Karel Svoboda

Contribution to the characterization of the sphere in E^3

Archivum Mathematicum, Vol. 15 (1979), No. 3, 171--177

Persistent URL: http://dml.cz/dmlcz/107037

ARCH. MATH. 3, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XV: 171—178, 1979

CONTRIBUTION TO THE CHARACTERIZATION OF THE SPHERE IN E^3

KAREL SVOBODA, Brno

(Received June 15, 1978)

The present paper contains a generalization of the results due to A. Švec [1] and M. Afwat [2].

1. Let M be a surface in the 3-dimensional Euclidean space E^3 and ∂M its boundary. Let $\{M; v_1, v_2, v_3\}$ be a field of orthonormal frames on M, $v_1, v_2 \in T(M)$, $T(M)$ being the tangent bundle of M. Then

(1)
$$\begin{aligned} dM &= \omega^1 v_1 + \omega^2 v_2, \\ dv_1 &= \omega_1^2 v_2 + \omega_1^3 v_3, \\ dv_2 &= -\omega_1^2 v_1 + \omega_2^3 v_3, \\ dv_3 &= -\omega_1^3 v_1 - \omega_2^3 v_2; \end{aligned}$$

(2)
$$\omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3 = 0,$$
$$d\omega^1 = -\omega^2 \wedge \omega_1^2, \qquad d\omega^2 = \omega^1 \wedge \omega_1^2,$$
$$d\omega_1^2 = -\omega_1^3 \wedge \omega_2^3, \qquad d\omega_1^3 = \omega_1^2 \wedge \omega_2^3, \qquad d\omega_2^3 = -\omega_1^2 \wedge \omega_1^3$$

on M. Following [1] we have

(3)
$$\omega_1^3 = a\omega^1 + b\omega^2, \qquad \omega_2^3 = b\omega^1 + c\omega^2;$$

(4)
$$\begin{aligned} da - 2b\omega_1^2 &= \alpha\omega^1 + \beta\omega^2, \\ db + (a - c)\,\omega_1^2 &= \beta\omega^1 + \gamma\omega^2, \\ dc + 2b\omega_1^2 &= \gamma\omega^1 + \delta\omega^2; \end{aligned}$$

(5)
$$\begin{aligned} d\alpha - 3\beta\omega_1^2 &= A\omega^1 + (B - bK)\,\omega^2, \\ d\beta + (\alpha - 2\gamma)\,\omega_1^2 &= (B + bK)\,\omega^1 + (C + aK)\,\omega^2, \\ d\gamma + (2\beta - \delta)\,\omega_1^2 &= (C + cK)\,\omega^1 + (D + bK)\,\omega^2, \\ d\delta + 3\gamma\omega_1^2 &= (D - bK)\,\omega^1 + E\omega^2, \end{aligned}$$

where

(6)
$$K = ac - b^2$$

is the Gauss curvature of M. Denote further

$$H = \frac{1}{2}(a + c) \tag{7}$$

the mean curvature of M and define

$$f = 2(H^2 - K) = \frac{1}{2}(a - c)^2 + 2b^2. \tag{8}$$

Let F be a real-valued function on M. Its covariant derivatives F_i, F_{ij} $(i, j = 1, 2)$ on M with respect to the given field of tangent frames are defined by

$$\begin{gathered} \mathrm{d}F = F_1\omega^1 + F_2\omega^2, \\ \mathrm{d}F_1 - F_2\omega_1^2 = F_{11}\omega^1 + F_{12}\omega^2, \qquad \mathrm{d}F_2 + F_1\omega_1^2 = F_{21}\omega^1 + F_{22}\omega^2. \end{gathered} \tag{9}$$

Using (9), we get for the functions K, H, f introduced by (6), (7), (8), respectively.

$$\begin{aligned} K_1 &= a\gamma - 2b\beta + c\alpha, \\ K_2 &= a\delta - 2b\gamma + c\beta. \\ K_{11} &= aC - 2bB + cA + 2(\alpha\gamma - \beta^2) + (ac - 2b^2)\,K, \\ K_{12} &= aD - 2bC + cB + \alpha\delta - \beta\gamma - b(a + c)\,K, \\ K_{22} &= aE - 2bD + cC + 2(\beta\delta - \gamma^2) + (ac - 2b^2)\,K; \end{aligned} \tag{10}$$

$$\begin{aligned} 2H_1 &= \alpha + \gamma, \\ 2H_2 &= \beta + \delta, \\ 2H_{11} &= A + C + cK, \\ 2H_{12} &= B + D, \\ 2H_{22} &= C + E + aK; \end{aligned} \tag{11}$$

$$\begin{aligned} f_{11} &= (a - c)(A - C) + 4bB + (\alpha - \gamma)^2 + 4\beta^2 + [-c(a - c) + 4b^2]\,K, \\ f_{12} &= (a - c)(B - D) + 4bC + (\alpha - \gamma)(\beta - \delta) + 4\beta\gamma + 2b(a + c)\,K, \\ f_{22} &= (a - c)(C - E) + 4bD + (\beta - \delta)^2 + 4\gamma^2 + [a(a - c) + 4b^2]\,K. \end{aligned} \tag{12}$$

To complete the preliminaries, we formulate the maximum principle in the form used in [1]:

Let M be a surface in E^3, $F : M \to \mathscr{R}$ a function with covariant derivatives F_i, $F_{ij} = F_{ji}$ $(i, j = 1, 2)$ given by (9). Let (a) $F \geqq 0$ on M; (b) $F = 0$ on ∂M; (c) F satisfy on M the equation

$$a_{11}F_{11} + 2a_{12}F_{12} + a_{22}F_{22} + a_1F_1 + a_2F_2 + a_0F = a,$$

where $a_0 \leqq 0$, $a \geqq 0$ and the quadratic form $a_{ij}x^ix^j$ is positive definite. Then $F = 0$ on M.

Note that the function f introduced by (8) satisfies obviously the conditions (a) and (b) supposing that ∂M consists of umbilical points ($a = c$, $b = 0$).

2. We are going to formulate the

Theorem 1. *Let M be a surface in E^3, ∂M its boundary and $\lambda, \mu : M \to \mathcal{R}$ functions on M satisfying*

$$\lambda + H\mu > 0, \tag{13}$$

$$\lambda^2 + 2H\lambda\mu + K\mu^2 > 0. \tag{14}$$

Let

(i) $K > 0$ *on* M;

(ii) *on* M,

$$2\lambda[(a - c)(H_{11} - H_{22}) + 4bH_{12}] + \mu[(a - c)(K_{11} - K_{22}) + 4bK_{12}] \geqq 0 \tag{15}$$

and

$$\frac{1}{3}(\lambda + \mu k_1) \leqq \lambda + \mu k_2 \leqq 3(\lambda + \mu k_1), \tag{16}$$

k_1, k_2 *being the principal curvatures of* M;

(iii) ∂M *consist of umbilical points.*

Then M is a part of a sphere in E^3.

Proof. Following [1], p. 32–33, we have, according to (10), (11), (12)

$$\begin{aligned} &f_{11} + f_{22} - 4Kf = \\ &= 2[(a - c)(H_{11} - H_{22}) + 4bH_{12}] + (\alpha - \gamma)^2 + (\beta - \delta)^2 + 4(\beta^2 + \gamma^2) \end{aligned} \tag{17}$$

and

$$\begin{aligned} &cf_{11} - 2bf_{12} + af_{22} - 4HKf = \\ &= (a - c)(K_{11} - K_{22}) + 4bK_{12} + \\ &+ a(\delta^2 + 2\gamma^2 + 3\beta^2 - 2\alpha\gamma) - 2b(\alpha + \gamma)(\beta + \delta) + c(\alpha^2 + 2\beta^2 + 3\gamma^2 - 2\beta\delta). \end{aligned} \tag{18}$$

Multiplying (17) by λ, (18) by μ and adding these equations we obtain

$$\begin{aligned} &(\lambda + \mu c)f_{11} - 2b\mu f_{12} + (\lambda + \mu a)f_{22} - 4(\lambda + H\mu)Kf = \\ &= 2\lambda[(a - c)(H_{11} - H_{22}) + 4bH_{12}] + \mu[(a - c)(K_{11} - K_{22}) + 4bK_{12}] + \Phi, \end{aligned} \tag{19}$$

where

$$\begin{aligned} \Phi = {} & \lambda[(\alpha - \gamma)^2 + (\beta - \delta)^2 + 4(\beta^2 + \gamma^2)] + \\ & + \mu[a(\delta^2 + 2\gamma^2 + 3\beta^2 - 2\alpha\gamma) - 2b(\alpha + \gamma)(\beta + \delta) + c(\alpha^2 + 2\beta^2 + 3\gamma^2 - 2\beta\delta)]. \end{aligned} \tag{20}$$

It is easy to see that the coefficients of λ, μ (20) are invariant on M. Therefore, it is possible to examine the expression Φ in a generic point $m \in M$ and choose the field of moving frames arround m in such a way that $b = 0$ at m. Then a, c are principal curvatures and, according to (13), (14), $\lambda + \mu a > 0$, $\lambda + \mu c > 0$ at m. Taking regard of these relations, we have, from (20),

$$\Phi = \frac{1}{\lambda + \mu c} [(\lambda + \mu c)\alpha - (\lambda + \mu a)\gamma]^2 + \frac{1}{\lambda + \mu a} [(\lambda + \mu a)\delta - (\lambda + \mu c)\beta]^2 +$$

$$+ 2(\lambda + H\mu)\left\{\frac{1}{\lambda + \mu a} [3(\lambda + \mu a) - (\lambda + \mu c)]\beta^2 + \frac{1}{\lambda + \mu c} [3(\lambda + \mu c) - (\lambda + \mu a)]\gamma^2\right\}$$

and thus $\Phi \geqq 0$ at m, according to (16). Using the inequalities mentioned in the theorem and applying the maximum principle, we obtain $f = 0$ on M.

Remark. Taking $\lambda = 0$, $\mu = 1$, resp. $\lambda = 1$, $\mu = 0$, we get the theorem 4.2, resp. 4.3, of [1]. Further, supposing $\lambda \geqq 0$, $\mu \geqq 0$ on M, the relation (20) can be written in the form

$$\Phi = \lambda[(\alpha - \gamma)^2 + (\beta - \delta)^2 + 4(\beta^2 + \gamma^2)] +$$
$$+ \mu\{c^{-1}(c\alpha - a\gamma)^2 + a^{-1}(a\delta - c\beta)^2 + 2H[a^{-1}(3a - c)\beta^2 + c^{-1}(3c - a)\gamma^2]\};$$

and thus, for Φ being non-negative, it is sufficient to consider

$$\frac{1}{3} \leqq \frac{k_2}{k_1} \leqq 3$$

instead of (16).

As a consequence of the preceeding result we get the

Theorem 2. *Let M be a surface in E^3, ∂M its boundary and $\lambda, \mu : M \to \mathscr{R}$ functions on M satisfying* (13) *and* (14). *Let*

(i) $K > 0$ *on* M;

(ii) *there exist a net of lines of curvature on M with the unit tangent vector fields* V_1, V_2;

(iii) *on* M,

(21) $$2\lambda(k_1 - k_2)(V_1V_1 - V_2V_2)H + \mu(k_1 - k_2)(V_1V_1 - V_2V_2)K \geqq 0$$

and

(22) $$\frac{4}{11}(\lambda + \mu k_1)^2 \leqq (\lambda + \mu k_2)^2 \leqq \frac{11}{4}(\lambda + \mu k_1)^2,$$

k_1, k_2 being the principal curvatures of M;

(iv) *∂M consist of umbilical points.*

Then M is a part of a sphere in E^3.

Proof. Let us choose the tangent frames on M in such a way that $v_1 = V_1$, $v_2 = V_2$. Then $b = 0$ on M and a, c are the principal curvatures of M.

Let φ be a real-valued function on M. Then, from (9),

$$V_1\varphi = \varphi_1, \qquad V_2\varphi = \varphi_2$$

and

$$V_1V_1\varphi = \varphi_{11} + \varphi_2\omega_1^2(V_1), \qquad V_2V_2\varphi = \varphi_{22} - \varphi_1\omega_1^2(V_2).$$

From (4),

$$(a - c)\,\omega_1^2 = \beta\omega^1 + \gamma\omega^2$$

and hence

$$(23) \qquad \begin{aligned} (a - c)\,V_1V_1\varphi &= (a - c)\,\varphi_{11} + \beta\varphi_2, \\ (a - c)\,V_2V_2\varphi &= (a - c)\,\varphi_{22} - \gamma\varphi_1. \end{aligned}$$

Applying (23) to the functions H, K, we obtain from (19) putting $b = 0$,

$$\begin{aligned} &(\lambda + \mu c) f_{11} + (\lambda + \mu a) f_{22} - 4(\lambda + H\mu)\,Kf = \\ &= 2\lambda(a - c)(V_1V_1 - V_2V_2)\,H + \mu(a - c)(V_1V_1 - V_2V_2)\,K + \Psi, \end{aligned}$$

where

$$(24) \qquad \begin{aligned} \Psi = {}& \lambda[(\alpha - \gamma)^2 + (\beta - \delta)^2 + 4(\beta^2 + \gamma^2) - \gamma(\alpha + \gamma) - \beta(\beta + \delta)] + \\ &+ \mu[a(\delta^2 + \gamma^2 + 3\beta^2 - 2\alpha\gamma - \beta\delta) + c(\alpha^2 + \beta^2 + 3\gamma^2 - 2\beta\delta - \alpha\gamma)]. \end{aligned}$$

By an easy calculation we get

$$\begin{aligned} \Psi = {}& (\lambda + \mu a)\left[\delta - \left(\frac{1}{2} + \frac{\lambda + \mu c}{\lambda + \mu a}\right)\beta\right]^2 + (\lambda + \mu c)\left[\alpha - \left(\frac{1}{2} + \frac{\lambda + \mu a}{\lambda + \mu c}\right)\gamma\right]^2 + \\ &+ (\lambda + \mu a)\left[\frac{11}{4} - \frac{(\lambda + \mu c)^2}{(\lambda + \mu a)^2}\right]\beta^2 + (\lambda + \mu c)\left[\frac{11}{4} - \frac{(\lambda + \mu a)^2}{(\lambda + \mu c)^2}\right]\gamma^2 \end{aligned}$$

and hence $\Psi > 0$ because of $\lambda + \mu a > 0$ and $\lambda + \mu c > 0$. Thus the inequalities (13), (14), (21), (22) ensure that, using the maximum principle, our assertion is true.

Remark. For $\lambda = 1$, $\mu = 0$ and $\lambda = 0$, $\mu = 1$ we get the results of the theorem 1 and 2 of [2], respectively. Again, considering $\lambda \geqq 0$, $\mu \geqq 0$ on M, (24) has the form

$$\begin{aligned} \Psi = {}& \lambda\left[\left(\alpha - \frac{3}{2}\gamma\right)^2 + \left(\delta - \frac{3}{2}\beta\right)^2 + \frac{7}{4}(\beta^2 + \gamma^2)\right] + \\ &+ \mu a\left\{\left[\delta - \left(\frac{1}{2} + \frac{c}{a}\right)\beta\right]^2 + \left(\frac{11}{4} - \frac{c^2}{a^2}\right)\beta^2\right\} + \\ &+ \mu c\left\{\left[\alpha - \left(\frac{1}{2} + \frac{a}{c}\right)\gamma\right]^2 + \left(\frac{11}{4} - \frac{a^2}{c^2}\right)\gamma^2\right\} \end{aligned}$$

and thus the inequality

$$\frac{4}{11} \leqq \frac{k_2^2}{k_1^2} \leqq \frac{11}{4}$$

implies that $\Psi \geqq 0$ on M.

3. To the end of this paper we introduce the following results concerning the generalized Weingarten surfaces.

Corollary 1. *Let M be a surface in E^3, ∂M its boundary and $F(H, K)$ a function on M such that*

$$F_H + 2HF_K > 0, \tag{25}$$

$$F_H^2 + 4HF_HF_K + 4KF_K^2 > 0. \tag{26}$$

Let M satisfy the conditions (i), (iii) *of the theorem 1 and, on M,*

$$(a - c)(F_{11} - F_{22}) + 4bF_{12} -$$
$$- (a - c)[F_{HH}(H_1^2 - H_2^2) + 2F_{HK}(H_1K_1 - H_2K_2) + F_{KK}(K_1^2 - K_2^2)] -$$
$$- 4b[F_{HH}H_1H_2 + 2F_{HK}(H_1K_2 + H_2K_1) + F_{KK}K_1K_2] \geqq 0$$

and

$$\frac{1}{3}(F_H + 2k_1F_K) \leqq F_H + 2k_2F_K \leqq 3(F_H + 2k_1F_K),$$

k_1, k_2 being the principal curvatures of M. Then M is a part of a sphere in E^3.

Proof. It is sufficient to put $\lambda = \frac{1}{2}F_H$, $\mu = F_K$ in the theorem 1 and take into account the relations

$$F_{ij} = F_{HH}H_iH_j + F_{HK}(H_iK_j + H_jK_i) + F_{KK}K_iK_j + F_HH_{ij} + F_KK_{ij} \quad (i, j = 1, 2) \tag{27}$$

for the covariant derivatives of F.

Corollary 2. *Let M be a surface in E^3, ∂M its boundary and $F(H, K)$ a function on M such that* (25) *and* (26) *are fulfilled. Let M satisfy the conditions* (i), (ii) *and* (iv) *of the theorem 2 and, on M,*

$$(k_1 - k_2)(V_1V_1 - V_2V_2)F -$$
$$- (k_1 - k_2)(V_1 + V_2)H[F_{HH}(V_1 - V_2)H + F_{HK}(V_1 - V_2)K] -$$
$$- (k_1 - k_2)(V_1 + V_2)K[F_{HK}(V_1 - V_2)H + F_{KK}(V_1 - V_2)K] \geqq 0$$

and

$$\frac{4}{11}(F_H + 2k_1F_K)^2 \leqq (F_H + 2k_2F_K)^2 \leqq \frac{11}{4}(F_H + 2k_1F_K)^2,$$

k_1, k_2 being the principal curvatures of M. Then M is a part of a sphere in E^3.

Proof. The result follows from the theorem 2 for $\lambda = \frac{1}{2}F_H$, $\mu = F_K$ when using (23) and (27).

Remark. Supposing $F_H \geqq 0$, $F_K \geqq 0$, we get from this corollary the theorem 3 of [2].

REFERENCES

[1] A. Švec: *Contributions to the global differential geometry of surfaces.* **Rozpravy ČSAV 1, 87** (1977), p. 1—94.

[2] M. Afwat: *On generalized Weingarten surfaces.* Čas. pro pěst. matem. 3, 101 (1976), p. 263—270.

K. Svoboda
University of Technology
602 00 Brno, Gorkého 13
Czechoslovakia