psi(x_{k+2})\,y'(x_{k+2})]^2 - [g(x_k)\,\psi(x_k)\,y'(x_k)]^2 = \\ = \int_{x_k}^{x_{k+2}} \left[\frac{y(x)}{\psi(x)}\right]^2 D_x\{[(g\psi')' + f\psi]\,\psi^3 g\}\,dx.$$

The left member cannot be positive, in view of (3.6), since $W(x)$ is positive and non-increasing. But the right member cannot be negative, since $D_x\{[(g\psi')' + f\psi]\psi^3 g\} \geqq 0$. Hence, they must both be zero. Therefore $D_x\{[(g\psi')' + f\psi]\psi^3 g\} = 0$ $x_k < x < x_{k+2}$. Thus, the function $((g\psi')' + f\psi)\psi^3 g$ is a constant.

Remark 5. Under certain circumstances equality can be deleted from (3.4) when N is finite. It has been shown [4] that if equality occurs in (3.4) for some pair of indices n, k, where $n \leqq N - 1$, $k = 1, 2, \ldots$, then in our notation,

$$\left| \frac{W(x_k)}{g(x_k)\,\psi(x_k)\,y'(x_k)} \right|$$

is eventually constant, i.e. constant for all sufficiently large k. This implies that

$$\frac{W(x_k)}{g(x_k)\,\psi(x_k)\,y'(x_k)} = \frac{W(x_{k+2m})}{g(x_{k+2m})\,\psi(x_{k+2m})\,y'(x_{k+2m})}$$

for a fixed such k and all $m = 1, 2, \ldots$ A knowledge of the asymptotics of the situation will often show this to be impossible. This would imply strict inequality in (3.4) except possibly for $n = N$.

Example 3. The differential equation (2.8) satisfies hypotheses of Theorem 3.2 for $|\nu| \geqq 1/2$ with $\psi = 1/\sqrt{x}$. Hence, there holds

$$(-1)^n \Delta^n \left\{ \frac{W(x_k)}{\sqrt{c_{\nu k}}\, C'_\nu(c_{\nu k})} \right\} > 0,$$

where $c_{\nu 1}, c_{\nu 2}, \ldots$ are consecutive zeros of $C_\nu(x)$.

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