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# LATTICES OF GENERATING SYSTEMS

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## 0. INTRODUCTION

In [6] and [3], every closure operator $\varphi$ on the set of all subsets of a lattice $L$ such that $\varphi\{a\} = \{b \in L; b \leqq a\}$ for each $a \in L$, was called an embedding operator and the set of all $A \subseteq L$ satisfying $\varphi A = A$ a generating system on $L$. These concepts were investigated in [4] on arbitrary posets. In [5], there were proved some properties of the lattice of all embedding operators on a poset $P$. This one is dual to the lattice Gs $(P)$ of all generating systems on $P$ which we call the gs-lattice on $P$.

In this paper some statements concerning gs-lattices in general are formulated. For an arbitrary set $\{P_i; i \in I\}$ of nonempty posets, a poset $P$ is found such that Gs $(P) \cong \prod_{i \in I}$ Gs $(P_i)$. We say that a poset $P$ is simple whenever there are only those generating systems in Gs$(P)$ which were constructed in [2] as a solution of a certain embedding problem. An elementary description of the gs-lattice on each simple poset is given and the class of all gs-lattices on simple posets is characterized. It is shown that every poset, in the gs-lattice on which each completely $\vee$-irreducible element has a complement, is simple and that the class of all gs-lattices with this property is (up to isomorphism) exactly the class of all complete atomic Boolean algebras.

## 1. THE CONCEPT OF A GS-LATTICE

We denote by $\emptyset$ the empty set, by $\subseteq$ the relation of inclusion and by $\subset$ that of a proper inclusion. We say that a set $\mathfrak{A}$ is a *system* whenever every element of $\mathfrak{A}$ is a set. If $\bigcap \mathfrak{B} \in \mathfrak{A}$ for all $\mathfrak{B}$, $\emptyset \subset \mathfrak{B} \subseteq \mathfrak{A}$, then we call the system $\mathfrak{A}$ *multiplicative*. In case $\mathfrak{A} = \emptyset$ we put $\bigcup \mathfrak{A} = \emptyset$. The standard partial ordering on each system is the inclusion.

Let $P$ be a poset. We denote by $\leqq$ the partial order, by $<$ the relation "less than" and by $\prec$ the covering relation on $P$. $P$ is said to be a *chain*, an *antichain* if every two different elements of $P$ are comparable, incomparable, respectively. Each set

$Q \subseteq P$ is considered partially ordered by the restriction of $\leqq$ from $P$ to $Q$. If this is the case then we call $P$ an *extension* of $Q$.

We denote by $\bigvee_P A$ the l. u. bound and by $\bigwedge_P A$ the g. l. bound of $A$ in $P$. Instead of $\bigvee_P\{a, b\}$ we write $a \vee b$. We define $\bigvee_P \emptyset$ iff $P$ has a least element $o$; then we put $\bigvee_P \emptyset = o$. We say that an element $a \in P$ is *completely* $\vee$*-irreducible* in P if $a = \bigvee_P A \Rightarrow a \in A$ for all $A \subseteq P$. The set of all completely $\vee$-irreducible elements in $P$ will be denoted by $\mathbf{IR}_P$ and the set of all elements of $P$ having the dual property by $\mathbf{IR}_P^d$.

If $a \leqq b \Rightarrow \iota a \leqq \iota b$ for all $a, b \in P$ then we call the map $\iota : P \to Q$ *isotone*; if the converse implication is also true then we say that $\iota$ is an *embedding* of $P$ into $Q$ for arbitrary posets $P$ and $Q$. Clearly, each embedding is an injection. If $\iota$ is an embedding and also a surjection then we call $\iota$ an *isomorphism* of $P$ onto $Q$, $Q$ the *isomorphic image* of $P$ and write $P \cong Q$.

Whenever $a \leqq \bigvee_L A \Rightarrow$ there exists $b \in A$ such that $a \leqq b$ for all $A \subseteq L$ holds for an element $a$ in a complete lattice $L$ then we say that $a$ is *completely* $\vee$*-primitive* in $L$. The set of all completely $\vee$-primitive elements in $L$ will be denoted by $\mathbf{P}_L$ and that of all dual atoms in $L$ by $\mathbf{A}_L^d$.

We consider every ordinal number $\mu$ to be the set of all ordinals less than $\mu$ ordered in the natural way.

The elements of the cartesian product $A_1 \times A_2 \times \ldots \times A_m$ of sets will be denoted by $(a_1, a_2, \ldots, a_m)$. If $L_1, L_2, \ldots, L_m$ are complete lattices then $L_1 \times L_2 \times \ldots \times L_m$ means the direct product of them. We denote by $(a_i)_{i \in I}$ an element of the direct product $\prod_{i \in I} L_i$ of complete lattices. In case $L_i = L$ for all $i \in I$ we write $L^I$ instead of $\prod_{i \in I} L_i$. We identify the complete lattice $2^I$ with the system of all subsets of the set $I$.

If $\mathscr{L}$ is a class of complete lattices then $\Pi\mathscr{L}$ denotes the least superclass of $\mathscr{L}$ closed under direct products and isomorphic images. One can easily see that $\Pi\mathscr{L}$ is exactly the class of all complete lattices $L$ for which there exists a system $\{L_i; i \in I\} \subseteq \mathscr{L}$ satisfying $L \cong \prod_{i \in I} L_i$.

The definitions of those basic lattice-theoretical notions which we use and do not define here can be found in [1].

**1.1. Definition.** Let $P$ be a poset and $a \in P$. Then we denote

$$\omega_P a = \{b; b \in P \text{ and } b \leqq a\}, \qquad \varepsilon_P a = \{b; b \in P \text{ and } a \leqq b\},$$

$$\omega_P^- a = \omega_P a - \{a\}, \qquad \bar{\varepsilon}_P a = P - \varepsilon_P a.$$

We put $\alpha_P A = \bigcup \alpha_P[A]$ for $\alpha = \omega, \varepsilon$ and all $A \subseteq P$.

**1.2. Definition.** Let $A$ be an arbitrary subset of a poset $P$. If $A = \omega_P A$, $A = \varepsilon_P A$ then we call $A$ an *initial*, *final segment* in $P$, respectively.

We denote by $\mathfrak{O}_P$ (or, if no confusion arises, by $\mathfrak{O}$) the system of all initial segments in $P$.

**1.3. Definition.** Let $P$ be a poset.

We say that $\mathfrak{G}$ is a *generating system* on $P$ whenever $\{P\} \cup \omega_P[P] \subseteq \mathfrak{G} \subseteq \mathfrak{D}_P$ and $\mathfrak{G}$ is multiplicative.

The system of all generating systems on $P$ is said to be a *gs-lattice* on $P$ and denoted by Gs($P$).

**1.4. Theorem.** *Let P be a poset. Then the assertions* (i), (ii), (iii) *hold.*

(i) *Every generating system on P is a complete lattice.*

(ii) Gs($P$) *is a complete lattice.*

(iii) *Both in an arbitrary generating system on P and in* Gs($P$) *the l. u. bound of each nonempty subset is its intersection.*

Proof. The statements follow by theorem 10 [1], by the multiplicativity of Gs($P$) and by the fact that $\mathfrak{D}_P$ is a greatest element in Gs ($P$).

**1.5. Definition.** The class of all complete lattices isomorphic to Gs ($P$) for some poset $P$ will be denoted by **G**.

**1.6. Definition.** Let $P$ be a poset. We denote by $\mathfrak{N}_P$ (by $\mathfrak{N}$) the least element in Gs($P$). The complete lattice $\mathfrak{N}_P$ is called a *normal* or a *MacNeille completion* of $P$.

**1.7. Lemma.** *Let P be a poset. Then the assertions* (i) – (iv) *are true.*

(i) $\varepsilon_P : P \to \mathfrak{D}_P$ *is an embedding.*

(ii) $\mathfrak{G} \cap \varepsilon_P[P] \subseteq \mathbf{IR}^d_{\mathfrak{G}}$ *for each* $\mathfrak{G} \in$ Gs ($P$).

(iii) $\varepsilon_P[P] \subseteq \mathfrak{G} \Rightarrow \mathfrak{G} = \mathfrak{D}_P$ *for each* $\mathfrak{G} \in$ Gs($P$).

(iv) $\varepsilon_P[P] = \mathbf{IR}^d_{\mathfrak{D}}$.

Proof. (1) $a \leqq b \Leftrightarrow \varepsilon_P a \subseteq \varepsilon_P b$ for all $a, b \in P$ is true trivially.

(2) Consider $\mathfrak{G} \in$ Gs($P$), $\varepsilon_P a \in \mathfrak{G}$ and $\mathfrak{A} \subseteq \mathfrak{G}$ such that $\varepsilon_P a = \bigwedge_{\mathfrak{G}} \mathfrak{A}$. If $\mathfrak{A} = \emptyset$ then $\varepsilon_P a = P$ which is a contradiction. In case $\mathfrak{A} \neq \emptyset$ we have $\varepsilon_P a = \bigcap \mathfrak{A}$ by 1.4(iii). Then $a \notin \bigcap \mathfrak{A}$ and there is $A \in \mathfrak{A}$ with the property $a \notin A$. This and $\varepsilon_P a \subseteq A$ give $\varepsilon_P a = A \in \mathfrak{A}$ which proves (ii).

(3) The statement (iii) follows immediately by 1.4(iii) and by the fact that $A = \bigcap \varepsilon_P[P - A]$ for each $A \in \mathfrak{D}_P - \{P\}$. This fact and (ii) imply (iv).

**1.8. Lemma.** *If P is a poset,* $\mathfrak{G} \in$ Gs($P$) *and* $\mathfrak{A} \subseteq \mathbf{IR}^d_{\mathfrak{G}} - \mathfrak{N}_P$ *then* $\mathfrak{G} - \mathfrak{A} \in$ Gs($P$).

Proof. Clearly, it is sufficient to prove the multiplicativity of $\mathfrak{G} - \mathfrak{A}$. If $\emptyset \subset \mathfrak{B} \subseteq \mathfrak{G} - \mathfrak{A}$ then $\bigcap \mathfrak{B} \in \mathfrak{G}$ and either $\bigcap \mathfrak{B} \in \mathfrak{B} \subseteq \mathfrak{G} - \mathfrak{A}$ or $\bigcap \mathfrak{B} \notin \mathfrak{B}$. In the second case $\bigcap \mathfrak{B} \notin \mathbf{IR}^d_{\mathfrak{G}}$ according to 1.4(iii). Hence $\bigcap \mathfrak{B} \notin \mathfrak{A}$ and, further, $\bigcap \mathfrak{B} \in \mathfrak{G} - \mathfrak{A}$.

**1.9. Corollary.** ([5], Corollary 1 of Theorem 4) *Each complete lattice* $L \in \mathbf{G}$ *is dually atomic and the set* $\mathbf{A}^d_L$ *generates a complete sublattice of L isomorphic to* $2^{\mathbf{A}^d_L}$.

Proof. If $P$ is an arbitrary poset, $\mathfrak{G} \in$ Gs($P$) and $\mathfrak{G} \subset \mathfrak{D}_P$ then there exists $a \in P$ satisfying $\varepsilon_P a \notin \mathfrak{G}$ by 1.7 (iii). We obtain $\mathfrak{G} \subseteq \mathfrak{H} \prec \mathfrak{D}_P$ for $\mathfrak{H} = \mathfrak{D}_P - \{\varepsilon_P a\}$ and $\mathfrak{H} \in$ Gs($P$) according to 1.8. This says that Gs($P$) is dually atomic and that $\mathbf{A}^d_{\mathrm{Gs}(P)} =$

$= \{\mathfrak{D}_P - \{A\};\ A \in \varepsilon_P[P] - \mathfrak{N}_P\}$. The remaining part of the statement is a consequence of 1.8 and of the selfduality of $2^{\varepsilon_P[P] - \mathfrak{N}_P}$.

**1.10. Definition.** Let $P$ be a poset and $\mathfrak{A} \subseteq \mathfrak{D}_P$. We denote by $\langle \mathfrak{A} \rangle$ the least multiplicative system $\mathfrak{B}$ with the properties $P \in \mathfrak{B}$, $\mathfrak{A} \subseteq \mathfrak{B}$.

If $\mathfrak{G} \in \mathrm{Gs}(P)$ and $\mathfrak{A} = \{A_1, A_2, \ldots, A_m\}$ then it is possible to write $\langle \mathfrak{G}, A_1, A_2, \ldots, \ldots, A_m \rangle$ instead of $\langle \mathfrak{G} \cup \mathfrak{A} \rangle$.

**1.11. Lemma.** *Let $P$ be a poset, $\mathfrak{A} \subseteq \mathfrak{D}_P$ and $A \in \mathfrak{D}_P$. Then the assertions* (i), (ii) *hold.*

(i) $\langle \mathfrak{A} \rangle = \{\bigcap \mathfrak{B};\ \emptyset \subset \mathfrak{B} \subseteq \{P\} \cup \mathfrak{A}\}$.

(ii) $\langle \mathfrak{N}_P, A \rangle$ *is the least* $\mathfrak{G} \in \mathrm{Gs}(P)$ *satisfying* $A \in \mathfrak{G}$.

**1.12. Lemma.** *Let $P$ be a poset, $I \neq \emptyset$ and $\mathfrak{A}_i \subseteq \mathfrak{D}_P$ for each $i \in I$. Then*

$$\langle \bigcup_{i \in I} \mathfrak{A}_i \rangle = \{\bigcap_{i \in I} A_i;\ \ A_i \in \langle \mathfrak{A}_i \rangle \ \ \textit{for all } i \in I\}.$$

Proof. Let us put $\mathfrak{C} = \{\bigcap_{i \in I} A_i;\ A_i \in \langle \mathfrak{A}_i \rangle$ for all $i \in I\}$. Clearly, $P \in \mathfrak{C}$, $\bigcup_{i \in I} \mathfrak{A}_i \subseteq \mathfrak{C}$, and $\bigcup_{i \in I} \mathfrak{A}_i \subseteq \mathfrak{D} \Rightarrow \mathfrak{C} \subseteq \mathfrak{D}$ for every multiplicative system $\mathfrak{D}$. That is why it is sufficient to verify the multiplicativity of $\mathfrak{C}$ only. Choose $\mathfrak{B}$, $\emptyset \subset \mathfrak{B} \subseteq \mathfrak{C}$, arbitrarily. Then there is $C_i^B \in \langle \mathfrak{A}_i \rangle$ such that $B = \bigcap_{i \in I} C_i^B$ for all $i \in I$, $B \in \mathfrak{B}$. If we put $C_i = \bigcap_{B \in \mathfrak{B}} C_i^B$ then $C_i \in \langle \mathfrak{A}_i \rangle$ for each $i \in I$ and, obviously, $\bigcap \mathfrak{B} = \bigcap_{i \in I} C_i \in \mathfrak{C}$.

**1.13. Corollary.** *The assertions* (i), (ii) *hold for an arbitrary poset $P$.*

(i) $\bigvee_{\mathrm{Gs}(P)} \mathbf{A} = \{\bigcap_{\mathfrak{H} \in \mathbf{A}} A_{\mathfrak{H}};\ A_{\mathfrak{H}} \in \mathfrak{H}$ *for each* $\mathfrak{H} \in \mathbf{A}\}$ *for every nonempty system* $\mathbf{A} \subseteq \subseteq \mathrm{Gs}(P)$.

(ii) $\langle \mathfrak{G}, A \rangle \subseteq \mathfrak{G} \cup \omega_{\mathfrak{D}} A$ *for all* $\mathfrak{G} \in \mathrm{Gs}(P)$, $A \in \mathfrak{D}_P$.

Proof. The statement (i) follows by 1.12 and by $\bigvee_{\mathrm{Gs}(P)} \mathbf{A} = \langle \bigcup \mathbf{A} \rangle$ for each nonempty system $\mathbf{A} \subseteq \mathrm{Gs}(P)$. Regarding 1.12 we obtain $\langle \mathfrak{G}, A \rangle = \{C \cap D;\ C \in \mathfrak{G}$ and $D \in \{P, A\}\}$; this gives (ii).

**1.14. Lemma.** *Let $P$ be a poset, $\mathfrak{G} \in \mathrm{Gs}(P)$ and $A \in \mathfrak{D}_P$. If $A \notin \mathfrak{G}$ then $\langle \mathfrak{G}, A \rangle - \{A\} \in \mathrm{Gs}(P)$.*

Proof. Suppose that $A \notin \mathfrak{G}$ and put $\mathfrak{C} = \langle \mathfrak{G}, A \rangle - \{A\}$. $\mathfrak{C}$ is multiplicative: Let us take $\mathfrak{A}$, $\emptyset \subset \mathfrak{A} \subseteq \mathfrak{C}$, arbitrarily. Then $\mathfrak{A} \subseteq \langle \mathfrak{G}, A \rangle \Rightarrow \bigcap \mathfrak{A} \in \langle \mathfrak{G}, A \rangle$. $\mathfrak{A} \subseteq \subseteq \mathfrak{G} \cup \omega_{\mathfrak{D}}^- A$ by 1.13(ii). If $\mathfrak{A} \cap \omega_{\mathfrak{D}}^- A = \emptyset$ then $\mathfrak{A} \subseteq \mathfrak{G}$ and $\bigcap \mathfrak{A} \in \mathfrak{G} \subseteq \mathfrak{C}$. Otherwise $\bigcap \mathfrak{A} \subset A$ and $\bigcap \mathfrak{A} \in \mathfrak{C}$, too.

**1.15. Lemma.** $\mathbf{IR}_{\mathrm{Gs}(P)} = \{\langle \mathfrak{N}_P, A \rangle;\ A \in \mathfrak{D}_P - \mathfrak{N}_P\}$ *for every poset $P$.*

Proof. Let $P$ be an arbitrary poset. Clearly, $\mathfrak{G} = \bigvee_{\mathrm{Gs}(P)} \{\langle \mathfrak{N}_P, A \rangle;\ A \in \mathfrak{G} - \mathfrak{N}_P\}$ for each $\mathfrak{G} \in \mathrm{Gs}(P)$. If $\mathfrak{G} \in \mathbf{IR}_{\mathrm{Gs}(P)}$ then $\mathfrak{G} = \langle \mathfrak{N}_P, A \rangle$ for some $A \in \mathfrak{G} - \mathfrak{N}_P \subseteq \subseteq \mathfrak{D}_P - \mathfrak{N}_P$.

Put $\mathfrak{G} = \langle \mathfrak{N}_P, A \rangle$ for an $A \in \mathfrak{D}_P - \mathfrak{N}_P$ and suppose that $\mathfrak{G} = \bigvee_{Gs(P)} \mathbf{A}$ where $\mathbf{A} \subseteq Gs(P)$. It holds $\mathbf{A} \neq \emptyset$ trivially and for each $\mathfrak{H} \in \mathbf{A}$ there is $A_{\mathfrak{H}} \in \mathfrak{H}$ satisfying $A = \bigcap_{\mathfrak{H} \in \mathbf{A}} A_{\mathfrak{H}}$ according to 1.13(i). By this, $\mathfrak{H} \subseteq \mathfrak{G} \subseteq \mathfrak{N}_P \cup \omega_{\mathfrak{D}} A$ (see 1.13(ii)) and by $A \subseteq A_{\mathfrak{H}}$ it follows that $A_{\mathfrak{H}} \in \mathfrak{N}_P$ or $A_{\mathfrak{H}} = A$ for every $\mathfrak{H} \in \mathbf{A}$. If $A_{\mathfrak{H}} \in \mathfrak{N}_P$ for each $\mathfrak{H} \in \mathbf{A}$ then $A \in \mathfrak{N}_P \subseteq \mathfrak{G}$ and we have a contradiction. Thus there exists $\mathfrak{H}_0 \in \mathbf{A}$ with $A_{\mathfrak{H}_0} = A$. Then $\mathfrak{G} \subseteq \mathfrak{H}_0$ and, with respect to the validity of the converse inclusion, $\mathfrak{G} = \mathfrak{H}_0 \in \mathbf{A}$.

**1.16. Corollary.** *If $L \in \mathbf{G}$ then every element of $L$ is the l. u. bound of a set of completely $\vee$-irreducible elements.*

## 2. DIRECT PRODUCT IN THE CLASS G

$\mathfrak{N}_P = \{P\} \cup \{\bigcap \omega_P[X];\ \emptyset \subset X \subseteq P\}$ is an easy consequence of $\mathfrak{N}_P = \langle \omega_P[P] \rangle$ and 1.11(i).

**2.1. Lemma.** *Let us take a poset $P$, a final segment $Q$ in $P$, $A \in \mathfrak{D}_Q - \{\emptyset\}$ and $B = (P - Q) \cup A$. Then the assertions (i), (ii), (iii), are true.*

(i) $B \in \mathfrak{N}_P \Rightarrow A \in \mathfrak{N}_Q$.

(ii) $B \in \bar{\varepsilon}_P[P] \Rightarrow A \in \bar{\varepsilon}_Q[Q]$.

(iii) $B \in \omega_P^-[P] \Rightarrow A \in \omega_Q^-[Q]$.

Proof. Suppose that $B \in \mathfrak{N}_P$. $B = P$ implies $A = Q \in \mathfrak{N}_Q$. If $B \subset P$ then $B = \bigcap \omega_P[X]$ for a set $X$, $\emptyset \subset X \subseteq P$. Since $A \neq \emptyset$ there is $a \in A \subseteq \bigcap \omega_P[X]$ and we obtain $X \subseteq \varepsilon_P a$; this and $\varepsilon_P a \subseteq Q$ give $X \subseteq Q$. Then $A = B \cap Q = \bigcap \omega_P[X] \cap Q = \bigcap \omega_Q[X] \in \mathfrak{N}_Q$.

If $B \in \bar{\varepsilon}_P[P]$ then there exists $a \in P$ satisfying $B = \bar{\varepsilon}_P a$. By $P - Q \subseteq B$ and $a \notin B$ we obtain $a \in Q$. Then $A = B \cap Q = \bar{\varepsilon}_Q a \in \bar{\varepsilon}_Q[Q]$.

If $B \in \omega_P^-[P]$ then $B = \omega_P^- a$ for an element $a \in P$. As $Q$ is a final segment in $P$, $\emptyset \subset A \subseteq Q$ and $A \subseteq \omega_P a$, we have $a \in Q$ and $A = \omega_P^- a \cap Q = \omega_Q^- a \in \omega_Q^-[Q]$.

**2.2. Lemma.** *Let $P$ be a poset, $Q$ a final segment in $P$, $A \in \mathfrak{D}_Q - \{\emptyset\}$ and let $B = (P - Q) \cup A$ satisfy $\omega_P A = B$. Then the assertions* (i), (ii), (iii) *hold.*

(i) $A \in \mathfrak{N}_Q \Rightarrow B \in \mathfrak{N}_P$.

(ii) $A \in \bar{\varepsilon}_Q[Q] \Rightarrow B \in \bar{\varepsilon}_P[P]$.

(iii) $A \in \omega_Q^-[Q] \Rightarrow B \in \omega_P^-[P]$.

Proof. Let us assume that $A \in \mathfrak{N}_Q$. $A = Q$ implies $B = P \in \mathfrak{N}_P$. If $A \subset Q$ then $A = \bigcap \omega_Q[X]$ for a nonempty set $X \subseteq Q$. $A \subseteq \bigcap \omega_P[X]$ is true evidently. For each $b \in P - Q$ there is an $a \in A$ such that $b < a$ because $\omega_P A = B$. Hence $b \in \bigcap \omega_P[X]$ and also $P - Q \subseteq \bigcap \omega_P[X]$. We have proved $B \subseteq \bigcap \omega_P[X]$. This inclusion and the obvious validity of its converse give $B \in \mathfrak{N}_P$.

If $A \in \varepsilon_Q[Q]$ then there is an $a \in Q$ with $A = \varepsilon_Q a$. As $a \nleqq b$ for all $b \in P - Q$, we get $P - Q \subseteq \varepsilon_P a$. By this and by $\varepsilon_P a \cap Q = \varepsilon_Q a$ we obtain $\varepsilon_P a = (\varepsilon_P a \cap Q) \cup \cup (\varepsilon_P a \cap (P - Q)) = A \cup (P - Q) = B$ which proves $B \in \varepsilon_P[P]$.

If $A \in \omega_Q^-[Q]$ then $A = \omega_Q^- a$ for some $a \in Q$. For every $b \in P - Q$ there exists $c \in A$ such that $b \leqq c$. As simultaneously $c < a$, it holds $b < a$ and we have $P - Q \subseteq \subseteq \omega_P^- a$. This and $\omega_Q^- a = \omega_P^- a \cap Q$ imply $B = \omega_P^- a \in \omega_P^-[P]$.

**2.3. Definition.** Let $I$ be a chain and $\{P_i; i \in I\}$ a system of nonempty posets. We denote by $\sum_{i \in I} P_i$ the disjoint union $\bigcup_{i \in I} P_i$ partially ordered in the following way. For arbitrary elements $a, b \in \bigcup_{i \in I} P_i$ there are $j, k \in I$ such that $a \in P_j$, $b \in P_k$. We put $a \leqq b$ if $j = k$ and $a \in \omega_{P_k} b$ or if $j < k$.

The poset $\sum_{i \in I} P_i$ is called an *ordinal sum* of $\{P_i;\ i \in I\}$. One can write $P_0 + P_1$ instead of $\sum_{i \in 2} P_i$.

**2.4. Lemma.** *Let* $P = \sum_{i \in I} P_i$, $A \in \mathfrak{D}_P$, $j \in I$ *and* $A_j = P_j \cap A$. *Then* (i), (ii) *are true.*

(i) $\emptyset \subset A_j \Rightarrow P_i \subseteq A$ *for each* $i < j$.

(ii) $\emptyset \subset A_j \subset P_j \Rightarrow A = \sum_{i<j} P_i + A_j$.

**2.5. Lemma.** *Let* $P = \sum_{i \in I} P_i$, $A \in \mathfrak{D}_P$, $j \in I$ *and* $A_j = P_j \cap A$. *If* $\emptyset \subset A_j \subset P_j$ *then the assertions* (i), (ii), (iii) *hold.*

(i) $A \in \mathfrak{N}_P \Leftrightarrow A_j \in \mathfrak{N}_{P_j}$.

(ii) $A \in \varepsilon_P[P] \Leftrightarrow A_j \in \varepsilon_{P_j}[P_j]$.

(iii) $A \in \omega_P^-[P] \Leftrightarrow A_j \in \omega_{P_j}^-[P_j]$.

Proof. If we put $Q = \sum_{i<j} P_i$ and $R = P - Q$ then $P = Q + R$ and $R = P_j + + (R - P_j)$.

(1) $A_j \in \mathfrak{N}_{P_j} \Leftrightarrow A_j \in \mathfrak{N}_R$: Since $A_j \subset P_j$, it holds $A_j \in \mathfrak{N}_{P_j}$ iff $A_j = \bigcap \omega_{P_j}[X]$ for a set $X$, $\emptyset \subset X \subseteq P_j$. This is equivalent to $A_j = \bigcap \omega_R[X] \in \mathfrak{N}_R$ regarding $\omega_{P_j} a = \omega_R a$ for each $a \in X$ and $P_j \subseteq \omega_R a$ for each $a \in R - P_j$.

(2) $A_j \in \alpha_{P_j}[P_j] \Leftrightarrow A_j \in \alpha_R[R]$ for $\alpha = \varepsilon, \omega^-$: $\alpha_{P_j} a = \alpha_R a$ for all $a \in P_j$ and $A_j \subset \subset P_j \subseteq \alpha_R a$ for all $a \in R - P_j$.

(3) $A_j \in \mathfrak{N}_R \Leftrightarrow A \in \mathfrak{N}_P$ follows immediately by 2.1(i) and 2.2(i).

(4) $A_j \in \alpha_R[R] \Leftrightarrow A \in \alpha_P[P]$ for $\alpha = \varepsilon, \omega^-$ is a consequence of 2.1(ii), (iii) and 2.2(ii), (iii).

By (1), (3) we obtain (i) and (2), (4) imply the statements (ii), (iii).

**2.6. Lemma.** *Let* $A$ *be an initial segment in* $P = \sum_{i \in I} P_i$ *with the property* $P_i \cap A \in \in \{\emptyset, P_i\}$ *for each* $i \in I$. *Denote by* ($\alpha$) *the following condition. There is* $k \in I$ *such that* $P_k$ *has a least element* $o$, $A = \omega_P^- o$ *and* $A$ *has not a greatest element.*

*Then* $A \in (\varepsilon_P[P] \cap \omega_P^-[P]) - \mathfrak{N}_P$ *if* $(\alpha)$ *is true and* $A \in \mathfrak{N}_P$ *otherwise.*

Proof. It holds $P = A + R$ for $R = \sum_{i \in J} P_i$ where $J = \{i;\ i \in I,\ P_i \cap A = \emptyset\}$.

If $R = \emptyset$ or if $A$ has a greatest element then, clearly, $A \in \mathfrak{N}_P$.

Suppose that $R \neq \emptyset$ and $A$ has not a greatest element. By the assumption that $R$ has not a least element we obtain $A = \bigcap \omega_P[R] \in \mathfrak{N}_P$. If $R$ has a least element $o$ then $J$ has a least element $k$ and $o$ is a least one in $P_k$. As $o$ is comparable with all elements of $P$, we have $A = \varepsilon_P o = \omega_P^- o \in \varepsilon_P[P] \cap \omega_P^-[P]$. Let us admit that $A \in \mathfrak{N}_P$. Then $A = \bigcap \omega_P[X]$ for some set $X \subseteq P$. For each $a \in X$ it holds $a \notin A$ because $A$ has not a greatest element and $a$ is an upper bound of $A$. Hence $o \in \omega_P a$ and we obtain $o \in \bigcap \omega_P[X] = A$, a contradiction.

**2.7. Corollary.** *Let P be a poset. Then* $\emptyset \in (\varepsilon_P[P] \cap \omega_P^-[P]) - \mathfrak{N}_P$ *if P has a least element and* $\emptyset \in \mathfrak{N}_P$ *otherwise.*

**2.8. Definition.** If $I$ is a chain and $\Gamma = \{P_i;\ i \in I\}$ a system of nonempty posets then we put $I_0(\Gamma) = \{i;\ P_i$ has a greatest element and there is $i''$ satisfying $i \prec i''$, $P_{i''}$ has a least element$\}$. Let $J_0(\Gamma)$ be a set disjoint with $I$ for which there is a bijection $' : I_0(\Gamma) \to J_0(\Gamma)$. Let the chain $J(\Gamma) = J_0(\Gamma) \cup I$ be an extension of $I$ with the property $i < i' < i''$ for all $i \in I_0(\Gamma)$, $i \prec i''$ in $I$.

The ordinal sum of the system $\{P_j;\ j \in J(\Gamma)\}$, where $P_j$ is an antichain $\{a_j, b_j\}$ for each $j \in J_0(\Gamma)$, is said to be an *ordinal m-sum* of $\Gamma$ and denoted by $\overset{m}{\sum_{i \in I}} P_i$. One can write $P_0 \overset{m}{+} P_1$ instead of $\overset{m}{\sum_{i \in 2}} P_i$.

**2.9. Lemma.** *Let A be an initial segment in* $P = \overset{m}{\sum_{i \in I}} P_i$ *satisfying* $P_i \cap A \in \{\emptyset, P_i\}$ *for each* $i \in I$. *Then* $A \notin \mathfrak{N}_P$ *if and only if there is* $k \in I$ *such that* $P_k$ *has a least element o and* $A = \omega_P^- o$.

Proof. Let us denote $\Gamma = \{P_i;\ i \in I\}$.

If there is $k \in J_0(\Gamma)$ with $\emptyset \subset P_k \cap A \subset P_k$ then $A \in \{\omega_P a_k, \omega_P b_k\} \subseteq \mathfrak{N}_P$. Suppose that $P_i \cap A \in \{\emptyset, P_i\}$ for each $i \in J(\Gamma)$. Regarding 2.6, it is sufficient to prove the equivalence $(\alpha) \Leftrightarrow$ there is $k \in I$ such $P_k$ has a least element $o$ and $A = \omega_P^- o$.

$(\alpha)$ implies that $P_k$ has a least element $o$ and $A = \omega_P^- o$ for some $k \in J(\Gamma)$. Since $P_i$ has not a least element for each $i \in J_0(\Gamma)$, it holds $k \in I$.

If there exists $k \in I$ such that $P_k$ has a least element $o$ and $A = \omega_P^- o$ then $A$ has not a greatest element: Let us admit that $i$ is a greatest element in $A$. Then we can find $l$, $l \prec k$ in $J(\Gamma)$ such that $i$ is the greatest one in $P_l$. As $l \in I$ is obvious, we have $l \prec k$ in $I$, $P_l$ has a greatest and $P_k$ a least element. Thus there is $l' \in J_0(\Gamma)$ with $l < l' < k$ in $J(\Gamma)$, a contradiction.

**2.10. Theorem.** *If* $P = \overset{m}{\sum_{i \in I}} P_i$ *then* $\mathrm{Gs}(P) \cong \prod_{i \in I} \mathrm{Gs}(P_i)$.

**Proof.** Let us put $\iota\mathfrak{G} = (\mathfrak{G}_i)_{i\in I}$ where

$$\mathfrak{G}_i = \begin{cases} \{P_i \cap A;\ A \in \mathfrak{G}\} - \{\emptyset\} \text{ if } P_i \text{ has a least element } o \text{ and } \omega_P^- o \notin \mathfrak{G}, \\ \{P_i \cap A;\ A \in \mathfrak{G}\} \text{ otherwise} \end{cases}$$

for an arbitrary $\mathfrak{G} \in \mathrm{Gs}(P)$.

(1) $\mathfrak{G}_i \in \mathrm{Gs}(P_i)$ for each $i \in I$: By $P \in \mathfrak{G}$ and $\emptyset \subset P_i = P_i \cap P$ it follows that $P_i \in \mathfrak{G}_i$. Because of $\emptyset \subset P_i \cap \omega_P a = \omega_{P_i} a$ and $\omega_P a \in \mathfrak{G}$ for all $a \in P_i$, it holds $\omega_{P_i}[P_i] \subseteq \mathfrak{G}_i$. The inclusion $\mathfrak{G}_i \subseteq \mathfrak{O}_{P_i}$ is true trivially. If $\emptyset \subset \mathfrak{A}_i \subseteq \mathfrak{G}_i$ then there is $\mathfrak{A}$, $\emptyset \subset \mathfrak{A} \subseteq \mathfrak{G}$, with the property $\mathfrak{A}_i = \{P_i \cap A;\ A \in \mathfrak{A}\}$. By this we obtain $\bigcap \mathfrak{A}_i = \bigcap \{P_i \cap A;\ A \in \mathfrak{A}\} = P_i \cap \bigcap \mathfrak{A} \in \{P_i \cap A;\ A \in \mathfrak{G}\}$. If $P_i$ has a least element $o$ and $\omega_P^- o \notin \mathfrak{G}$ then $\emptyset \notin \mathfrak{A}_i$ and we have $o \in A$ for each $A \in \mathfrak{A}_i$. Hence $\emptyset \subset \bigcap \mathfrak{A}_i$ and also $\bigcap \mathfrak{A}_i \in \mathfrak{G}_i$.

(2) $\iota$ is an embedding of $\mathrm{Gs}(P)$ into $\prod_{i\in I} \mathrm{Gs}(P_i)$: Regarding (1) and the fact that $\iota$ is isotone it is sufficient to prove $\mathfrak{G} \nsubseteq \mathfrak{H} \Rightarrow$ there is $k \in I$ having the property $\mathfrak{G}_k \nsubseteq \mathfrak{H}_k$ for all $\mathfrak{G}, \mathfrak{H} \in \mathrm{Gs}(P)$.

Thus, let $A \in \mathfrak{G} - \mathfrak{H}$ for some $\mathfrak{G}, \mathfrak{H} \in \mathrm{Gs}(P)$. Then $I \neq \emptyset$, $A \notin \mathfrak{N}_P$ and, by 2.9, one of the following possibilities arises.

(a) There is $k \in I$ such that $P_k$ has a least element $o$ and $A = \omega_P^- o$.

(b) $\emptyset \subset P_k \cap A \subset P_k$ for an index $k \in I$.

In case (a) we have $\omega_P^- o \in \mathfrak{G} - \mathfrak{H}$ and it follows that $\emptyset \in \mathfrak{G}_k - \mathfrak{H}_k$. If (b) is true then $P_k \cap A \in \mathfrak{G}_k$. If we admit $P_k \cap A \in \mathfrak{H}_k$ then there is $B \in \mathfrak{H}$ satisfying $P_k \cap B = P_k \cap A$. By this and by 2.4(ii) we obtain $A = B \in \mathfrak{H}$ which is a contradiction.

(3) $\iota$ is a surjection: Let us denote $\Gamma = \{P_i;\ i \in I\}$ and $Q_i = \sum_{j \in \omega_{J(\Gamma)}^- i} P_j$ for each $i \in I$. Choose $(\mathfrak{H}_i)_{i\in I} \in \prod_{i\in I} \mathrm{Gs}(P_i)$ arbitrarily and put

$$\mathfrak{G} = \mathfrak{N}_P \cup \{Q_i + A;\ A \in \mathfrak{H}_i - \{P_i\},\ i \in I\}.$$

$\mathfrak{G} \in \mathrm{Gs}(P)$: The inclusions $\{P\} \cup \omega_P[P] \subseteq \mathfrak{G} \subseteq \mathfrak{O}_P$ hold obviously. We prove that $\mathfrak{G}$ is multiplicative. Let $\mathfrak{A}$, $\emptyset \subset \mathfrak{A} \subseteq \mathfrak{G}$, be arbitrary and let $A = \bigcap \mathfrak{A}$. With respect to $\mathfrak{N}_P \subseteq \mathfrak{G}$, 2.9 it is sufficient to investigate the possibilities (a), (b) from (2). If (a) is true then $A = \varepsilon_P o$. Thus it follows $A \in \mathfrak{A} \subseteq \mathfrak{G}$ by 1.7(iv), 1.4(iii). In case (b) denote $\mathfrak{B} = \{B;\ B \in \mathfrak{A} \text{ and } P_k \nsubseteq B\}$ and $\mathfrak{B}_k = \{P_k \cap B;\ B \in \mathfrak{B}\}$. Then, clearly $\mathfrak{B} \neq \emptyset \neq \mathfrak{B}_k$. For an arbitrary $B_k \in \mathfrak{B}_k$ we can find $B \in \mathfrak{B}$ such that $B_k = P_k \cap B$. If $B \in \mathfrak{N}_P$ then $B_k \in \mathfrak{N}_{P_k} - \{P_k\} \subset \mathfrak{H}_k - \{P_k\}$ regarding 2.5(i) and $B_k \subset P_k$. If $B \notin \mathfrak{N}_P$ then there are $i \in I$, $C \in \mathfrak{H}_i - \{P_i\}$ with the property $B = Q_i + C$. This and $B = Q_k + B_k$, $\emptyset \subset B_k \subset P_k$ give $i = k$ and $B_k = C \in \mathfrak{H}_k - \{P_k\}$ by 2.4. Hence $\mathfrak{B}_k \subseteq \mathfrak{H}_k - \{P_k\}$ and $A_k = P_k \cap A = \bigcap \mathfrak{B}_k \in \mathfrak{H}_k - \{P_k\}$; we have $A = Q_k + A_k \in \mathfrak{G}$.

$(\mathfrak{G}_i)_{i\in I} = (\mathfrak{H}_i)_{i\in I}$: Let $i \in I$ and $A \in \mathfrak{G}_i$ be arbitrary.

$A = P_i$ implies $A \in \mathfrak{H}_i$.

By $\emptyset \subset A \subset P_i$ we obtain $A = P_i \cap B$ for some $B \in \mathfrak{G}$. If $B \in \mathfrak{N}_P$ then $A \in \mathfrak{N}_{P_i} \subseteq \subseteq \mathfrak{H}_i$ according to 2.5(i). If $B = Q_j + C$ for some $j \in I$, $C \in \mathfrak{H}_j - \{P_j\}$ then $j = i$ and $A = C \in \mathfrak{H}_i$ regarding 2.4.

Assume that $A = \emptyset$. Then either $P_i$ has not a least element or $P_i$ has a least element $o$ and $Q_i = \omega_P^- o \in \mathfrak{G}$. In the first case $A \in \mathfrak{N}_{P_i} \subseteq \mathfrak{H}_i$ by 2.7. In the second one $Q_i \notin \mathfrak{N}_P$ according to 2.9. Thus, there are $j \in I$ and $C \in \mathfrak{H}_j - \{P_j\}$ with the property $Q_i = Q_j + C$. Hence $j = i$ and $C = A$ so that $A \in \mathfrak{H}_i$.

If we consider an arbitrary element $A \in \mathfrak{H}_i$ then one of the cases $A = P_i, \emptyset \subset A \subset \subset P_i, A = \emptyset$ arises. $A = P_i \in \mathfrak{G}_i$ with respect to (1). If $\emptyset \subset A \subset P_i$ then $B = Q_i + A$ and $A = P_i \cap B \in \mathfrak{G}_i$. By $A = \emptyset$ it follows that $Q_i \in \mathfrak{G}$ and by this $A \in \mathfrak{G}_i$.

**2.11. Corollary.** $\mathbf{G} = \Pi\mathbf{G}$.

## 3. THE CONCEPT OF A SIMPLE POSET

**3.1. Definition.** Let $P$ be a poset. We say that an ordered pair $(a, a')$ of elements of $P$ is a *twin-pair* in $P$ whenever $a \nleq x \Leftrightarrow x \leq a'$ for each $x \in P$.

We put $\mathbf{U}_P = \mathbf{V}_P \cup \mathbf{W}_P$ where $\mathbf{V}_P$ is the set of all first members of twin-pairs in $P$ and $\mathbf{W}_P$ is the set of all such elements of $P$ which are comparable with all elements of $P$. Clearly, $\mathbf{V}_P = \{a;\ a \in P \text{ and } \varepsilon_P a \in \omega_P[P]\}$ and $\mathbf{W}_P = \{a;\ a \in P \text{ and } \varepsilon_P a = \omega_P^- a\}$.

**3.2. Lemma.** $\mathbf{V}_P = \mathbf{U}_P \cap \mathbf{IR}_P$ *for every poset* $P$.

Proof: Let $a \in \mathbf{V}_P$ be arbitrary. One can find $a' \in P$ such that $(a, a')$ is a twin-pair in $P$. Suppose that $B \subseteq P$ satisfies $\bigvee_P B = a$. If $a \notin B$ then $a \nleq b$ and thus $b \leq a'$ for all $b \in B$. This implies $a = \bigvee_P B \leq a'$. But then $a' \nleq a'$ by the definition of a twin-pair which is a contradiction. Hence $a \in B$ and we have proved $a \in \mathbf{IR}_P$, $\mathbf{V}_P \subseteq \mathbf{IR}_P$. That is why $\mathbf{V}_P \subseteq \mathbf{U}_P \cap \mathbf{IR}_P$.

Let us admit that there is an element $a \in (\mathbf{U}_P \cap \mathbf{IR}_P) - \mathbf{V}_P$. Then $\varepsilon_P a = \omega_P^- a$ regarding $a \in \mathbf{U}_P - \mathbf{V}_P = \mathbf{W}_P$ and because of $a \in \mathbf{IR}_P$, $\bigvee_P \omega_P^- a = a$ is not true. Thus, there exists an upper bound $b$ of $\omega_P^- a$ with the property $a \nleq b$. If $b < a$ then $b \in \omega_P^- a$ and, further, $\varepsilon_P a = \omega_P^- a = \omega_P b$. That means $a \in \mathbf{V}_P$ which is a contradiction. In case $b \nless a$ it holds $b \in \varepsilon_P a - \omega_P^- a$; this contradicts $a \in \mathbf{W}_P$.

**3.3. Definition.** We say that $(R, C)$ is a *suitable pair* in a poset $P$ if the assertions (i), (ii) hold.

(i) $\mathbf{IR}_P \subseteq R \subseteq P$.

(ii) $\mathbf{U}_P \cap R \subseteq C \subseteq R$.

We denote by $\mathrm{S}(P)$ the set of all suitable pairs in $P$ ordered in the following way. $(R_1, C_1) \leq (R_2, C_2)$ if $R_1 \subseteq R_2$ and $C_1 \subseteq C_2$ for arbitrary $(R_1, C_1), (R_2, C_2) \in \mathrm{S}(P)$.

**3.4. Theorem.** *If $P$ is a poset then* $\mathrm{S}(P) \cong 2^H \times 3^I \times 2^J$ *where* $H = \mathbf{U}_P - \mathbf{IR}_P$, $I = P - (\mathbf{U}_P \cup \mathbf{IR}_P)$ and $J = \mathbf{IR}_P - \mathbf{U}_P$.

Proof. For each $(R, C) \in \mathrm{S}(P)$ put $\iota(R, C) = ((k_a)_{a \in H}, (m_a)_{a \in I}, (n_a)_{a \in J})$ in such a way that

$$k_a = \begin{Bmatrix} 0 \text{ for } a \notin R \\ 1 \text{ for } a \in R \end{Bmatrix}, \quad m_a = \begin{Bmatrix} 0 \text{ for } a \notin R \\ 1 \text{ for } a \in R - C \\ 2 \text{ for } a \in C \end{Bmatrix} \text{ and } n_a = \begin{Bmatrix} 0 \text{ for } a \notin C \\ 1 \text{ for } a \in C \end{Bmatrix}.$$

$\iota$ is an embedding of $\mathrm{S}(P)$ into $2^H \times 3^I \times 2^J$: It is evident that $\iota$ is isotone. Let us thus suppose that $(R_1, C_1) \nleq (R_2, C_2)$ for some $(R_1, C_1), (R_2, C_2) \in \mathrm{S}(P)$.

If there is $a \in R_1 - R_2$ then $a \notin \mathbf{IR}_P$ and either $a \in \mathbf{U}_P$ or $a \notin \mathbf{U}_P$. In the first case we have $a \in \mathbf{U}_P - \mathbf{IR}_P$; by this we obtain $k_a = 1$ in $\iota(R_1, C_1)$, $k_a = 0$ in $\iota(R_2, C_2)$. In the second one $a \in P - (\mathbf{U}_P \cup \mathbf{IR}_P)$, $m_a > 0$ in $\iota(R_1, C_1)$ and $m_a = 0$ in $\iota(R_2, C_2)$.

Let there exist $a \in C_1 - C_2$. Since $\mathbf{U}_P \cap \mathbf{IR}_P \subseteq \mathbf{U}_P \cap R_2 \subseteq C_2$ and $a \notin C_2$, it holds $a \notin \mathbf{U}_P \cap \mathbf{IR}_P$. Thus, exactly one of the assertions $a \in \mathbf{U}_P - \mathbf{IR}_P$, $a \in P - (\mathbf{U}_P \cup \mathbf{IR}_P)$, $a \in \mathbf{IR}_P - \mathbf{U}_P$ is true. In the first case $a \in C_1 \Rightarrow a \in R_1$, $a \notin C_2 \Rightarrow a \notin \mathbf{U}_P \cap R_2$ and, as $a \in \mathbf{U}_P$, it holds $a \notin R_2$. Hence $a \in R_1 - R_2$ and we have $k_a = 1$ in $\iota(R_1, C_1)$, $k_a = 0$ in $\iota(R_2, C_2)$. In the second one it holds $m_a = 2$ in $\iota(R_1, C_1)$, $m_a < 2$ in $\iota(R_2, C_2)$ and in the third one $n_a = 1$ in $\iota(R_1, C_1)$, $n_a = 0$ in $\iota(R_2, C_2)$.

We have shown that each possibility gives $\iota(R_1, C_1) \nleq \iota(R_2, C_2)$ which proves the statement.

$\iota$ is a surjection: Let us put $R = \mathbf{IR}_P \cup \{a \in H;\ k_a = 1\} \cup \{a \in I;\ m_a \geq 1\}$ and $C = (\mathbf{U}_P \cap \mathbf{IR}_P) \cup \{a \in H;\ k_a = 1\} \cup \{a \in I;\ m_a = 2\} \cup \{a \in J;\ n_a = 1\}$ for an arbitrary element $\pi = ((k_a)_{a \in H}, (m_a)_{a \in I}, (n_a)_{a \in J}) \in 2^H \times 3^I \times 2^J$.

$\mathbf{IR}_P \subseteq R \subseteq P$ is true obviously. This, $\mathbf{U}_P \cap R = (\mathbf{U}_P \cap \mathbf{IR}_P) \cup \{a \in H; k_a = 1\} \subseteq C$ and $C \subseteq R$ imply $(R, C) \in \mathrm{S}(P)$. It is now easy to verify that $\iota(R, C) = \pi$.

In the following we shall need some corollaries and nonessential modifications of statements from [2]. For a better understanding of the text we introduce all of them consecutively.

**3.5. Lemma.** ([2], 2.10(i), 2.11) *Let $P$ be a poset and $\mathfrak{G} \in \mathrm{Gs}(P)$. Then the assertions* (i), (ii) *hold.*

(i) $\omega_P: P \to \mathfrak{G}$ *is an embedding.*

(ii) $\mathbf{IR}_{\mathfrak{G}}$ *and* $\mathbf{P}_{\mathfrak{G}}$ *are subsets of* $\omega_P[P]$.

**3.6. Theorem.** ([2], 4.7, 4.10, 4.13) *Let $P$ be a poset and $R$, $C$ subsets in $P$. Then* $(R, C) \in \mathrm{S}(P)$ *if and only if there is* $\mathfrak{G} \in \mathrm{Gs}(P)$ *satisfying* $\mathbf{IR}_{\mathfrak{G}} = \omega_P[R]$, $\mathbf{P}_{\mathfrak{G}} = \omega_P[C]$.

**3.7. Lemma.** ([2], 3.4, 3.5) *Let $P$ be a poset,* $\mathfrak{G} \in \mathrm{Gs}(P)$ *and $a \in P$. Then the assertions* (i), (ii) *are true.*

(i) $\omega_P a \in \mathbf{IR}_{\mathfrak{G}} \Leftrightarrow \omega_P^- a \in \mathfrak{G}$.

(ii) $\omega_P a \in \mathbf{P}_{\mathfrak{G}} \Leftrightarrow \varepsilon_P a \in \mathfrak{G}$.

**3.8. Corollary.** *If $P$ is a poset and* $\mathfrak{G}, \mathfrak{H} \in \mathrm{Gs}(P)$ *then* $\mathfrak{G} \subseteq \mathfrak{H} \Rightarrow \mathbf{IR}_{\mathfrak{G}} \subseteq \mathbf{IR}_{\mathfrak{H}}$, $\mathbf{P}_{\mathfrak{G}} \subseteq \mathbf{P}_{\mathfrak{H}}$.

Proof. Suppose that $\mathfrak{G} \subseteq \mathfrak{H}$. If $A \in \mathbf{IR}_{\mathfrak{G}}$ then there is $a \in P$ such that $A = \omega_P a$ according to 3.5(ii). By this and by 3.7(i) it follows that $\omega_P^- a \in \mathfrak{G}$ and this gives $\omega_P^- a \in \mathfrak{H}$. Then $A = \omega_P a \in \mathbf{IR}_{\mathfrak{H}}$ by 3.7(i) again. The inclusion $\mathbf{P}_{\mathfrak{G}} \subseteq \mathbf{P}_{\mathfrak{H}}$ can be proved similarly using 3.7(ii) instead of 3.7(i).

**3.9. Corollary.** *$\mathbf{IR}_{\mathfrak{N}} = \omega_P[\mathbf{IR}_P]$ and $\mathbf{P}_{\mathfrak{N}} = \omega_P[\mathbf{V}_P]$ for every poset $P$.*

Proof. If $P$ is a poset then there exists $(R, C) \in \mathrm{S}(P)$ with the properties $\omega_P[R] = \mathbf{IR}_{\mathfrak{N}}$, $\omega_P[C] = \mathbf{P}_{\mathfrak{N}}$ by 3.6. $(R_0, C_0) = (\mathbf{IR}_P, \mathbf{V}_P)$ is the least element in $\mathrm{S}(P)$ regarding 3.2. From this and 3.6 it follows $\mathbf{IR}_{\mathfrak{G}} = \omega_P[R_0]$, $\mathbf{P}_{\mathfrak{G}} = \omega_P[C_0]$ for some $\mathfrak{G} \in \mathrm{Gs}(P)$. According to $\mathfrak{N}_P \subseteq \mathfrak{G}$ and 3.8 we obtain $\omega_P[R] = \mathbf{IR}_{\mathfrak{N}} \subseteq \mathbf{IR}_{\mathfrak{G}} = \omega_P[R_0]$, $\omega_P[C] = \mathbf{P}_{\mathfrak{N}} \subseteq \mathbf{P}_{\mathfrak{G}} = \omega_P[C_0]$. Then $(R, C) \leqq (R_0, C_0)$ by 3.5(i) and, immediately, $(R, C) = (R_0, C_0)$.

**3.10. Corollary.** *Let $P$ be a poset. Then the assertions* (i), (ii) *are true*

(i) $\varepsilon_P a \notin \mathfrak{N}_P \Leftrightarrow a \in P - \mathbf{V}_P$.

(ii) *There is a bijection of $P - \mathbf{V}_P$ onto $\mathbf{A}^d_{\mathrm{Gs}(P)}$.*

Proof. It follows by 3.9 and 3.5(i) that $\omega_P a \in \mathbf{P}_{\mathfrak{N}} \Leftrightarrow a \in \mathbf{V}_P$. This and 3.7(ii) give $\varepsilon_P a \in \mathfrak{N}_P \Leftrightarrow a \in \mathbf{V}_P$ which is equivalent to (i).

The proof of 1.9 and (i) imply $\mathbf{A}^d_{\mathrm{Gs}(P)} = \{\mathfrak{D}_P - \{\varepsilon_P a\};\ a \in P - \mathbf{V}_P\}$. By this and by 1.7(i) we obtain (ii).

**3.11. Definition.** If $P$ is a poset and $Q \subseteq P$ then we put $\mathfrak{H}^Q_P = \{A;\ A \in \mathfrak{D}_P$ and $\omega_P^- a \subseteq A \Rightarrow a \in A$ for all $a \in P - Q\}$.

**3.12. Lemma.** *Let $P$ be a poset and $Q \subseteq P$. Then $\mathfrak{H}^Q_P = \langle \mathfrak{H}^Q_P \rangle$.*

Proof. $P \in \mathfrak{H}^Q_P$ holds trivially. $\mathfrak{H}^Q_P$ is multiplicative: Let us take $\mathfrak{A}$, $\emptyset \subset \mathfrak{A} \subseteq \mathfrak{H}^Q_P$, arbitrarily. If $\omega_P^- a \subseteq \bigcap \mathfrak{A}$ for an element $a \in P - Q$ then $\omega_P^- a \subseteq A$ and $a \in A$ for each $A \in \mathfrak{A}$. Thus $a \in \bigcap \mathfrak{A}$.

**3.13. Lemma.** ([2], 3.11, 3.12, 3.13) *Let $P$ be a poset and $\mathbf{IR}_P \subseteq R \subseteq P$. Then the assertions* (i), (ii), (iii) *hold.*

(i) $\mathfrak{H}^R_P \in \mathrm{Gs}(P)$.

(ii) $\mathbf{IR}_{\mathfrak{H}^R_P} = \omega_P[R] = \mathbf{P}_{\mathfrak{H}^R_P}$.

(iii) $\mathbf{IR}_{\mathfrak{G}} \subseteq \omega_P[R] \Leftrightarrow \mathfrak{G} \subseteq \mathfrak{H}^R_P$ *for all* $\mathfrak{G} \in \mathrm{Gs}(P)$.

**3.14. Definition.** Let $P$ be a poset and $(R, C) \in \mathrm{S}(P)$. We put $\mathfrak{J}_P(R, C) = \mathfrak{H}^R_P - \varepsilon_P[R - C]$.

**3.15. Theorem.** *Let $P$ be a poset and $(R, C) \in \mathrm{S}(P)$. Then the assertions* (i)–(iv) *are true.*

(i) $\mathfrak{J}_P(R, C) \in \mathrm{Gs}(P)$.

(ii) $\mathbf{IR}_{\mathfrak{J}_P(R,C)} = \omega_P[R]$.

(iii) $\mathbf{P}_{\mathfrak{J}_P(R,C)} = \omega_P[C]$.

(iv) $\mathbf{IR}_{\mathfrak{G}} \subseteq \omega_P[R]$ *and* $\mathbf{P}_{\mathfrak{G}} \subseteq \omega_P[C] \Leftrightarrow \mathfrak{G} \subseteq \mathfrak{J}_P(R, C)$ *for all* $\mathfrak{G} \in \mathrm{Gs}(P)$.

Proof. (1) If $\varepsilon_P a \in \mathfrak{N}_P$ then $a \in \mathbf{V}_P$ by 3.10(i) and $a \in C$ as $\mathbf{V}_P \subseteq C$ by 3.2. Hence $\varepsilon_P[R - C] \cap \mathfrak{N}_P = \emptyset$ and we obtain $\mathfrak{Z}_P(R, C) = \mathfrak{H}_P^R - \varepsilon_P[R - C] \in \mathrm{Gs}(P)$ using 3.13(i), 1.7(ii) and 1.8.

(2) $\varepsilon_P[R - C] \cap \omega_P^-[P] = \emptyset$: Let $a \in R - C$ be arbitrary. Then $a \in R$, $a \notin \mathbf{U}_P \cap \cap R$ and for this reason $a \notin \mathbf{U}_P \supseteq \mathbf{W}_P$. Hence $\omega_P^- a \subset \varepsilon_P a$. If we admit $\varepsilon_P a \in \omega_P^-[P]$ then $\varepsilon_P a = \omega_P^- b$ for an element $b \in P$. This is equivalent to $a \nleq x \Leftrightarrow x < b$ for all $x \in P$ and it gives $a = b$. But then $\varepsilon_P a = \omega_P^- a$ which is a contradiction.

(3) $R = \{a \in P;\ \omega_P a \in \mathbf{IR}_{\mathfrak{H}_P^R}\} = \{a \in P;\ \omega_P^- a \in \mathfrak{H}_P^R\} = \{a \in P;\ \omega_P^- a \in \mathfrak{Z}_P(R, C)\}$ according to 3.13(ii), 3.5(i), 3.7(i) and (2). By $\omega_P^- a \in \mathfrak{Z}_P(R, C) \Leftrightarrow a \in R$ and by 3.7(i) we obtain (ii). Similarly, 3.13(ii), 3.5(i) and 3.7(ii) imply $R = \{a \in P;\ \varepsilon_P a \in \mathfrak{H}_P^R\}$ so that $C = \{a \in P;\ \varepsilon_P a \in \mathfrak{Z}_P(R, C)\}$ regarding 1.7(i). This and 3.7(ii) give (iii).

(4) Let us take $\mathfrak{G} \in \mathrm{Gs}(P)$ arbitrarily. $\mathfrak{G} \subseteq \mathfrak{Z}_P(R, C)$ implies $\mathbf{IR}_{\mathfrak{G}} \subseteq \omega_P[R]$, $\mathbf{P}_{\mathfrak{G}} \subseteq \subseteq \omega_P[C]$ according to 3.8 and (ii), (iii).

If $\mathbf{IR}_{\mathfrak{G}} \subseteq \mathbf{IR}_{\mathfrak{Z}_P(R,C)}$ then $\mathbf{IR}_{\mathfrak{G}} \subseteq \omega_P[R]$ by (ii); this and 3.13(iii) give $\mathfrak{G} \subseteq \mathfrak{H}_P^R$. If, moreover, $\mathbf{P}_{\mathfrak{G}} \subseteq \mathbf{P}_{\mathfrak{Z}_P(R,C)}$ then $\mathbf{P}_{\mathfrak{G}} \subseteq \omega_P[C]$ with respect to (iii). By this and by 3.5(i) we obtain $\omega_P a \in \mathbf{P}_{\mathfrak{G}} \Leftrightarrow a \in C$. Then $\varepsilon_P a \in \mathfrak{G} \Leftrightarrow a \in C$ by 3.7(ii) and, clearly, $\mathfrak{G} \cap \varepsilon_P[R - C] = \emptyset$. We have proved $\mathfrak{G} \subseteq \mathfrak{H}_P^R - \varepsilon_P[R - C] = \mathfrak{Z}_P(R, C)$.

**3.16. Corollary.** $\mathfrak{Z}_P$: $\mathrm{S}(P) \to \mathrm{Gs}(P)$ *is an embedding for each poset* $P$.

Proof. Let a poset $P$ and $(R_1, C_1), (R_2, C_2) \in \mathrm{S}(P)$ be arbitrary. Regarding 3.5(i) it holds $(R_1, C_1) \leqq (R_2, C_2)$ iff $\omega_P[R_1] \subseteq \omega_P[R_2]$ and $\omega_P[C_1] \subseteq \omega_P[C_2]$. This assertion is equivalent to $\mathbf{IR}_{\mathfrak{Z}_P(R_1,C_1)} \subseteq \omega_P[R_2]$ and $\mathbf{P}_{\mathfrak{Z}_P(R_1,C_1)} \subseteq \omega_P[C_2]$ by 3.15(ii), (iii) and this is true iff $\mathfrak{Z}_P(R_1, C_1) \subseteq \mathfrak{Z}_P(R_2, C_2)$ by 3.15(iv).

**3.17. Definition.** A poset $P$ is said to be *simple* whenever $\mathfrak{Z}_P$: $\mathrm{S}(P) \to \mathrm{Gs}(P)$ is a surjection.

We denote by $\mathscr{P}_S$ the class of all simple posets and by $\mathbf{G}_S$ the class of all complete lattices isomorphic to $\mathrm{Gs}(P)$ for some $P \in \mathscr{P}_S$.

**3.18. Corollary.** *If* $P$ *is a simple poset then* $\mathrm{Gs}(P) \cong 2^H \times 3^I \times 2^J$ *where* $H = = \mathbf{U}_P - \mathbf{IR}_P$, $I = P - (\mathbf{U}_P \cup \mathbf{IR}_P)$ *and* $J = \mathbf{IR}_P - \mathbf{U}_P$.

Proof. This is a consequence of 3.16, 3.4.

**3.19. Theorem.** *Let* $P$ *be a poset. Then the assertions* (i), (ii) *are equivalent.*

(i) $P \in \mathscr{P}_S$.

(ii) $\mathfrak{D}_P \subseteq \mathfrak{N}_P \cup \varepsilon_P[P] \cup \omega_P^-[P]$.

Proof. Assume that there is $A \in \mathfrak{D}_P - (\mathfrak{N}_P \cup \varepsilon_P[P] \cup \omega_P^-[P])$. If we denote $\mathfrak{G} = \langle \mathfrak{N}_P, A \rangle$ and $\mathfrak{H} = \mathfrak{G} - \{A\}$ then $\mathfrak{H} \in \mathrm{Gs}(P)$ by 1.14. Let us admit that $\mathfrak{H} \in \in \mathfrak{Z}_P[\mathrm{S}(P)]$. Then $\mathfrak{H} = \mathfrak{Z}_P(R, C)$ for some $(R, C) \in \mathrm{S}(P)$ and $\mathbf{IR}_{\mathfrak{H}} = \omega_P[R]$, $\mathbf{P}_{\mathfrak{H}} = = \omega_P[C]$ according to 3.15(ii), (iii). By this and by 3.7(i), (ii), 3.5(i) we obtain $R = = \{a \in P;\ \omega_P^- a \in \mathfrak{H}\}$, $C = \{a \in P;\ \varepsilon_P a \in \mathfrak{H}\}$. This and $\omega_P^- a \in \mathfrak{H} \Leftrightarrow \omega_P^- a \in \mathfrak{G}$, $\varepsilon_P a \in \in \mathfrak{H} \Leftrightarrow \varepsilon_P a \in \mathfrak{G}$ for each $a \in P$ imply $\mathbf{IR}_{\mathfrak{G}} = \omega_P[R]$, $\mathbf{P}_{\mathfrak{G}} = \omega_P[C]$ regarding 3.7(i), (ii). Then $\mathfrak{G} \subseteq \mathfrak{H}$ by 3.15(iv) which is a contradiction; hence $\mathfrak{H} \notin \mathfrak{Z}_P[\mathrm{S}(P)]$ and also $P \notin \mathscr{P}_S$.

Suppose that $P \notin \mathscr{P}_S$. Then there exists $\mathfrak{G} \in \mathrm{Gs}(P) - \mathfrak{Z}_P[\mathrm{S}(P)]$ and we can find $(R, C) \in \mathrm{S}(P)$ such that $\mathbf{IR}_{\mathfrak{G}} = \omega_P[R]$, $\mathbf{P}_{\mathfrak{G}} = \omega_P[C]$ by 3.6. From this it follows

(a) $$\varepsilon_P a \in \mathfrak{G} \Leftrightarrow \varepsilon_P a \in \mathfrak{Z}_P(R, C), \quad \omega_P^- a \in \mathfrak{G} \Leftrightarrow \omega_P^- a \in \mathfrak{Z}_P(R, C)$$

according to 3.15(ii), (iii), 3,7(i), (ii) on the one hand and $\mathfrak{G} \subseteq \mathfrak{Z}_P(R, C)$ by 3.15(iv) on the other hand. As we suppose $\mathfrak{G} \neq \mathfrak{Z}_P(R, C)$, there is $A \in \mathfrak{Z}_P(R, C) - \mathfrak{G}$. By this, (a) and $\mathfrak{N}_P \subseteq \mathfrak{G}$ we get $A \in \mathfrak{D}_P - (\mathfrak{N}_P \cup \varepsilon_P[P] \cup \omega_P^-[P])$.

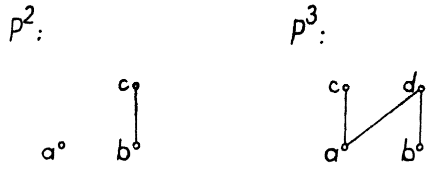


Figure 1

**3.20. Example.** By means of 3.19 one can easily see that the posets $P^2$, $P^3$ from Fig. 1 are simple. Regarding 3.18 and $\mathbf{U}_{P^2} = \{a, b\}$, $\mathbf{IR}_{P^2} = \{a, b, c\}$, $\mathbf{U}_{P^3} = \{a, b, c\} = \mathbf{IR}_{P^3}$ it holds $\mathrm{Gs}(P^2) \cong 2^{\emptyset} \times 3^{\emptyset} \times 2^{\{c\}} \cong 2$ and $\mathrm{Gs}(P^3) \cong 2^{\emptyset} \times 3^{\{d\}} \times 2^{\emptyset} \cong 3$.

**3.21. Theorem.** $\mathbf{G}_S = \Pi\{2, 3\}$.

Proof. $\mathbf{G}_S \subseteq \Pi\{2, 3\}$ according to 3.18.

If $L \in \Pi\{2, 3\}$ then there are ordinal numbers $\mu$, $\nu$ satisfying $L \cong 2^\mu \times 3^\nu$. Let us put $\varkappa = \mu + \nu$, $P_i = P^2$ for $i < \mu$, $P_i = P^3$ for $\mu \leqq i < \varkappa$ and $P = \sum_{i \in \varkappa}^{m} P_i$. Then $\mathrm{Gs}(P) \cong \prod_{i \in \varkappa} \mathrm{Gs}(P_i) \cong 2^\mu \times 3^\nu \cong L$ by 2.10 and 3.20.

$P \in \mathscr{P}_S$: Choose $A \in \mathfrak{D}_P$ arbitrarily. With respect to 2.9 it holds $A \in \mathfrak{N}_P$ in all cases except (a), (b) from 2.10(2). The possibility (a) does never arise because $P_i$ has not a least element for all $i \in \varkappa$. If (b) is true then there is $k \in \varkappa$ such that $\emptyset \subset P_k \cap A \subset P_k$. By $P_k \in \{P^2, P^3\} \subseteq \mathscr{P}_S$ and by 3.19 it follows $P_k \cap A \in \mathfrak{N}_{P_k} \cup \varepsilon_{P_k}[P_k] \cup \omega_{P_k}^-[P_k]$. This gives $A \in \mathfrak{N}_P \cup \varepsilon_P[P] \cup \omega_P^-[P]$ regarding 2.5 and then $P \in \mathscr{P}_S$ by 3.19.

The following example is a negative answer to the question whether $\mathrm{Gs}(P) \in \mathbf{G}_S \Rightarrow P \in \mathscr{P}_S$ for each poset $P$.

**3.22. Example.** Consider the poset $Q$ from Fig. 2 and put $A = \{a, b\}$, $B = \{a, b, d\}$, $C = \{a, c, d\}$, $D = \{a, b, c, d\}$, $E = \{a, b, c, d, e\}$. One can easily verify that $\mathrm{Gs}(Q)$ is the complete lattice from Fig. 2 where, for example, the generating system $\mathfrak{N}_Q \cup \{A, C\}$ is denoted by $A, C$.

$\mathrm{Gs}(Q) \in \mathbf{G}_S$ obviously and, at the same time, $Q \notin \mathscr{P}_S$ by 3.19 because $A \in \mathfrak{D}_P - (\mathfrak{N}_P \cup \varepsilon_P[P] \cup \omega_P^-[P])$.

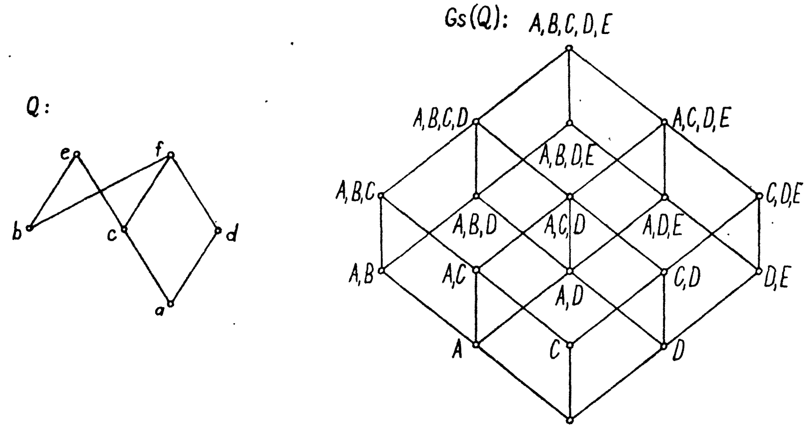


Figure 2

## 4. COMPLEMENTATION IN THE CLASS G

**4.1. Lemma.** *Let $P$ be a poset, $A \in \mathfrak{D}_P - \mathfrak{N}_P$ and $\mathfrak{G} = \langle \mathfrak{N}_P, A \rangle$. Then $\mathfrak{G}$ has a complement in* Gs($P$) *if and only if $A \in \varepsilon_P[P - \mathbf{V}_P]$.*

Proof. If $A \notin \varepsilon_P[P - \mathbf{V}_P]$ then $A \notin \varepsilon_P[P]$ according to $A \notin \mathfrak{N}_P$ and 3.10(i). This and 1.7(iv) give $A \notin \mathbf{IR}^d_{\mathfrak{D}}$. Then there is a system $\mathfrak{B}$, $\emptyset \subset \mathfrak{B} \subseteq \mathfrak{D}_P$, satisfying $A = \bigcap \mathfrak{B}$, $A \notin \mathfrak{B}$ with respect to $A \subset P$ and 1.4(iii). Let us admit that $\mathfrak{G}$ has a complement $\mathfrak{H}$ in Gs($P$). Then $\mathfrak{D}_P = \mathfrak{G} \vee \mathfrak{H} = \{C \cap D;\ C \in \mathfrak{G}$ and $D \in \mathfrak{H}\}$ by 1.13(i). Especially, for each $B \in \mathfrak{B}$ there are $C_B \in \mathfrak{G}$, $D_B \in \mathfrak{H}$ such that $B = C_B \cap D_B$. By this, $A \subset B \subseteq C_B$ and 1.13(ii) we have $C_B \in \mathfrak{N}_P \subseteq \mathfrak{H}$. We obtain consecutively $B \in \mathfrak{H}$, $\mathfrak{B} \subseteq \mathfrak{H}$ and $A \in \mathfrak{H}$. But then $A \in \mathfrak{G} \cap \mathfrak{H} = \mathfrak{N}_P$ which is a contradiction.

If $A \in \varepsilon_P[P - \mathbf{V}_P]$ then there is $a \in P - \mathbf{V}_P$ such that $A = \varepsilon_P a$. Put $\mathfrak{H}_a = \mathfrak{H}_P^{P-\{a\}}$ and $\mathfrak{H} = \langle \mathfrak{N}_P \cup \mathfrak{H}_a \rangle$.

$\mathfrak{G} \vee \mathfrak{H} = \mathfrak{D}_P$: It is sufficient to prove that $\mathfrak{D}_P \subseteq \mathfrak{G} \vee \mathfrak{H}$. For the sake of this let us take $B \in \mathfrak{D}_P$ arbitrarily. If $\omega_P^- a \subseteq B \Rightarrow a \in B$ then $B \in \mathfrak{H}_a \subseteq \mathfrak{H} \subseteq \mathfrak{G} \vee \mathfrak{H}$. In case $\omega_P^- a \subseteq B$, $a \notin B$ denote $B_a = B \cup \{a\}$. Then $B_a \in \mathfrak{H}_a \subseteq \mathfrak{H}$ and $B = \varepsilon_P a \cap B_a \in \mathfrak{G} \vee \mathfrak{H}$.

$\mathfrak{G} \cap \mathfrak{H} = \mathfrak{N}_P$: We prove the inclusion $\mathfrak{G} \cap \mathfrak{H} \subseteq \mathfrak{N}_P$. Thus, let $B \in \mathfrak{G} \cap \mathfrak{H}$ be arbitrary. Since $B \in \mathfrak{G}$ and $\mathfrak{G} = \langle \mathfrak{N}_P, \varepsilon_P a \rangle$, there are $C_1 \in \mathfrak{N}_P$ and $D_1 \in \langle \{\varepsilon_P a\} \rangle = \{P, \varepsilon_P a\}$ with the property $B = C_1 \cap D_1$ by 1.12. If $B = C_1$ then $B \in \mathfrak{N}_P$. If $B \subset C_1$ then $D_1 = \varepsilon_P a$, $a$ is the least element in $C_1 - B$ and, clearly, $\omega_P^- a \subseteq B$. Regarding 1.12 and 3.12, $\mathfrak{H} = \{C \cap D;\ C \in \mathfrak{N}_P, D \in \mathfrak{H}_a\}$. By this and by $B \in \mathfrak{H}$ we obtain $B = C_2 \cap D_2$ where $C_2 \in \mathfrak{N}_P$, $D_2 \in \mathfrak{H}_a$. Since $\omega_P^- a \subseteq B \subseteq D_2$, we have $a \in D_2$; this and $a \notin B$ give $a \notin C_2$. Then $B \subseteq C_1 \cap C_2$, $a$ is a least element in $C_1 - B$ and $a \notin C_2$. It is now obvious that $B = C_1 \cap C_2 \in \mathfrak{N}_P$.

**4.2. Definition.** We denote by $\mathbf{G}_C$ the class of all complete lattices $L \in \mathbf{G}$ such that each element of $\mathbf{IR}_L$ has a complement in $L$.

The class of all posets $P$ satisfying $\mathrm{Gs}(P) \in \mathbf{G}_C$ will be denoted by $\mathscr{P}_C$.

**4.3. Theorem.** *Let $P$ be a poset. Then the assertions* (i), (ii), (iii) *are equivalent.*

(i) $P \in \mathscr{P}_C$.

(ii) $\mathfrak{D}_P \subseteq \mathfrak{N}_P \cup \varepsilon_P[P]$.

(iii) $\mathrm{Gs}(P) \cong 2^{P - \mathbf{V}_P}$.

Proof. (i) $\Rightarrow$ (ii): If there is $A \in \mathfrak{D}_P - (\mathfrak{N}_P \cup \varepsilon_P[P])$ then $\langle \mathfrak{N}_P, A \rangle \in \mathbf{IR}_{\mathrm{Gs}(P)}$ by 1.15 and $\langle \mathfrak{N}_P, A \rangle$ has not a complement in $\mathrm{Gs}(P)$ according to 4.1. Hence $P \notin \mathscr{P}_C$.

(ii) $\Rightarrow$ (iii): If $\mathfrak{D}_P \subseteq \mathfrak{N}_P \cup \varepsilon_P[P]$ then $\mathfrak{D}_P - \mathfrak{N}_P = \varepsilon_P[P - \mathbf{V}_P]$ regarding 3.10(i). By this, 1.7(i), (iv) and 1.8 it follows that the map $\iota$: $2^{P - \mathbf{V}_P} \to \mathrm{Gs}(P)$ defined by $\iota X = \mathfrak{N}_P \cup \varepsilon_P[X]$ is an isomorphism.

(iii) $\Rightarrow$ (i) holds trivially.

**4.4. Theorem.** $\mathbf{G}_C = \Pi\{2\}$.

Proof. $\mathbf{G}_C \subseteq \Pi\{2\}$ is true by 4.3. The validity of the converse inclusion can be verified by the method used in the proof of 3.21.

**4.5. Definition.** We denote by $\mathscr{P}_T$ the class of all posets with a trivial (one-element) gs-lattice.

**4.6. Theorem.** $P \in \mathscr{P}_T \Leftrightarrow \mathfrak{D}_P \subseteq \mathfrak{N}_P$ *for each poset $P$.*

## REFERENCES

[1] G. Szász: *Einführung in die Verbandstheorie,* Verlag der Ungarischen Akademie der Wissenschaften, Budapest, 1962.

[2] J. Dalík: *An embedding problem and its application in linguistics,* Arch. Math. Brno Fasc. 3, Tom XVI, 1978, p. 123—138.

[3] R. P. Dilworth and J. E. McLaughlin: *Distributivity in lattices,* Duke Math. Journal. vol. 19 (1952), p. 683—693.

[4] N. Funayama: *Imbedding partly ordered sets into infinitely distributive complete lattices,* Tohoku Math. Journal 8, N1 (1956), p. 54—62.

[5] G. B. Robison and E. S. Wolk: *The imbedding operators on a partially ordered set,* Proc. Amer. Math. Soc. 8 (1957), p. 551—559.

[6] M. Ward: *The closure operators of a lattice,* Ann. of Math. vol. 43 (1942), p. 191—196.

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