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Archivum Mathematicum, Vol. 15 (1979), No. 4, 203--204

Persistent URL: `http://dml.cz/dmlcz/107042`

ARCH. MATH. 4, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XV: 203—204, 1979

A CHARACTERIZATION OF DISTRIBUTIVE LATTICES BY TOLERANCE LATTICES

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(Received June 6, 1978)

The purpose of this short paper is to show how lattices of compatible tolerances can be used for the classification of varieties.

Let $\mathfrak{A} = (A, F)$ be an algebra. A binary relation T on A is called a *compatible tolerance* provided it is reflexive, symmetric and compatible (it means that $\langle a_i, b_i \rangle \in T$ for $i = 1, \ldots, n$ always imply $\langle f(a_1, \ldots, a_n), f(b_1, \ldots, b_n) \rangle \in T$ for every n-ary $f \in F$, $n \geqq 1$). Likewise in [2], denote by $LT(\mathfrak{A})$ the set of all compatible tolerances on an algebra $\mathfrak{A}$. As it was proved in [2], $LT(\mathfrak{A})$ is an algebraic lattice with respect to the set inclusion for every algebra $\mathfrak{A}$. If $\mathscr{V}$ is a variety of algebras, we say that $\mathscr{V}$ has (infinitely meet) distributive tolerances provided $LT(\mathfrak{A})$ is (infinitely meet) distributive lattice for every $\mathfrak{A} \in \mathscr{V}$.

Theorem. *Let $\mathscr{V}$ be a variety of lattices. Then the following conditions are equivalent:*
(a) *$\mathscr{V}$ is a variety of distributive lattices,*
(b) *$\mathscr{V}$ has distributive tolerances,*
(c) *$\mathscr{V}$ has infinitely meet-distributive tolerances.*

Proof. (a) $\Rightarrow$ (c) follows directly by Theorem 16 in [2] and (c) $\Rightarrow$ (b) is trivial. Accordingly, it remains only to prove (b) $\Rightarrow$ (a). Let $\mathscr{V}$ not be a variety of distributive lattices. As it is known, then $\mathscr{V}$ contains either the non-distributive modular five element lattice M_5 or the non-modular five element lattice, i.e. the pentagon N_5.

Suppose $M_5 \in \mathscr{V}$. Then clearly also the lattice $\mathfrak{L}$ on Fig. 1 is contained in $\mathscr{V}$. We shall show that $\mathfrak{L}$ has a non-distributive lattice $LT(\mathfrak{L})$.

Call $B \subseteq \mathfrak{L}$ to be a block of the tolerance T provided $x, y \in B$ always implies $\langle x, y \rangle \in T$ and B is a maximal subset of $\mathfrak{L}$ with this property.

Now, we can consider the three tolerances T_1, T_2, T_3 on $\mathfrak{L}$ determined by the blocks:

T_1 has blocks $B_1 = \{1, x, a\}$, $B_2 = \{0, a, b, c, x\}$,
T_2 has blocks $C_1 = \{1, x, b\}$ and B_2,
T_3 has blocks $D_1 = \{1, x, c\}$ and B_2.

It is clear that $T_1 \wedge T_2 \wedge T_3$, T_1, T_2, T_3, $T_1 \vee T_2 \vee T_3$ form the non-distributive sublattice M_5 of $LT(\mathfrak{L})$. Hence $LT(\mathfrak{L})$ is not distributive.

Suppose $N_5 \in \mathscr{V}$. Then clearly the lattice $\mathfrak{L}^*$ on Fig. 2 is contained in $\mathscr{V}$. Consider $T_1, T_2, T_3 \in LT(\mathfrak{L}^*)$ determined by the blocks:

T_1 has blocks $B_1 = \{1, x, a\}$, $B_2 = \{0, a, b, c, x\}$.
T_2 has blocks $C_1 = \{1, x, b, c\}$, B_2,
T_3 has blocks $D_1 = \{1, x, c\}$ and B_2.

It is clear that $T_3 \subseteq T_2$, further T_1 is noncomparable with T_2 and T_3 and $T_1 \wedge T_2 = T_1 \wedge T_3$, $T_1 \vee T_2 = T_1 \vee T_3$. Hence, T_1, T_2, T_3 generate the non-modular sublattice N_5 of $LT(\mathfrak{L}^*)$. Accordingly, $\mathscr{V}$ has not distributive tolerances in any case.

Q.E.D.

ACKNOWLEDGEMENT

I would like to express my thanks to Dr. H.-J. Bandelt for his suggestion for the formulation of this result.

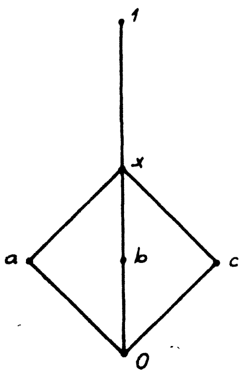

Fig. 1

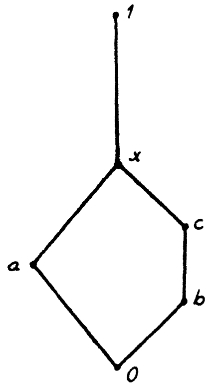

Fig. 2

REFERENCES

[1] Chajda I., Zelinka B.: *Tolerance relations on lattices,* Časopis pro pěst. matem. 99 (1974), 394—399.

[2] Chajda I., Zelinka B.: *Lattices of tolerances,* Časopis pro pěst. matem. 102 (1977), 10—24.

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