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ON PRIME AND PRINCIPAL IDEALS ON GRAPHS

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In the paper [4] we applied the concept of an ideal on a graph to analyzing structural properties of undirected graphs. The concept of an ideal on a graph is a generalization of the concept of convex sublattices of a lattice. In the recent paper [2] Duda and Chajda presented a generalization of ideals in lattices. This generalization gave us hints for developing the analysis of graphs further by means of ideals on graphs. We shall present the concepts of prime and principal ideals on a graph and prove results analogous to that given by Balbes [1] concerning prime ideals on semilattices.

We shall consider finite undirected and connected graphs $G = (P(G), L(G))$ only without loops and multiple lines, where $P(G)$ is the set of points of G and $L(G)$ its set of lines. SP is a mapping $P(G) \times P(G) \rightarrow 2^{P(G)}$ defined as follows:

$$SP(x, y) = \{z \mid z \in P(G) \textit{ and } z \textit{ is on a shortest path joining } x \textit{ and } y \textit{ in } G\}.$$

In particular, $\{x, y\} \subseteqq SP(x, y)$ and $SP(x, x) = \{x\}$, $x, y \in P(G)$. Let U and W be two subsets of $P(G)$, then $SP(U, W) = \{z \mid z \in SP(u, w) \textit{ for some } u \textit{ and } w,\ u \in U \textit{ and } w \in W\}$. A set $U \subset P(G)$ is called an ideal of G, if $U \neq \emptyset$ and $SP(U, U) = U$ (cf. Nebeský [3]). By $SP^n(x, y)$ we denote the set $SP(SP^{n-1}(x, y), SP^{n-1}(x, y))$. As we consider finite graphs only, there is for any pair $x, y \in P(G)$ a value of n such that $SP^n(x, y)$ is an ideal on G. We shall denote by $SU(x, y)$ the ideal constructed from a pair $x, y \in P(G)$ by means of sequential application of the SP-operation.

If U and J are ideals on G and $U \cap J \neq \emptyset$, then clearly $U \cap J$ is an ideal on G and it is the greatest ideal contained in U and J. Obviously, the ideals on G constitute a join-semilattice $\mathscr{I}(G)$, where G is the greatest element and $U \vee J$ denotes the least ideal on G that contains U and J. If we denote by $\emptyset$ the least ideal on G, $\mathscr{I}(G)$ is a lattice.

We shall say that an ideal U on a graph G is *prime* if and only if the condition (P) holds for any two points $a, b \in P(G) \setminus U$: (P) $SP(a, b) \cap U = \emptyset$.

Theorem 1. *If U is a prime ideal on a graph G, then $P(G) \setminus U$ is a prime ideal, too.*

Proof. According to the definitions of a prime ideal and an ideal, $SP(P(G)\backslash U, P(G)\backslash U) \subseteq P(G)\backslash U$, whence $P(G)\backslash U$ is an idal on G. As U is an ideal $SP(a, b) \cap U = SP(a, b)$ for each two element $a, b \in U$, whence $P(G)\backslash U$ is prime.

Theorem 2. *An ideal U of a graph G is prime if and only if the following condition holds: If $SP(a, b) \cap U \neq \emptyset$, then a or b belongs to U for any two points $a, b \in P(G)$.*

The proof follows directly from the definition of a prime ideal.

By using the same proof technique as used by Duda and Chajda in [2, Thm. 4], we obtain

Theorem 3. *Let U be a prime ideal of a graph G and let $I, J \in \mathscr{I}(G)$. If $\emptyset \neq I \cap J \subset U$, then $I \subset U$ or $J \subset U$.*

We denote by $(I]$ and $[I)$ principal and dual principal ideal, respectively, induced by an element I in the lattice $\mathscr{I}(G)$. As well known, $(I]$ is prime if and only if for any $J_{n+1} \in (I]$, where $\emptyset \neq J_{n+1} = J_1 \cap \ldots \cap J_n$, at least one J_i belongs to $(I]$, $i = 1, \ldots, n$ (see e.g. Balbes [1]).

Theorem 4. *Let U be an ideal of a graph G. U is a prime ideal on G if and only if $(U]$ is a prime ideal in $\mathscr{I}(G)$.*

Proof. 1° Let U be a prime ideal of G, $\emptyset \neq J_{n+1} = J_1 \cap \ldots \cap J_n$ and $J_{n+1} \in (U]$. We denote $J_2 \cap J_3 \cap \ldots \cap J_n = J_2'$. As $J_1 \cap J_2' \subset U$, J_1 or J_2' is contained in U according to Theorem 3. If $J_1 \subset U$, $(U]$ is prime in $\mathscr{I}(G)$, and if $J_2' \subset U$, a similar way of deduction can be applied to J_2' as above. After applying the deduction step n times, we see that at least one of the ideals J_i is contained in U, whence $(U]$ is a prime ideal in $\mathscr{I}(G)$.

2° Let $(U]$ be a prime ideal in $\mathscr{I}(G)$ and let $SP(a, b) \cap U \neq \emptyset$. According to the properties of $\mathscr{I}(G)$, there is a greatest element $J \in \mathscr{I}(G)$ such that $SU(a, b) \cap U = J \neq \emptyset$. Obviously, $SP(a, b) \cap U \subset J$. If a or b belongs to $J \subset U$, there is nothing to prove, and hence we assume that $a, b \notin J \subset U$. We shall show that under these assumptions $SP(a, b) \cap J = \emptyset$, which is a contradiction.

If there is a point $x \in J$ such that (t, x) $(s, x) \in L(G)$, where $t, s \in SU(a, b)\backslash J$, then $SU(t, x) \cap SU(s, x) = \{x\} \subset J \subset U$. As $(U]$ is prime, $SU(t, x)$ or $SU(s, x)$ is contained in $(U]$, and thus t or s belongs to J, which is a contradiction.

Let $t \in SU(a, b)\backslash J$, $x \in J$ and q be a point on a shortest path from t to x such that $(q, y) \in L(G)$ and $y \in J$. As J is an ideal on G, J contains all the points between y and x on this shortest path, and from the same reason, all the points between t and q on this shortest path do not belong to J.

Assume that t and s are points of $SU(a, b)\backslash J$ such that (t, y), $(s, x) \in L(G)$ where $y, x \in J$. If $SU(t, x) \cap SU(s, x) = SU(x, x)$, we obtain a contradiction as above. Hence, for any such a point t as defined above it holds: $SU(s, x) \subset SU(t, x)$. Let $t_{11}, t_{12}, \ldots, t_{1k_1}$ be the points of $SP(t, x) \cap (SU(a, b)\backslash J)$ being joined by a line to at least some point of J. If $s \notin T_1 = \{t_{11}, \ldots, t_{1k_1}\}$, we consider the points of T_1.

According to the observation above, $SU(s, x) \subset SU(t_{1i}, x)$ for each i, $i = 1, \ldots, k_1$. By T_2 we denote the set of points $\{t_{21}, t_{22}, \ldots, t_{2k_2}\}$, where each of the points is joined by a line to at least some of the points in J and belongs to at least one of the sets $SP(t_{1i}, x) \cap (SU(a, b) \setminus J)$, $i = 1, \ldots, k_1$. If $s \notin T_2$, we continue the process described before. The facts, that $(s, x) \in L(G)$, $SU(s, x) \subset SU(t, x)$ for each t defined above and the finiteness of G, imply together that after a finite number of steps we shall find a T_n such that $s \in T_n$. According to the determination process of the points in $T_1, T_2, \ldots, T_n$, we can conclude that $s \in SP(t, x)$. Moreover, on the shortest path from t to x, on which s is, all the points between t and s do not belong to J according to the observation proved above. This proof can be repeated for any pair of points s and t defined above. As s is on the shortest path from t to x, the shortest path joining s and t is shorter than the shortest path from t to x. The basic assumption was that $J \cap SP(a, b) = \emptyset$, and hence there is at least one pair of points s and t through which the shortest path between a and b touch J. But the considerations above show that the shortest path joining s and t does not go through the points of J and hence $J \cap SP(a, b) \neq \emptyset$, which is the final contradiction. Thus we can conclude that at least a or b belongs to U, whence U is a prime ideal of G.

Following Balbes [1], we call a graph G prime if for each meet $J_1 \cap \ldots \cap J_n = J_{n+1} \neq \emptyset$ in $\mathscr{I}(G)$ and for each $I \in \mathscr{I}(G)$ it holds: $I \vee J_{n+1} = (I \vee J_1) \cap (I \vee J_2) \cap \ldots \cap (I \vee J_n)$. According to Theorem 4, the following theorem can now be proved by the dual of the proof technique used by Balbes [1, Thm. 2.2], and hence we omit the proof.

Theorem 5. *A graph G is prime if and only if for any pair I, $J \in \mathscr{I}(G)$, I, $J \neq \emptyset$, and $(I] \cap [J) = \emptyset$, there exists a prime ideal U of G such that $I \subset (U]$ and $(I] \cap [J) = \emptyset$.*

Thus the lattice $\mathscr{I}(G)$ offers a bridge for translating the results obtained in [1] to the case of prime graphs. We do not perform the translation work here but refer only to the paper [1].

We shall call a graph *principal* if for each collection C of points of G there is at least one point p such that $\bigcup_x \{SP(x, p) \mid x \in C\}$ is an ideal of G. Principal graphs seem not to have as important role among graphs as their analogy in the paper [2] of Duda and Chajda. We present only a characterization of principal graphs in the next theorem. If a pointset C generates an ideal $I = \bigcup_x \{SP(x, p) \mid x \in C\}$ for some point $p \in C$, then I is called a principal ideal generated by C.

Theorem 6. *A graph G is principal if and only if for any three points $x, y, z \in P(G)$ there is a point, say z, such that $SP(x, y) \subset SP(x, z) \cup SP(y, z)$.*

Proof. 1° Let G be a principal graph. Then for each $x, y, z \in P(G)$, $I = SP(x, z) \cup SP(y, z)$ is an ideal of G. As $x, y \in I$, $SP(x, y) \subset I$ according to the definition of an ideal on a graph.

2° Let G satisfy the condition of the theorem for any triple $x, y, z \in P(G)$. If G contains a cycle of points in D, then D induces a complete subgraph of G, as in the other case we can always choose three points from the cycle such that the condition assumed be valid does not hold. Let $\mathscr{D} = \{D_1, \ldots, D_n\}$ be the collection of all maximal subgraphs of G. According to the observation above, D_i and D_j can have at most one common point when $i \neq j$. We map G onto a graph $G_{\mathscr{D}}$ the points of which are labelled by the symbols $D_1, \ldots, D_n$ and where a line joins two points D_i and D_j whenever D_i and D_j have a common point in G. According to the property of G proved before and the connectivity of G, $G_{\mathscr{D}}$ is a tree.

As one can easily see, each ideal of a tree is a subtree of T and conversely. If $C \subset \subset P(T)$, then the paths joining the points of C is a subtree of T and obviously it is a principal ideal generated by C on T. Hence each tree is a principal graph. One can also easily see that each complete graph is a principal graph.

Let $C \subset P(G)$. C is mapped under the mapping $f_{\mathscr{D}}$ induced by $\mathscr{D}$ onto points D_h of $G_{\mathscr{D}}$ having the property: $C \cap P(D_h) \neq \emptyset$ in G, where $P(D_h)$ is the set of points of the maximal complete subgraph D_h in G. As $G_{\mathscr{D}}$ is a tree, the points $f_{\mathscr{D}}(C)$ generate a principal ideal in $G_{\mathscr{D}}$. If we substitute a point in a tree by a complete graph, the graph such obtained is further a principal ideal. Hence C generates a principal ideal on G and G is a principal graph.

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