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SOME PROPERTIES OF AN ORDERING RELATION ON CERTAIN CLASSES OF FUNCTORS

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Let $\mathcal{P}$ and $\mathcal{T}$ be the class of all partially ordered sets and topological spaces in the sense of Čech, respectively. Consider a mapping $F : \mathcal{P} \to \mathcal{T}$ such that for every $(A, \leqq) \in \mathcal{P}$, $F(A, \leqq)$ is a topological space with the underlying set A and with a topology convexly compatible (or convexly weakly compatible) with the ordering $\leqq$, (for these notions, cf. [3]). Such a mapping will be called an α-mapping (or a β-mapping, respectively) provided that F is a covariant functor of the category $\mathfrak{P}$ of all partially ordered sets with isomorphisms as morphisms to the category $\mathfrak{T}$ of all topological spaces with homeomorphisms as morphisms, putting $F(\varphi) = \varphi$ for every $\varphi \in \operatorname{Mor} \mathfrak{P}$. Denote by $\alpha(\mathcal{P}, \mathcal{T})$ and $\beta(\mathcal{P}, \mathcal{T})$ the class of all α- and β-mappings, respectively. On these classes there can be defined an ordering relation in a natural way. The aim of this paper is to investigate some properties of the partially ordered classes $\alpha(\mathcal{P}, \mathcal{T})$, $\beta(\mathcal{P}, \mathcal{T})$. The idea of the investigation came from [2].

1. PRELIMINARIES

For the sake of completeness let us recall some definitions introduced in [3]. Denote by 2^P the system of all subsets of a set P.

1.1. Definition. *Let P be a given set. A mapping $u : 2^P \to 2^P$ is said to be a topology on P, if the following three axioms are satisfied:*

(1) $u\emptyset = \emptyset$,

(2) $M \subset P \Rightarrow M \subset uM$,

(3) $M_1 \subset M_2 \subset P \Rightarrow uM_1 \subset uM_2$.

If u is a topology on P, the pair (P, u) is called a topological space. The system of all topologies on P is denoted by $T(P)$.

1.2. Definition. *A set $O \subset P$ is said to be a neighborhood of an element $x \in P$ in the space (P, u), if $x \notin u(P - O)$. The notation $D_u(x)$ is used for the system of all neighborhoods of x in (P, u).*

The following statement enables one to introduce a topology into a set P (cf. [1], 4.1).

1.3. Theorem. *Let P be a set and let $D(x)$ be a nonvoid family of subsets of P, assigned to each element $x \in P$, satisfying:*

(1) $O \in D(x) \Rightarrow x \in O$,

(2) $O \subset O_1,\ O \in D(x) \Rightarrow O_1 \in D(x)$.

If we define a mapping $u : 2^P \to 2^P$ in such a way that $x \in uM (M \subset P)$ if and only if $P - M \notin D(x)$, then u is a topology on P and for each $x \in P$ it is $D_u(x) = D(x)$.

1.4. Definition. *Let (P, u), (Q, v) be topological spaces, φ a mapping of P to G. Then φ is called a homeomorphism of (P, u) onto (Q, v) if φ is one-to-one, onto and $\varphi(uM) = v(\varphi(M))$ for every $M \subset P$.*

It is easy to verify that the following theorem holds.

1.5. Theorem. *Let (P, u), (Q, v) be topological spaces. A one-to-one mapping φ of P onto Q is a homeomorphism of (P, u) onto (Q, v) if and only if $D_v(\varphi(x)) = \{\varphi(O) : O \in D_u(x)\}$ for every $x \in P$.*

1.6. Definition. *Let $(A, \leqq)$ be a partially ordered set. A topology u on A is said to be convexly compatible with the ordering $\leqq$, if it has the following property:*

(α) *If $a, b \in A$ and if U is a neighborhood of a with $b \notin U$, then there exists a convex neighborhood V of a such that $b \notin V$.*

1.7. Definition. *Let $(A, \leqq)$ be a partially ordered set. A topology u on A is called convexly weakly compatible with the ordering $\leqq$, if it has the following property:*

(β) *If a and b are comparable elements of A and U is a neighborhood of a with $b \notin U$, then there exists a convex neighborhood V of a such that $b \notin V$.*

Let $(A, \leqq)$ be a partially ordered set. Denote by $\alpha(A, \leqq)$ and $\beta(A, \leqq)$ the set of all topologies on A, which are convexly compatible and convexly weakly compatible with the ordering $\leqq$, respectively. Clearly $\alpha(A, \leqq) \subset \beta(A, \leqq) \subset T(A)$. For $u, v \in T(A)$ set $u \leqq v$ if and only if $uM \subset vM$ for every $M \subset A$. Then $T(A)$, and hence also $\alpha(A, \leqq)$ and $\beta(A, \leqq)$, turn out to be partially ordered sets. The following theorems hold (1.8 is easy to verify; for 1.9 and 1.10, cf. [4]).

1.8. Theorem. *The set $T(A)$ of all topologies on a set A is a complete lattice with respect to the relation $\leqq$ defined above. A topology u is a meet of $\{u_i : i \in I\} \subset T(A)$ if and only if one of the following two conditions is fulfilled:*

(a) $uM = \cap\{u_i M : i \in I\}$ *for every* $M \subset A$,

(b) $D_u(x) = \cup\{D_{u_i}(x) : i \in I\}$ *for every* $x \in A$,

and dually for the join. The least element of $T(A)$ is a topology u^0 such that $u^0 M = M$ for every $M \subset A$. The greatest topology u^1 satisfies $u^1 \emptyset = \emptyset$, $u^1 M = A$ for every $\emptyset \neq M \subset A$.

1.9. Theorem. *Let $(A, \leqq)$ be a partially ordered set. The set $\beta(A, \leqq)$ is a closed sublattice of the complete lattice $T(A)$.*

1.10. Theorem. *Let $(A, \leqq)$ be a partially ordered set. The set $\alpha(A, \leqq)$ is a complete lattice. The meet of a nonempty subset $\{u_i : i \in I\}$ of $\alpha(A, \leqq)$ in $\alpha(A, \leqq)$ is the same as in the complete lattice $T(A)$. The join w of $\{u_i : i \in I\}$ in $\alpha(A, \leqq)$ can be described as follows: for each $a \in A$,*

$$D_w(a) = \{O \in D_v(a) : O \supset \cap\{[V] : V \in D_v(a)\}\},$$

where v is the join of $\{u_i : i \in I\}$ in $T(A)$ and $[V]$ is the convex hull of V in $(A, \leqq)$.

Adopt the following convention: The meet and the join in $T(A)$ will be denoted by the symbols $\wedge$, $\vee$, respectively; the symbol $\vee^{\alpha}$ will be used for the join in $\alpha(A, \leqq)$.

We shall need the following theorems (cf. [4]):

1.11. Theorem. *The lattice $\beta(A, \leqq)$ is completely distributive.*

1.12. Theorem. *If* card $A \geqq 2$, *then the lattices $\alpha(A, \leqq)$, $\beta(A, \leqq)$ have* card A *atoms.*

1.13. Theorem. *Let ξ be a cardinal number and let $(A, \leqq)$ be an antichain of the cardinality ξ. Then the lattices $\alpha(A, \leqq)$, $\beta(A, \leqq)$ have $\xi(\xi - 1)$ dual atoms.*

2. THE PARTIAL ORDERING ON THE CLASSES $\alpha(\mathscr{P}, \mathscr{T})$, $\beta(\mathscr{P}, \mathscr{T})$

Let us denote by $\mathscr{P}$ the class of all partially ordered sets and by $\mathscr{T}$ the class of all topological spaces.

2.1. Definition. *An α-mapping is a mapping F of $\mathscr{P}$ into $\mathscr{T}$ such that the following conditions are fulfilled for each $(A, \leqq) \in \mathscr{P}$:*

(i) *$F(A, \leqq)$ is a topological space with the underlying set A and with a topology which is convexly compatible with the ordering $\leqq$ on A.*

(ii) *If φ is an isomorphism of $(A, \leqq)$ onto a partially ordered set $(A_1, \leqq_1)$, then φ is a homeomorphism of $F(A, \leqq)$ onto $F(A_1, \leqq_1)$.*

A β-mapping is a mapping of $\mathscr{P}$ into $\mathscr{T}$ satisfying (i*), (ii) *for every $(A, \leqq) \in \mathscr{P}$, where* (i*) *is obtained from* (i) *replacing "convexly compatible" by "convexly weakly compatible".*

We shall denote by $\alpha(\mathscr{P}, \mathscr{T})$ and $\beta(\mathscr{P}, \mathscr{T})$ the class of all α- and β-mappings, respectively. Clearly $\alpha(\mathscr{P}, \mathscr{T}) \subset \beta(\mathscr{P}, \mathscr{T})$. Elements of $\beta(\mathscr{P}, \mathscr{T})$ will usually be denoted by capital Latin letters F, G, H and for the topology of $F(A, \leqq)$ and $G(A, \leqq)$ and $H(A, \leqq)$ the notation $f(A, \leqq)$ and $g(A, \leqq)$ and $h(A, \leqq)$ respectively, will be used.

The classes $\alpha(\mathcal{P}, \mathcal{T})$, $\beta(\mathcal{P}, \mathcal{T})$ can be partially ordered as follows:

2.2. Definition. *If $F, G \in \alpha(\mathcal{P}, \mathcal{T})$ or $\beta(\mathcal{P}, \mathcal{T})$, we put $F \leqq G$ if and only if $f(A, \leqq) \leqq g(A, \leqq)$ for every $(A, \leqq) \in \mathcal{P}$.*

At first we will show that every subclass of $\alpha(\mathcal{P}, \mathcal{T})$ has supremum and infimum in $\alpha(\mathcal{P}, \mathcal{T})$ and analogously for $\beta(\mathcal{P}, \mathcal{T})$.

Let F^0, F^1 be mappings $\mathcal{P} \to \mathcal{T}$ defined as follows: for every $(A, \leqq) \in \mathcal{P}$ it is $F^0(A, \leqq) = (A, u^0)$, $F^1(A, \leqq) = (A, u^1)$, where u^0 is the least and u^1 the greatest topology on A. It is easy to verify that the following lemma holds.

2.3. Lemma. *Let F^0, F^1 be mappings as above. Then $F^0, F^1 \in \alpha(\mathcal{P}, \mathcal{T})$ and F^0 is the least, F^1 the greatest element of $\beta(\mathcal{P}, \mathcal{T})$.*

2.4. Lemma. *Let $\{F_i : i \in I\}$ be an arbitrary nonempty subclass of $\alpha(\mathcal{P}, \mathcal{T})$. Define a mapping $F : \mathcal{P} \to \mathcal{T}$ in the following way:*

$$(A, \leqq) \in \mathcal{P} \Rightarrow F(A, \leqq) = (A, \vee^{\alpha}\{f_i(A, \leqq) : i \in I\}).$$

Then $F \in \alpha(\mathcal{P}, \mathcal{T})$ and $F = \sup \{F_i : i \in I\}$ in the class $\alpha(\mathcal{P}, \mathcal{T})$.

Proof. It is obvious that F satisfies (i). From the fact that each F_i fulfils the condition (ii) from 2.1 it follows that F fulfils this condition as well. Evidently, $F = \sup \{F_i : i \in I\}$ in $\alpha(\mathcal{P}, \mathcal{T})$.

The proofs of the following two lemmas are straightforward.

2.5. Lemma. *Let $\emptyset \neq \{F_i : i \in I\} \subset \alpha(\mathcal{P}, \mathcal{T})$. Define a mapping $G : \mathcal{P} \to \mathcal{T}$ as follows:*

$$(A, \leqq) \in \mathcal{P} \Rightarrow G(A, \leqq) = (A, \wedge\{f_i(A, \leqq) : i \in I\}).$$

Then $G \in \alpha(\mathcal{P}, \mathcal{T})$, $G = \inf \{F_i : i \in I\}$ in the class $\alpha(\mathcal{P}, \mathcal{T})$.

2.6. Lemma. *Let $\emptyset \neq \{F_i : i \in I\} \subset \beta(\mathcal{P}, \mathcal{T})$. Define mappings $F, G : \mathcal{P} \to \mathcal{T}$ as follows:*

$$(A, \leqq) \in \mathcal{P} \Rightarrow F(A, \leqq) = (A, \vee\{f_i(A, \leqq) : i \in I\}),$$
$$G(A, \leqq) = (A, \wedge\{f_i(A, \leqq) : i \in I\}).$$

Then $F, G \in \beta(\mathcal{P}, \mathcal{T})$, $F = \sup \{F_i : i \in I\}$ in $\beta(\mathcal{P}, \mathcal{T})$, $G = \inf \{F_i : i \in I\}$ in $\beta(\mathcal{P}, \mathcal{T})$.

Further we deal with the modularity and distributivity of the classes $\alpha(\mathcal{P}, \mathcal{T})$, $\beta(\mathcal{P}, \mathcal{T})$.

2.7. Theorem. *The partially ordered class $\alpha(\mathcal{P}, \mathcal{T})$ does not satisfy the modular identity.*

Proof. Let $(A, \leqq)$ be a partially ordered set represented by the diagram in Fig. 1. Define topologies u, v, w on the set $A = \{o, i, a, b, c\}$ as follows:

$D_u(a) = \{O \subset A : O \supset \{o, a\} \text{ or } O \supset \{a, c\}\}$,
$D_v(a) = \{O \subset A : O \supset \{a, i\}\}$,
$D_w(a) = \{O \subset A : O \supset \{o, a\}\}$,
$D_u(z) = D_v(z) = D_w(z) = \{A\}$ for $z \in A$, $z \neq a$.

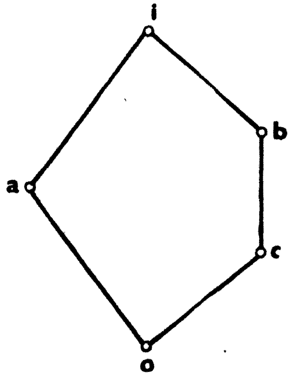

Fig. 1

Then evidently the topologies u, v, w are convexly compatible with the ordering on A and it holds $u < w$, $u \vee^\alpha (v \wedge w) \neq (u \vee^\alpha v) \wedge w$.

Define mappings $F, G, H : \mathscr{P} \to \mathscr{T}$ in the following way:

(1) If a partially ordered set $(A_1, \leqq_1)$ is isomorphic to $(A, \leqq)$ and φ is the unique isomorphism of $(A, \leqq)$ onto $(A_1, \leqq_1)$, set $F(A_1, \leqq_1) = (A_1, u_1)$, $G(A_1, \leqq_1) = (A_1, v_1)$, $H(A_1, \leqq_1) = (A_1, w_1)$, where u_1, v_1, w_1 are the topologies on A_1 such that $x \in A_1 \Rightarrow D_{u_1}(x) = \{O \subset A_1 : \varphi^{-1}(O) \in D_u(\varphi^{-1}(x))\}$, $D_{v_1}(x) = \{O \subset A_1 : \varphi^{-1}(O) \in \in D_v(\varphi^{-1}(x))\}$, $D_{w_1}(x) = \{O \subset A_1 : \varphi^{-1}(O) \in D_w(\varphi^{-1}(x))\}$.

(2) If a partially ordered set $(A_1, \leqq_1)$ is not isomorphic to $(A, \leqq)$, set $F(A_1, \leqq_1) = = G(A_1, \leqq_1) = H(A_1, \leqq_1) = (A_1, u^0)$, where u^0 is the least topology on A_1.

Obviously $F, G, H \in \alpha(\mathscr{P}, \mathscr{T})$, $F < H$. Denoting the supremum (infimum) in $\alpha(\mathscr{P}, \mathscr{T})$ by the symbol $\vee(\wedge)$, we have $(F \vee (G \wedge H))(A, \leqq) = (A, u \vee^\alpha (v \wedge w))$, $((F \vee G) \wedge H)(A, \leqq) = (A, (u \vee^\alpha v) \wedge w)$, hence $F \vee (G \wedge H) \neq (F \vee G) \wedge H$.

Using 1.11 and 2.6, we have the following theorem.

2.8. Theorem. *The partially ordered class $\beta(\mathscr{P}, \mathscr{T})$ is completely distributive.*

3. COVERING RELATION

Let F, G be α-mappings, $F < G$. If there is no element $H \in \alpha(\mathscr{P}, \mathscr{T})$ such that $F < H < G$, then we shall say that F is covered by G or that G covers F and we shall write $F \prec^\alpha G$. If $F \prec^\alpha G$, then the mapping G will be also called an atom over F

and the mapping F a dual atom under G in $\alpha(\mathscr{P}, \mathscr{T})$. The class of all atoms over F and dual atoms under F in $\alpha(\mathscr{P}, \mathscr{T})$ will be denoted by $\mathscr{A}_\alpha(F)$ and $\mathscr{A}'_\alpha(F)$, respectively. A similar terminology and notation will be used also for β-mappings.

In this section a necessary and sufficient condition for an α-mapping G to cover an α-mapping F in $\alpha(\mathscr{P}, \mathscr{T})$ is given. An analogous result is proved for β-mappings. It is shown that the classes $\mathscr{A}_\alpha(F)$, $\mathscr{A}_\beta(F)$, $\mathscr{A}'_\alpha(F)$, $\mathscr{A}'_\beta(F)$ may be empty, but it can also happen, that they are proper classes.

3.1. Lemma. *If $F, G \in \alpha(\mathscr{P}, \mathscr{T})$, $F \prec^\alpha G$ and for some partially ordered sets $(A_1, \leqq_1)$, $(A_2, \leqq_2)$ it is $f(A_1, \leqq_1) < g(A_1, \leqq_1)$, $f(A_2, \leqq_2) < g(A_2, \leqq_2)$, then $(A_1, \leqq_1)$ and $(A_2, \leqq_2)$ are isomorphic.*

Proof. Suppose the assumptions of 3.1 hold but $(A_1, \leqq_1)$, $(A_2, \leqq_2)$ are not isomorphic. Define a mapping $H : \mathscr{P} \to \mathscr{T}$ as follows:

If a partially ordered set $(A, \leqq)$ is isomorphic to $(A_1, \leqq_1)$, we put $H(A, \leqq) = F(A, \leqq)$, in the opposite case we set $H(A, \leqq) = G(A, \leqq)$. Then evidently $H \in \alpha(\mathscr{P}, \mathscr{T})$ and it is $F < H < G$, contrary to $F \prec^\alpha G$.

3.2. Lemma. *If $F, G \in \beta(\mathscr{P}, \mathscr{T})$, $F \prec^\beta G$ and for some partially ordered sets $(A_1, \leqq_1)$, $(A_2, \leqq_2)$ it is $f(A_1, \leqq_1) < g(A_1, \leqq_1)$, $f(A_2, \leqq_2) < g(A_2, \leqq_2)$, then $(A_1, \leqq_1)$ and $(A_2, \leqq_2)$ are isomorphic.*

The proof is analogous to that of 3.1.

Let $(A, \leqq)$ be a partially ordered set and let u, v be topologies on A with $u < v$. Consider the following condition for $(A, \leqq)$, u, v and $\gamma \in \{\alpha, \beta\}$:

(p_γ) *If $w \in \gamma(A, \leqq)$ and $u < w < v$, then there exists an isomorphism of $(A, \leqq)$ onto $(A, \leqq)$ which is not a homeomorphism of (A, w) onto (A, w).*

3.3. Lemma. *Let $F, G \in \alpha(\mathscr{P}, \mathscr{T})$, $F \prec^\alpha G$. If $(A, \leqq)$ is a partially ordered set with $f(A, \leqq) < g(A, \leqq)$, then for $(A, \leqq)$, $f(A, \leqq)$, $g(A, \leqq)$, the condition (p_α) is fulfilled.*

Proof. Suppose that $f(A, \leqq) < g(A, \leqq)$ and that for some topology $w \in \alpha(A, \leqq)$ with $f(A, \leqq) < w < g(A, \leqq)$, every isomorphism of $(A, \leqq)$ onto $(A, \leqq)$ is a homeomorphism of (A, w) onto (A, w).

Define a mapping $H : \mathscr{P} \to \mathscr{T}$ as follows:

(1) If $(A_1, \leqq_1)$ is a partially ordered set isomorphic to $(A, \leqq)$, take an arbitrary fixed isomorphism φ_1 of $(A, \leqq)$ onto $(A_1, \leqq_1)$ and set $H(A_1, \leqq_1) = (A_1, w_1)$, where w_1 is a topology on A_1 defined in the following way:

$$x \in A_1 \Rightarrow D_{w_1}(x) = \{O \subset A_1 : \varphi_1^{-1}(O) \in D_w(\varphi_1^{-1}(x))\}.$$

(2) If $(A_1, \leqq_1)$ is a partially ordered set which is not isomorphic to $(A, \leqq)$, put $H(A_1, \leqq_1) = F(A_1, \leqq_1)$.

To prove $H \in \alpha(\mathscr{P}, \mathscr{T})$, it is sufficient to show that the condition (ii) of 2.1 is fulfilled. Let φ be an isomorphism of $(A_1, \leqq_1)$ onto $(A_2, \leqq_2)$. Two possibilities can occur: the partially ordered sets $(A_1, \leqq_1)$, $(A_2, \leqq_2)$ are isomorphic to $(A, \leqq)$ or none of $(A_1, \leqq_1)$, $(A_2, \leqq_2)$ is isomorphic to $(A, \leqq)$. In the first case we have $H(A_1, \leqq_1) = (A_1, w_2)$, $H(A_2, \leqq_2) = (A_2, w_2)$, where w_i $(i \in \{1, 2\})$ is a topology on A_i such that there exists an isomorphism φ_i of $(A, \leqq)$ onto $(A_i, \leqq_i)$ which is a homeomorphism of (A, w) onto (A_i, w_i). Then $\varphi_2^{-1} \circ \varphi \circ \varphi_1$ is an isomorphism of $(A, \leqq)$ onto $(A, \leqq)$ and hence by assumption $\varphi_2^{-1} \circ \varphi \circ \varphi_1$ is a homeomorphism of (A, w) onto (A, w). Consequently, $\varphi_2 \circ \varphi_2^{-1} \circ \varphi \circ \varphi_1 \circ \varphi_1^{-1} = \varphi$ is a homeomorphism of (A_1, w_1) onto (A_2, w_2). In the second case, $H(A_i, \leqq_i) = F(A_i, \leqq_i)$ together with $F \in \alpha(\mathscr{P}, \mathscr{T})$ implies that φ is a homeomorphism of $H(A_1, \leqq_1)$ onto $H(A_2, \leqq_2)$.

Next we show that $F < H < G$. If $(A_1, \leqq_1)$ is a partially ordered set isomorphic to $(A, \leqq)$, then $h(A_1, \leqq_1)$ is a topology on A_1 such that there exists an isomorphism φ_1 of $(A, \leqq)$ onto $(A_1, \leqq_1)$ which is a homeomorphism of (A, w) onto $(A_1, h(A_1, \leqq_1))$. Since $F, G \in \alpha(\mathscr{P}, \mathscr{T})$, φ_1 is also a homeomorphism of $(A, f(A, \leqq))$ onto $(A_1, f(A_1, \leqq_1))$ and of $(A, g(A, \leqq))$ onto $(A_1, g(A_1, \leqq_1))$. The inequalities $f(A, \leqq) < < w < g(A, \leqq)$ imply that $f(A_1, \leqq_1) < h(A_1, \leqq_1) < g(A_1, \leqq_1)$. When $(A_1, \leqq_1)$ is a partially ordered set not isomorphic to $(A, \leqq)$, it is $f(A_1, \leqq_1) = h(A_1, \leqq_1) \leqq \leqq g(A_1, \leqq_1)$.

We have a contradiction and hence the proof is complete.

The proof of the following lemma is analogous to that of 3.3.

3.4. Lemma. *Let $F, G \in \beta(\mathscr{P}, \mathscr{T})$, $F \prec^{\beta} G$. If $(A, \leqq)$ is a partially ordered set with $f(A, \leqq) < g(A, \leqq)$, then for $(A, \leqq)$, $f(A, \leqq)$, $g(A, \leqq)$, the condition (p_β) is fulfilled.*

3.5. Lemma. *Let F, G be γ-mappings, $\gamma \in \{\alpha, \beta\}$, $F < G$ and suppose that the following two conditions are satisfied:*

(1) There exists a partially ordered set $(A, \leqq)$ with $f(A, \leqq) < g(A, \leqq)$ and for $(A, \leqq)$, $f(A, \leqq)$, $g(A, \leqq)$, the condition (p_γ) is fulfilled.

(2) If a partially ordered set $(A_1, \leqq_1)$ is not isomorphic to $(A, \leqq)$, then $f(A_1, \leqq_1) = = g(A_1, \leqq_1)$.

Then $F \prec^{\gamma} G$ holds.

Proof. We prove the part of the statement concerning α-mappings. The proof of the second part is analogous. Suppose α-mappings F, G with $F < G$ satisfy conditions (1), (2), but that it is not $F \prec^{\alpha} G$. Then there exists an α-mapping H with $F < H < G$. It follows the existence of partially ordered sets $(A_1, \leqq_1)$, $(A_2, \leqq_2)$ with $f(A_1, \leqq_1) < h(A_1, \leqq_1)$, $h(A_2, \leqq_2) < g(A_2, \leqq_2)$. By (2), the partially ordered sets $(A_1, \leqq_1)$, $(A_2, \leqq_2)$ are isomorphic to $(A, \leqq)$. Let φ_i $(i \in \{1, 2\})$ be an arbitrary fixed isomorphism of $(A, \leqq)$ onto $(A_i, \leqq_i)$. Since $F, H \in \alpha(\mathscr{P}, \mathscr{T})$, φ_1 is a homeomorphism of $(A, f(A, \leqq))$ onto $(A_1, f(A_1, \leqq_1))$ and also of $(A, h(A, \leqq))$ onto $(A_1, h(A_1, \leqq_1))$. The inequality $f(A_1, \leqq_1) < h(A_1, \leqq_1)$ then implies $f(A, \leqq) <$

$f < h(A, \leqq)$. The relation $h(A, \leqq) < g(A, \leqq)$ can be obtained analogously. Hence $f(A, \leqq) < h(A, \leqq) < g(A, \leqq)$ and (1) implies the existence of an isomorphism of $(A, \leqq)$ onto $(A, \leqq)$ which is not a homeomorphism of $(A, h(A, \leqq))$ onto $(A, h(A, \leqq))$. Since H is an α-mapping, we have a contradiction.

The following theorem is a straightforward consequence of Lemmas 3.1 – 3.5.

3.6. Theorem. *Let F, G be γ-mappings, $\gamma \in \{\alpha, \beta\}$, and let $F < G$. Then F is covered by G in $\gamma(\mathscr{P}, \mathscr{T})$ if and only if the following two conditions are satisfied:*

(1) *There exists a partially ordered set $(A, \leqq)$ with $f(A, \leqq) < g(A, \leqq)$ and for $(A, \leqq)$, $f(A, \leqq)$, $g(A, \leqq)$, the condition (p_γ) is fulfilled.*

(2) *If a partially ordered set $(A_1, \leqq_1)$ is not isomorphic to $(A, \leqq)$, then it is $(A_1, \leqq_1) = g(A_1, \leqq_1)$.*

3.7. Corollary. *Let $F \in \gamma(\mathscr{P}, \mathscr{T})$, $\gamma \in \{\alpha, \beta\}$, and let F be not the least element of $\gamma(\mathscr{P}, \mathscr{T})$. Then F is an atom of $\gamma(\mathscr{P}, \mathscr{T})$ if and only if the following two conditions are satisfied:*

(1) *There exists a partially ordered set $(A, \leqq)$ such that $f(A, \leqq)$ is not the least topology on A and either $f(A, \leqq)$ is an atom of $\gamma(A, \leqq)$ or for every topology $w \in \gamma(A, \leqq)$ different from the least one, with $w < f(A, \leqq)$, there exists an isomorphism of $(A, \leqq)$ onto $(A, \leqq)$ which is not a homeomorphism of (A, w) onto (A, w).*

(2) *If a partially ordered set $(A_1, \leqq_1)$ is not isomorphic to $(A, \leqq)$, then $f(A_1, \leqq_1)$ is the least topology on A_1.*

If we choose one partially ordered set from every maximal class of mutually isomorphic partially ordered sets, we obtain a proper class. Hence, by 3.7 and 1.12 we have:

3.8. Corollary. The class of all atoms of $\alpha(\mathscr{P}, \mathscr{T})$ and the class of all atoms of $\beta(\mathscr{P}, \mathscr{T})$ are proper classes.

3.9. Corollary. Let $F \in \gamma(\mathscr{P}, \mathscr{T})$, $\gamma \in \{\alpha, \beta\}$, and let F be not the greatest element of $\gamma(\mathscr{P}, \mathscr{T})$. Then F is a dual atom of $\gamma(\mathscr{P}, \mathscr{T})$ if and only if the following two conditions are satisfied:

(1) There exists a partially ordered set $(A, \leqq)$ such that $f(A, \leqq)$ is not the greatest topology on A, and either $f(A, \leqq)$ is a dual atom of $\gamma(A, \leqq)$ or for every topology $w \in \gamma(A, \leqq)$ different from the greatest one, with $f(A, \leqq) < w$, there exists an isomorphism of $(A, \leqq)$ onto $(A, \leqq)$ which is not a homeomorphism of (A, w) onto (A, w).

(2) If a partially ordered set $(A_1, \leqq_1)$ is not isomorphic to $(A, \leqq)$, then $f(A_1, \leqq_1)$ is the greatest topology on A_1.

Using 1.13, we have:

3.10. Corollary. The class of all dual atoms of $\alpha(\mathscr{P}, \mathscr{T})$ and the class of all dual atoms of $\beta(\mathscr{P}, \mathscr{T})$ are proper classes.

Next we shall show that the classes $\mathscr{A}_\alpha(F)$, $\mathscr{A}_\beta(F)$, $\mathscr{A}'_\alpha(F)$, $\mathscr{A}'_\beta(F)$ can be empty.

Let N be the set of all positive integers and let $A = \{x_i : i \in N\} \cup \{y_i : i \in N\}$. Define an ordering relation on A in such a way that the set $\{x_i : i \in N\}$ and $\{y_i : i \in N\}$ is the set of all minimal and maximal elements of A, respectively, and for $i \in N$, $y \in A$ it is $x_i < y$ if and only if $y \in \{y_i, y_{i+1}, \ldots, y_{2i-1}\}$. Further consider a topology u on A such that $D_u(a) = \{O \subset A : a \in O, \text{card } O = \aleph_0\}$ for every $a \in A$.

3.11. Lemma. *Let $(A, \leqq)$ be the partially ordered set and u the topology on A defined above. Then u is convexly compatible with the ordering $\leqq$ on A and there is no atom over u and no dual atom under u in both of the lattices $\alpha(A, \leqq)$, $\beta(A, \leqq)$.*

Proof. Every topology on A is convexly compatible with the ordering $\leqq$ on A. Hence it is sufficient to prove that if v_1 is a topology on A with $v_1 > u$, then there exists a topology w_1 on A such that $v_1 > w_1 > u$, and the dual condition.

If $v_1 > u$, then there exists $a_1 \in A$ such that $D_{v_1}(a_1) \subset D_u(a_1)$, $D_{v_1}(a_1) \neq D_u(a_1)$ and for every $z \in A$, $z \neq a_1$ it is $D_{v_1}(z) \subset D_u(z)$. Take an arbitrary fixed set $U \in D_u(a_1) - D_{v_1}(a_1)$ and define a topology w_1 on A as follows: $D_{w_1}(a_1) = D_u(a_1) - \{O \subset A : O \subset U, O \neq U\}$, $D_{w_1}(z) = D_u(z)$ for every $z \in A$, $z \neq a_1$. It is clear that $u \leqq w_1 \leqq v_1$. Since $U \in D_{w_1}(a_1) - D_{v_1}(a_1)$, and for arbitrary $b \in U$, $b \neq a_1$ it is $U - \{b\} \in D_u(a_1) - D_{w_1}(a_1)$, we have $u < w_1 < v_1$.

Assume $v_2 < u$. Then there exists $a_2 \in A$ such that $D_u(a_2) \subset D_{v_2}(a_2)$, $D_u(a_2) \neq D_{v_2}(a_2)$ and for every $z \in A$, $z \neq a_2$ it is $D_u(z) \subset D_{v_2}(z)$. Take an arbitrary fixed set $V \in D_{v_2}(a_2) - D_u(a_2)$ and define a topology w_2 on A in the following way: $D_{w_2}(a_2) = D_{v_2}(a_2) - \{O \subset A : O \subset V\}$, $D_{w_2}(z) = D_u(z)$ for every $z \in A$, $z \neq a_2$. Evidently $v_2 \leqq w_2 \leqq u$, but since $V \in D_{v_2}(a_2) - D_{w_2}(a_2)$ and for arbitrary $c \in A - V$ it is $V \cup \{c\} \in D_{w_2}(a_2) - D_u(a_2)$, we obtain $v_2 < w_2 < u$.

Define the mappings $F_1, F_2 : \mathscr{P} \to \mathscr{T}$ by the following rules:

(a) If a partially ordered set $(A_1, \leqq_1)$ is isomorphic to above-mentioned $(A, \leqq)$ and φ is the unique isomorphism of $(A, \leqq)$ onto $(A_1, \leqq_1)$, set $F_1(A_1, \leqq_1) = F_2(A_1, \leqq_1) = (A_1, u_1)$, where u_1 is the topology on A_1 such that $D_{u_1}(x) = \{O \subset A_1 : \varphi^{-1}(O) \in D_u(\varphi^{-1}(x))\}$ for every $x \in A_1$ and u as above.

(b) If a partially ordered set $(A_1, \leqq_1)$ is not isomorphic to $(A, \leqq)$, set $F_1(A_1, \leqq_1) = (A_1, u^1)$, $F_2(A_1, \leqq_1) = (A_1, u^0)$, where u^1 and u^0 is the greatest and the least topology on A_1, respectively.

Obviously $F_1, F_2 \in \alpha(\mathscr{P}, \mathscr{T})$ and the following theorem holds.

3.12 Theorem. *The classes $\mathscr{A}_\alpha(F_1)$, $\mathscr{A}_\beta(F_1)$, $\mathscr{A}'_\alpha(F_2)$, $\mathscr{A}'_\beta(F_2)$ are empty.*

Proof. We shall show, for example, that $\mathscr{A}_\alpha(F_1) = \emptyset$. Suppose this is not the case. Then there exists $G \in \alpha(\mathscr{P}, \mathscr{T})$ with $F_1 \prec^\alpha G$. By 3.6 it must be $u < g(A, \leqq)$. Using 3.11 we obtain that there exists a topology $w \in \alpha(A, \leqq)$ such that $u < w < g(A, \leqq)$. Again 3.6 ensures the existence of an isomorphism of $(A, \leqq)$ onto $(A, \leqq)$ which is not a homeomorphism of (A, w) onto (A, w). Since the unique isomorphism of $(A, \leqq)$ onto $(A, \leqq)$ is the identity mapping, we have a contradiction.

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