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ACYCLIC CHROMATIC INDEX OF A GRAPH WITH MAXIMUM VALENCY THREE

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We consider only finite graphs without loops and multiple edges. As a rule, we do not distinguish between isomorphic graphs.

Let E be the set of the edges of a graph G. A (regular) colouring of the edges is a decomposition of the set E into mutually disjoint classes (called colours) where no edges in the same class have vertices in common. An edge colouring of the graph G is said to be acyclic provided that every subgraph of G spanned by edges of two of the colours is acyclic (is a forest). Let $a(G)$ denote the minimum number of colours necessary for acyclic colouring of the set E. In [3] via maximum valency h in the graph G the upper bound of $a(G)$ is given. The purpose of this paper is to improve the bound $a(G) \leqq 5$ for $h = 3$.

Notation and terminology. Below $fe(fxy)$ will denote the colour of an edge $e(xy)$; X the set of colours of the edges which are adjacent (incident) to a vertex x and K_4 is a complete graph on four vertices. A cycle in which the edges are alternatively coloured by the colours i, j will be denoted by C_{ij}. Further undefined terms in this paper may be found in any of the standard texts in graph theory (see, for example, [1, 2, 5, 7, 8]).

Our main result is the following.

Theorem. *If the maximum valency of a graph G is at most three and $G \neq K_4$, then $a(G) \leqq 4$. If $G = K_4$, then $a(K_4) \doteq 5$.*

We devide our auxiliary results, needed for the prof of Theorem, into three sections. In section **1.** we embed the graph G with the maximum valency at most 3 into a cubic graph (a regular graph of degree 3 at each vertex) L (in the sequel by graph we mean cubic graph); in section **2.** we introduce a successive construction of graphs due to E. L. Johnson which makes us possible to construct any graph on $p + 2$ vertices ($p \geqq 6$) from a graph on p vertices; in section **3.** we introduce two Lemmas.

1. The embedding of a graph G into a graph L. For a cubic graph, the embedding is obvious. So we may assume that the graph G contains at least one vertex with valency $\neq 3$. Let K denote the graph obtained from K_4 by removing two arbitrary non-adjacent edges xy, zu. We consider two mutually disjoint copies of G. We say that a graph K is embedded between a vertex v of valency one in the graph G if the vertex v from first (second) copy of G is connected by the edges vx, vy(vz, vu). If we embed the graph K between every two vertices of valency one, and every couple of corresponding vertices of valency two (from both copies of G) is connected by an edge, then embedding of the graph G into cubic graph L is done. (The present construction, after small extension, simplifies the embedding given in [1, p. 16]).

2. The Johnson's construction. Johnson showed [6, p. 132] that any connected graph L^+ with $p + 2$ vertices is obtained by eliminating two non-adjacent edges

(1) $$ac, bd$$

of a suitable graph L and by adding two vertices u, v and the edges

(2) $$uv, au, ud, bv, vc$$

(On Fig. 1 the deleting edges (1) are denoted by a dashed line and the added edges (2) are denoted dark).

3. From an acyclic colouring of edges of the graph L it immediately follows

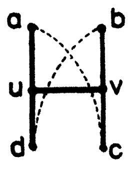


Fig. 1

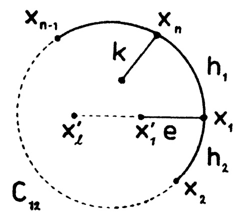


Fig. 2

Lemma 1. *Let the edges of the graph L be acyclically coloured by the set of colours* $\mathscr{S} = \{1, 2, 3, 4\}$ *and let* α *and* β *be two different colours of* $\mathscr{S}$. *We assume the removed edge* $ac \in L$ (see Fig. 1) *to be coloured by the colour* α. *If we colour the edges* au, $vc \in L^+$ *by the colour* α *and the edge* $uv \in L^+$ *by the colour* β, *then we form neither cycle* $C_{\alpha\beta}$ *nor any other cycle.*

Lemma 2. *Let the edges of a graph G be regularly coloured by the set of colours* $\mathscr{S}$ *and let an edge e be incident with two edges* h_1, h_2 *of a cycle* $C_{12} = x_1x_2 \dots x_{n-1}x_nx_1$ *as in* Fig. 2. *If we can recolour the edge e by one of the colours* fh_i, $i = 1, 2$ *in such a manner that we do not break the regularity of the colouring of the edges of the graph* $G - h_i$, *then we can regularly colour the edges of G by the set of colours* $\mathscr{S}$ *in such a fashion that we form a new cycle* C_{12}^+.

Proof of the Lemma 2. Let, for example, $fh_1 = 1$. We suppose that by putting $fe = 1$ we regularly colour the edges of the graph $G - h_1$. Let the edge k be attached to the vertex $x_n \in C_{12}$ and let $fk = \alpha$. If is $1 \neq \alpha \neq 2$ we recolour the edge h_1 by the fourth colour $\beta \in \mathcal{S}$ (so $\beta \notin \{1, 2, \alpha\}$), then the edges of G will be regularly coloured by the set of colours $\mathcal{S}$. It remains to show that by recolouring the edges e, h_1 (by colours $1, \beta$) we cannot form a new cycle C_{12}^+ in the graph G. A cycle C_{12}^+ can be formed only in the case if the endpoint of the edge of a path $x_n x_{n-1} \ldots x_2 x_1 x'_1 x'_2 \ldots x'_i$ whose edges are coloured alternatively by colours 2, 1 is identified with some vertex $x_j \in C_{12}$, i.e., that $x'_i = x_j$—which yields a contradiction to our assumption that the edges of G are regularly coloured. This proves the Lemma.

We are now ready to prove the Theorem.

Proof of the Theorem. We proceed by induction on the number p of vertices in the graph L. If $p = 6$ there are exactly two such graphs, namely, Kuratowski's graph and the graph in Fig. 3. It is easy to verify that their edges can be coloured acyclicaly by the set of colours $\mathcal{S}$.

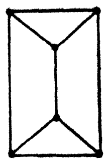


Fig. 3

An inductive assumption is that the edges of L with $p \geqq 8$ vertices are acyclically coloured by the set of colours $\mathcal{S}$. We extend the graph L by Johnson's construction given in section 2 to L^+ and then we prove that we can acyclically colour the edges of the graph L^+ by the set of colours $\mathcal{S}$, too.

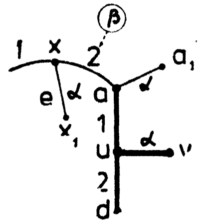
We restrict our further considerations to the colouring of the edges (2) (the colouring of the rest edges of graph L^+ can be induced from the colouring of the edges of the graph L). According to the colour of the edges (1) we distinguish two cases by the colour of the edges (2): A. $fac \neq fbd$, B. $fac = fbd$.

Case A. Let $fac = 1$ and $fbd = 2$. In Fig. 1 we put $fau = fvc = 1$, $fbv = fud = 2$, $fuv = \alpha(\alpha \in \mathcal{S}, \alpha \neq 1, 2)$. By Lemma 1 after such colouring in the graph L^+ we form neither cycle $C_{1\alpha}$ nor cycle $C_{2\alpha}$. Then the vertices $a, b; c, d(a, d; b, c)$ can be simultaneously connected in L by paths whose edges are alternatively coloured by the colours 1, 2. (In the opposite case we should have a cycle C_{12} in L.)

In the graph L^+ we consider a cycle

$$C_{12} = axy \ldots dua,$$

i.e., we suppose that in the graph L to vertices a, d there is attached a path P_{12} with length $\geqq 2$ the edges of which are alternatively coloured by the colours 1, 2. Let $faa_1 = fuv = \alpha$ (in the opposite case we recolour the edge uv)—see Fig. 4 where only the part different from L^+ is given. In the case that the vertices a_1, x_1 in L are identified, then the third edge adjacent to vertex a_1 is coloured by the colour 1 (2), and we can eliminate the cycle C_{12} from L^+ by Lemma 2. In the case that there exists at least one edge not coloured by 1 (2) and adjacent at least to one of the vertices a_1, x_1 the cycle C_{12} can be eliminated from L^+ by Lemma 2. Now we pass to the elimination of the cycle C_{12} in the case that $a_1 = x_1$ and the vertices $a_1, x_1 \in L$ are adjacent with the edges coloured by the colours 1, 2. According to the colour of the edge $xx_1 = e \in L$ we consider two subcases:



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Subcase a). $fe = \alpha$. For the elimination of the cycle C_{12} from L^+ it is enough to put $fax = \beta \in \mathcal{S}$—see Fig. 4 (we do not change the colouring of the rest of the edges of the graph L^+).

Subcase b). $fe = \beta$. On the path P_{12} we consider further vertex y and according to the colour of the edge $yy_1 = k \in L$ (we admit also $y = d$) two subcases must be considered:

i) $fk = \beta$. It is again sufficient to consider only the case that the vertices a_1, x_1, y_1 are mutually different and each of them incidents with the edges coloured by colours 1, 2 (in the opposite case we can eliminate the cycle C_{12} from L^+ by virtue of Lemma 2). For the elimination of a cycle C_{12} it is enough to put $fe = \alpha, fax = \beta$ (in Fig. 5 the recolouring edges are given in circles).

ii) $fk = \alpha$. The primary colouring of the edges e, k in the considered part of L^+ is given on Fig. 6. If we recolour two edges by the colours given in circles on Fig. 7

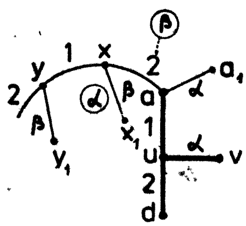

Fig. 5

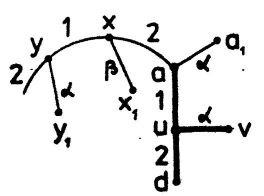

Fig. 6

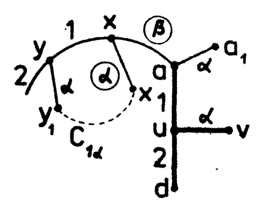

Fig. 7

(the colouring of the rest of the edges in the graph L^+ is not changed) then in the graph L^+ we can form a cycle

$$C_{1\alpha} = yxx_1 \dots y_1y.$$

In this case we recolour three edges in L^+ according to Fig. 8, i.e., we put $fe = faa_1 = \beta$, $fax = \alpha$. This procedure eliminates both cycles C_{12}, $C_{1\alpha}$ from L^+ and by virtue of Lemma 2 we do not create a new cycle $C_{1\alpha}^+$ in L^+.

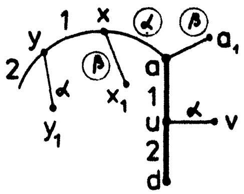

Fig. 8

The elimination of a cycle C_{12} from graph L^+ in the case A is finished.

Case B. Let $fac = fbd = 1$. We distinguish the following three subcases according to the colouring of the edges (2):

(a) $A \neq C(B \neq D)$;

(b) $A = C$, $B = D$ if $A \neq B$;

(c) $A = B = C = D$.

Subcase a) From the regularity of the colouring of the edges of the graph L with the set of colours $\mathscr{S}$ it follows that the vertices $a, c \in L$ incident with edges coloured with the colour $\alpha \in \mathscr{S}(\alpha \neq 1)$. From the assumption $A \neq C$ and removing the edges (1) we get $A = \{1, \alpha, \beta\}$, $C = \{1, \alpha, \gamma\}$. If we put $fbv = fud = 1$, $fau = \gamma$, $fuv = \alpha$, $fvc = \beta$ in Fig. 1, then by Lemma 1 we do not create a cycle $C_{1\alpha}$ in L^+ and the edges of L^+ are coloured in a desirable way. (In the case that $B \neq D$ we colour the edges (2) analogously as above.)

Subcase b) From the regularity of the colouring of the edges of the graph L with the set of colours $\mathscr{S}$, from the assumptions $A \neq B$, $A = C$, $B = D$, and from the colouring of the deleting edges (1) it follows that the edges (2) can be coloured according to Fig. 9. If we do not form a cycle (two mutually disjoint cycles) $C_{\beta\gamma}$ in L^+ by such a colouring, then the colouring of the edges of L^+ is finished (by virtue of Lemma 1 we do not form the cycles $C_{1\beta}$, $C_{1\gamma}$). In the opposite case we recolour three edges (2) by colours given in circles in Fig. 10. It is easy to see that by such a recolouring we eliminate the cycle (both cycles) $C_{\beta\gamma}$ and we colour the edges of L^+ in desirable way.

Subcase c) Let the vertex a in the graph L be coloured by the set of colours $A = \{1, \alpha, \beta\}$. If we colour the edges (2) according to Fig. 11 we do not create a cycle

$$C_{\alpha\gamma} = a_1auvc \dots a_1$$

in L^+, then the colouring of the edges of L^+ accomplished (by Lemma 1 we do not form a cycle $C_{1\alpha}$). In the opposite case we put $fuv = \beta$. If by such colouring we do not form a cycle

$$C_{\beta\gamma} = a_2auvc \dots a_2,$$

then the colouring of the edges of L^+ is obtained. It remains only to consider the case in which we form a cycle $C_{\beta\gamma}$ in L. We suppose that the edges a_1x, ya_2 in Fig. 11 are not coloured by the colour γ.

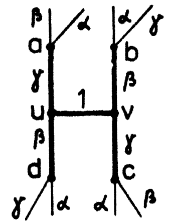

Fig. 9

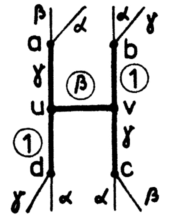

Fig. 10

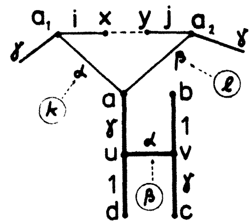

Fig. 11

We introduce the following notation. In Fig. 11 let $ij \to kl$ mean that if the edge $a_1x(ya_2)$ is coloured by the colour $i(j)$, then the edge $aa_1(aa_2)$ will be recoloured by the colour $k(l)$. From the regularity of the colouring of the edges in L by the set of the colours $\mathscr{S}$, it follows $ij \in \{11, \beta\alpha, 1\alpha, \beta 1\}$. For the elimination of a cycle $C_{\beta\gamma}$ from L^+ it is enough to put

(a) $11 \to \beta\alpha$;
(b) $1\alpha \to \beta 1$;
(c) $\beta\alpha \to \alpha 1$;
(d) $\beta 1 \to 1\alpha$.

We show that such recolouring is correct.

Case (a). If we recolour two edges, which incident with the vertex a, by the colours given in circles on Fig. 12, then from the graph L^+ we eliminate the cycle $C_{\beta\gamma}$ and by virtue of Lemma 2 we do not create a new cycle $C_{\beta\gamma}$.

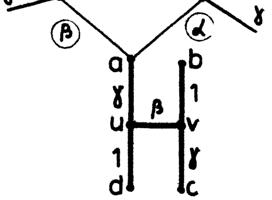

Fig. 12

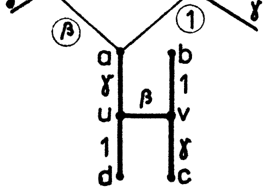

Fig. 13

Case (b). The elimination is illustrated on Fig. 13. By such a recolouring we eliminate a cycle $C_{\beta\gamma}$ and by virtue of Lemma 2 we do not form a new cycle $C^{+}_{\beta\gamma}$ in L^{+}. Since $D = \{\alpha, \beta\}$ in L we do not form a cycle $C_{1\gamma}$.

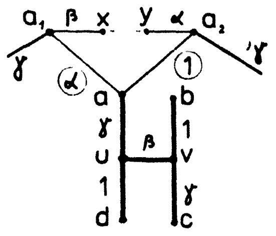

Fig. 14

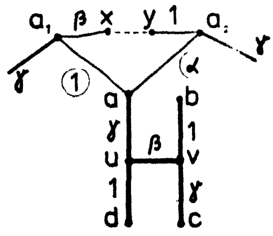

Fig. 15

Case (c). [d] We use Fig. 14 [Fig. 15] for illustration. In both cases by the indicated recolouring the cycle $C_{\beta\gamma}$ is eliminated and because $D = \{\alpha, \beta\}$ in L, no cycle $C_{1\gamma}$ is created in L^{+}.

Since these are the only possible cases, the upper bound given in Theorem is proved. Since the equality $a(K_4) = 5$ can be verified immediately, the proof of Theorem is complete.

Remarks. 1. The bound 4 in Theorem cannot be improved, every cubic graph different from K_4 has the acyclic chromatic index (below simply *ACI*) equal to four.

2. In the class of graphs with maximum valency 3 the following classification problem arises: which graphs have *ACI* 3, and which have *ACI* 4?

More about these topics will be given elsewhere.

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