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ASYMPTOTIC PROPERTIES OF SOLUTIONS OF THE DIFFERENTIAL EQUATION $\{A_{n-1}^{-1}(t)\dots[A_1^{-1}(t)\,y']'\dots\}' = A_n(t)\,y + F(t)$

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1. INTRODUCTION

The object under consideration is the nth order nonhomogeneous linear differential equation of the form

$$\{A_{n-1}^{-1}(t)\dots[A_1^{-1}(t)\,y']'\dots\}' = A_n(t)\,y + F(t). \tag{1,1}$$

As the asymptotic properties of solution of equation (1,1) for $F(t) \equiv 0$ were studied in [5], the form of the particular solution of equation (1,1) and of its certain derivatives will be constructed and their asymptotic properties studied under a convenient combination of conditions.

In the course of study it was found advantageous to use the so called Peano—Baker method, for it allowed to express the particular solution $y_1(t)$ of equation (1,1) in the form of infinite series converging on the interval I. The advantage of this method lies especially in the fact that when we express the solution in an approximate manner, it is possible to obtain a simple estimation of the error.

Let us make the following agreements:

1° If a is a real number, then the symbol $\int_{a_j}^{t} A_j(v)\,dv$ denotes:

a) Rieman's integral for $a_j = a$

b) $\lim_{\tau\to\infty} \int_{\tau}^{t} A_j(v)\,dv$ for $a_j = \infty$.

2° Throughout this paper it will be assumed that:

$$A_i(t) \in C_0(I), \quad i = 1, 2, \dots, n, \quad A_i(t) > 0 \quad \text{for } i = 1, 2, \dots, n-1,$$

$$A_i^{-1}(t) = \frac{1}{A_i(t)}, \quad F(t) \in C_0(I), \quad I = \langle t_0, \infty).$$

3° Provided no misunderstanding occurs, only $\int_{a_j}^{t} A_j, F$ will be used instead of $\int_{a_j}^{t} A_j(v)\,\mathrm{d}v, F(t)$, etc.

2. DEFINITIONS AND NOTATIONS

Definition 2.1. *Denote* $\mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}$ *and define the linear differential operators*

$$(2,1) \qquad L_s = A_s^{-1}\,\mathrm{D}A_{s-1}^{-1}\,\mathrm{D} \ldots \mathrm{D}A_1^{-1}\,\mathrm{D}, \qquad \textit{for } s = 1, 2, \ldots, n-1$$

$$(2,2) \qquad L_n = \mathrm{D}L_{n-1}.$$

The identical operator is denoted by the symbol L_0.

Definition 2.2. *Suppose that the function* $h(t) \in C_0(I)$. *Let the symbol* Q_j, $j = 1, 2, \ldots, n$ *denote the integral operator which maps the set of functions of* C_0 *into itself*

$$(2,3) \qquad Q_j(h) = \int_{a_j}^{t} A_j(v)\,h(v)\,\mathrm{d}v,$$

and let $Q(h)$ *be defined by*

$$(2,4) \qquad Q(h) = \int_{a_0}^{t} F(v)\,h(v)\,\mathrm{d}v.$$

For $j \neq k$, $j, k = 1, 2, \ldots, n$ *define the operator* Q_jQ_k

$$(2,5) \qquad Q_jQ_k(h) = \int_{a_j}^{t} A_j(v) \int_{a_k}^{v} A_k(u)\,h(u)\,\mathrm{d}u\,\mathrm{d}v$$

and the operator Q_jQ *by*

$$(2,6) \qquad Q_jQ(h) = \int_{a_j}^{t} A_j(v) \int_{a_0}^{v} F(u)\,h(u)\,\mathrm{d}u\,\mathrm{d}v.$$

The construction of further operators that occur in this paper will, similarly as above, be written in the form of products, such as $\mathrm{D}Q_j(h) = A_jh$.

Definition 2.3. *Let* $a \in I$ *be a real number and let* a_i, $i = 1, 2, \ldots, n$ *denote either* a *or* $+\infty$, $h(t) \in C_0(I)$. *Define the integral operators* $\mathscr{C}_j$, $j = 1, 2, \ldots, n$ *by relation*

$$(2,7) \qquad \mathscr{C}_j(h) = Q_jQ_{j+1} \ldots Q_nQ_{n+1} \ldots Q_{n+j-1}(h),$$

where $Q_{n+j} = Q_j$.

Furthermore, let us put

(2,8) $$\mathscr{C}_j^0(h) = h, \qquad \mathscr{C}_j^r(h) = \mathscr{C}_j \mathscr{C}_j^{r-1}(h), \qquad r = 1, 2, \ldots$$

Definition 2.4. *Let j, k be natural numbers $1 \leqq j,\ k \leqq n$, $h(t) \in C_0(I)$. Define the integral operators*

(2,9) $$\begin{aligned} &X_{j,j}(h) = h, \\ &X_{j,k}(h) = Q_j Q_{j+1} \ldots Q_{k-1}(h), \qquad \text{for } j < k, \\ &X_{j,k}(h) = 0, \qquad \text{for } j > k. \end{aligned}$$

Definition 2.5. *Let j be a natural number $1 \leqq j \leqq n - 1$. Define the functions*

(2,10) $$\Phi_j(t) = Q_j Q_{j+1} \ldots Q_{n-1} Q(1).$$

Furthermore, let us put

(2,11) $$\Phi_n(t) = Q(1).$$

Definition 2.6. *Let $h(t) \in C_0(I)$. The symbols q_j, $j = 1, 2, \ldots, n$ and q denote the operator*

(2,12) $$q_j(h) = \int_{a_j}^{t} | A_j(v) | \, h(v) \, dv, \qquad q(h) = \int_{a_0}^{t} | F(v) | \, h(v) \, dv.$$

By the product of operators $q_j q_k$ and $q_j q$, we understand the operator

(2,13) $$\begin{aligned} q_j q_k(h) &= \int_{a_j}^{t} | A_j(v) | \int_{a_k}^{v} | A_k(u) | \, h(u) \, du \, dv, \\ q_j q(h) &= \int_{a_j}^{t} | A_j(v) | \int_{a_0}^{v} | F(u) | \, h(u) \, du \, dv. \end{aligned}$$

Definition 2.7. *Let j be a natural number $1 \leqq j \leqq n$. Define the functions*

(2,14) $$\gamma_j(t) = | q_j q_{j+1} \ldots q_n q_{n+1} \ldots q_{n+j-1}(1) |,$$

while putting $q_{n+j} = q_j$.

Definition 2.8. *Let j be a natural number $1 \leqq j \leqq n - 1$. Define the functions*

(2,15) $$\varphi_j(t) = | q_j q_{j+1} \ldots q_{n-1} q(1) |.$$

Furthermore, we put

$$\varphi_n(t) = | q(1) |.$$

Definition 2.9. Let j, k be natural numbers. Define the functions

$$\begin{aligned} &\varkappa_{j,j}(t) = 1 \\ &\varkappa_{j,k}(t) = | q_j q_{j+1} \ldots q_{k-1}(1) |, \qquad \text{for } j < k \\ &\varkappa_{j,k}(t) = | q_j q_{j+1} \ldots q_n \ldots q_{n+k-1}(1) |, \qquad \text{for } j > k \end{aligned}$$

while putting $q_{n+j} = q_j$.

3. LEMMAS AND RELATIONS BETWEEN OPERATORS

Lemma 3.1. *Let the operators* $\mathscr{C}_j$ *and the functions* Φ_j *be defined by the formulae* (2,8), (2,10), *and* (2,11). *Then it holds*

(3,1) $\quad \mathscr{C}_j^r(\Phi_j) = Q_j \mathscr{C}_{j+1}^r(\Phi_{j+1}), \qquad$ for $j = 1, 2, \ldots, n-1, r = 0, 1, \ldots$

(3,2) $\quad \mathscr{C}_j^r(\Phi_j) = Q_j \mathscr{C}_{j+1}^{r-1}(\Phi_{j+1}), \qquad$ for $j = n, r = 1, 2, \ldots$

while putting $\mathscr{C}_{n+1} = \mathscr{C}_1$, $\Phi_{n+1} = \Phi_1$.

The proofs for existence of these relations will be given by using the method of complete induction. Both being analogous, only the proof of (3,1) will be given.

For $r = 1$ with regard to (2,7) and (2,10),

$$\mathscr{C}_j^1(\Phi_j) = Q_j Q_{j+1} \ldots Q_n Q_{n+1} \ldots Q_{n+j-1}(\Phi_j) =$$
$$= Q_j Q_{j+1} \ldots Q_n Q_{n+1} \ldots Q_{n+j-1} Q_j(\Phi_{j+1}) = Q_j \mathscr{C}_{j+1}(\Phi_{j+1}),$$

which is the relation (3,1) for $r = 1$.

And now suppose that (3,1) holds; then according to (2,8), (2,7) and (3,1),

$$\mathscr{C}_j^{r+1}(\Phi_j) = \mathscr{C}_j \mathscr{C}_j^r(\Phi_j) = Q_j Q_{j+1} \ldots Q_n Q_{n+1} \ldots Q_{n+j-1} Q_{n+j} \mathscr{C}_{j+1}^r(\Phi_{j+1}) =$$
$$= Q_j \mathscr{C}_{j+1} \mathscr{C}_{j+1}^r(\Phi_{j+1}) = Q_j \mathscr{C}_{j+1}^{r+1}(\Phi_{j+1}).$$

The proof is completed.

Lemma 3.2. It holds

$$L_i(\Phi_1) = \Phi_{i+1}, \qquad \text{for } i = 1, 2, \ldots, n-1,$$
$$L_n(\Phi_1) = F,$$
$$L_n \mathscr{C}_1(h) = A_n h.$$

These statements follow immediately from definitions 2.1, 2.3, and 2.5.

4. ASYMPTOTIC PROPERTIES OF SOLUTION OF DIFFERENTIAL EQUATION (1,1)

In this Section we shall deal with formal construction of the particular solution $y_1(t)$ of differential equation (1,1) and its derivation having the form $L_s y_1$. Using the notation as indicated above, the equation under consideration may be written as

(4,1) $$L_n y = A_n y + F.$$

Theorem 4.1. *Suppose that* 2° *holds; then the particular solution* $y_1(t)$ *of differential equation* (4,1) *can be expressed in a formal way as*

(4,2) $$y_1(t) = \sum_{r=0}^{\infty} \mathscr{C}_1^r(\Phi_1).$$

The proof will be given by verification. According to (2,7), (2,8), and Lemma 3.2 we obtain

$$L_n y_1 = L_n \sum_{r=0}^{\infty} \mathscr{C}_1^r(\Phi_1) = L_n \mathscr{C}_1^0(\Phi_1) + L_n \sum_{r=1}^{\infty} \mathscr{C}_1^r(\Phi_1) =$$

$$= L_n(\Phi_1) + \sum_{r=1}^{\infty} L_n \mathscr{C}_1 \mathscr{C}_1^{r-1}(\Phi_1) = F + \sum_{r=1}^{\infty} A_n \mathscr{C}_1^{r-1}(\Phi_1) =$$

$$= F + A_n \sum_{r=1}^{\infty} \mathscr{C}_1^{r-1}(\Phi_1) = F + A_n \sum_{r=0}^{\infty} \mathscr{C}_1^r(\Phi_1) = F + A_n y_1 .$$

Hence $L_n y_1 = F + A_n y_1$; thus the proof is complete.

Theorem 4.2. *Suppose that 2° holds; then*

$$L_s y_1(t) = \sum_{r=0}^{\infty} \mathscr{C}_{s+1}^r(\Phi_{s+1}), \qquad \text{for } s = 0, 1, \ldots, n-1. \tag{4,3}$$

The proof will be given by the method of complete induction. According to (2,1), (3,1), and (4,2) and for $s = 1$,

$$L_1 y_1 = A_1^{-1} D y_1 = A_1^{-1} D \sum_{r=0}^{\infty} \mathscr{C}_1^r(\Phi_1) = A_1^{-1} D \sum_{r=0}^{\infty} Q_1 \mathscr{C}_2^r(\Phi_2) = \sum_{r=0}^{\infty} \mathscr{C}_2^r(\Phi_2),$$

and thus (4,3) holds for $s = 1$.

Now suppose that (4,3) holds for $s = j - 2$. Then

$$L_{j-1} y_1 = A_{j-1}^{-1} D L_{j-2} y_1 = A_{j-1}^{-1} D \sum_{r=0}^{\infty} \mathscr{C}_{j-1}^r(\Phi_{j-1}).$$

If $j \neq n$, we obtain according to (3,1)

$$L_{j-1} y_1 = A_{j-1}^{-1} D \sum_{r=0}^{\infty} Q_{j-1} \mathscr{C}_j^r(\Phi_j) = \sum_{r=0}^{\infty} \mathscr{C}_j^r(\Phi_j).$$

If $j = n$, we obtain according to (3,2)

$$L_{n-1} y_1 = A_{n-1}^{-1} D \sum_{r=0}^{\infty} \mathscr{C}_{n-1}^r(\Phi_{n-1}) = A_{n-1}^{-1} D \sum_{r=0}^{\infty} Q_{n-1} \mathscr{C}_n^r(\Phi_n) = \sum_{r=0}^{\infty} \mathscr{C}_n^r(\Phi_n).$$

The proof is completed since (4,3) holds also for $s = j - 1$.

Note 4.3. From the previous relation we can easily verify that $y_1(t)$ is a formal solution to (4,1). Indeed, according to (3,2),

$$D L_{n-1} y_1 = L_n y_1 = D \sum_{r=0}^{\infty} \mathscr{C}_n^r(\Phi_n) = D[\Phi_n + \sum_{r=1}^{\infty} \mathscr{C}_n^r(\Phi_n)] =$$

$$= D[\Phi_n + \sum_{r=1}^{\infty} Q_n \mathscr{C}_{n+1}^{r-1}(\Phi_{n+1})] = D[\Phi_n + Q_n \sum_{r=0}^{\infty} \mathscr{C}_{n+1}^{r}(\Phi_{n+1})] =$$

$$= D[\Phi_n + Q_n \sum_{r=0}^{\infty} \mathscr{C}_1^r(\Phi_1)] = F + A_n y_1 .$$

Note 4.4. In the subsequent Section we shall study uniform convergence of the series (4,3). We shall prove that $y_1(t)$ is a solution of (4,1) on an interval $I_1 \in I$ if the series (4,3) are uniformly convergent on I_1 for $s = 0, 1, \ldots, n-1$.

5. UNIFORM CONVERGENCES OF FUNCTION SERIES

Theorem 5.1. *Suppose that* 2° *holds and* $a_i = \infty$ *for all* $i = 0, 1, \ldots, n$. *Assume*

(5,1) $$\gamma_{s+1}(t) < \infty, \qquad \varphi_{s+1}(t) < \infty, \qquad \text{for } s = 0, 1, \ldots, n-1.$$

If so, the series (4,3) *converge uniformly on the interval I.*

Proof. The uniform convergence of series (4,3) is proved by constructing their convergent majorants. Applying complete induction we can prove that

(5,2) $$|\mathscr{C}_{s+1}^r(\Phi_{s+1})| \leqq \varphi_{s+1}(t) \frac{\gamma_{s+1}^r(t)}{r!}, \qquad \text{for } s = 0, 1, \ldots, n-1.$$

Indeed, for $r = 1$,

$$\begin{aligned} |\mathscr{C}_{s+1}^1(\Phi_{s+1})| &= |Q_{s+1}Q_{s+2} \cdots Q_{n+s}Q_{n+s+1} \cdots Q_{n+n-1}Q(1)| \leqq \\ &\leqq |q_{s+1}q_{s+2} \cdots q_{n+s}q_{n+s+1} \cdots q_{n+n-1}q(1)| \leqq \\ &\leqq \gamma_{s+1}(t)\, \varphi_{s+1}(t), \end{aligned}$$

which is the inequality of (5,2) for $r = 1$.

Suppose that (5,2) holds. Then

$$|\mathscr{C}_{s+1}^{r+1}(\Phi_{s+1})| = |\mathscr{C}_{s+1}\mathscr{C}_{s+1}^r(\Phi_{s+1})| =$$

$$= |Q_{s+1}Q_{s+2} \cdots Q_{n+s}[\mathscr{C}_{s+1}^r(\Phi_{s+1})]| \leqq |q_{s+1}q_{s+2} \cdots q_{n+s}(|\mathscr{C}_{s+1}^r(\Phi_{s+1})|)| \leqq$$

$$\leqq \varphi_{s+1}(t) \left| \int_{\infty}^{t} \frac{\gamma_{s+1}^r(\tau)}{r!} |A_{s+1}(\tau)|\, q_{s+2} \cdots q_{n+s}(1) \right| =$$

$$= \varphi_{s+1}(t) \left| \int_{\infty}^{t} \frac{\gamma_{s+1}^r(\tau)}{r!} \gamma_{s+1}'(\tau)\, d\tau \right| = \varphi_{s+1}(t) \frac{\gamma_{s+1}^{r+1}(t)}{(r+1)!} .$$

Hence the inequality of (5,2) is proved.

For every $t \in I$

$$\sum_{r=0}^{\infty} |\mathscr{C}_{s+1}^r(\Phi_{s+1})| \leqq \varphi_{s+1}(t) \sum_{r=0}^{\infty} \frac{\gamma_{s+1}^r(t)}{r!} \leqq \varphi_{s+1}(t_0) \sum_{r=0}^{\infty} \frac{\gamma_{s+1}^r(t_0)}{r!} .$$

The series

$$\varphi_{s+1}(t_0)\sum_{r=0}^{\infty}\frac{\gamma_{s+1}^r(t_0)}{r!}$$

is the convergent majorant to series (4,3). The statement of Theorem 5.1 is proved.

Theorem 5.2. *Suppose that* 2° *holds. Be* $a_i = \infty$ *for all* $i = 0, 1, \dots, n$. *If*

(5,3) $$\int_{t_0}^{\infty} | A_i(s)\,ds | < \infty, \qquad \int_{t_0}^{\infty} | F(s)\,ds | < \infty, \qquad \text{for } i = 1, 2, \dots, n,$$

then all series of (4,3) *converge uniformly on the interval* I.

The proof of this Theorem being analogous to that of the foregoing one, may be omitted.

Theorem 5.3. *If the conditions given for Theorem 5.1 are fulfilled, then for* $t \in I$ *the following estimate holds*:

(5,4) $$| L_s y_1 - \sum_{r=0}^{n} \mathscr{C}_{s+1}^r(\Phi_{s+1}) | \leqq \varphi_{s+1}(t)\frac{\gamma_{s+1}^{n+1}(t)}{(n+1)!}\exp\{\gamma_{s+1}(t)\}.$$

Proof. Let us denote

$$R_{n+1}(t) = \sum_{r=n+1}^{\infty} \mathscr{C}_{s+1}^r(\Phi_{s+1}).$$

Then according to (5,2)

$$| R_{n+1}(t) | \leqq \varphi_{s+1}(t)\sum_{r=n+1}^{\infty}\frac{\gamma_{s+1}^r(t)}{r!} = \varphi_{s+1}(t)\left[\frac{\gamma_{s+1}^{n+1}(t)}{(n+1)!} + \frac{\gamma_{s+1}^{n+2}(t)}{(n+2)!} + \dots\right] =$$

$$= \varphi_{s+1}(t)\frac{\gamma_{s+1}^{n+1}(t)}{(n+1)!}\left[1 + \frac{\gamma_{s+1}(t)}{n+2} + \frac{\gamma_{s+1}^2(t)}{(n+2)(n+3)} + \dots\right] <$$

$$< \varphi_{s+1}(t)\frac{\gamma_{s+1}^{n+1}(t)}{(n+1)!}\exp\{\gamma_{s+1}(t)\},$$

which was to be proved.

Theorem 5.4. *Suppose that* 2° *holds. Let* $a_i = a$ *for all* $i = 0, 1, \dots, n$. *Then the series of* (4,3) *converge uniformly on the interval* $I_1 = \langle t_0, b\rangle$, *where* $b > t_0$ *is an arbitrary number.*

The proof of this Theorem is analogous to that of 5.1. Since

(5,5) $$| \mathscr{C}_{s+1}^r(\Phi_{s+1}) | \leqq \varphi_{s+1}(b)\frac{\gamma_{s+1}^r(b)}{r!},$$

the series $\sum_{r=0}^{\infty}\varphi_{s+1}(b)\frac{\gamma_{s+1}^r(b)}{r!}$ is a convergent majorant to the series (4,3), so that (4,3) converges uniformly on the interval I_1.

Theorem 5.5. *If the assumptions of Theorem 5.4 are fulfilled, then for $t \in I_1$ it holds*

$$(5,6) \qquad | L_s y_1 - \sum_{r=0}^{n} \mathscr{C}^r_{s+1}(\Phi_{s+1}) | \leqq \varphi_{s+1}(t) \frac{\gamma^{n+1}_{s+1}(t)}{(n+1)!} \exp \{\gamma_{s+1}(t)\}.$$

The proof of this Theorem, being analogous to that of 5.3, may be omitted.

Lemma 5.6. *Provided, $\gamma_j(t) < \infty$, $\varphi_j(t) < \infty$, then it holds*

$$(5,7) \qquad | \mathscr{C}^r_j(\Phi_j) | \leqq \sup_{s \in J} \varphi_j(s) \, . \, (\sup_{s \in J} \gamma_j(s))^r.$$

The proof will be given by using complete induction. For $r = 1$ it is

$$| \mathscr{C}^1_j(\Phi_j) | = | Q_j Q_{j+1} \dots Q_{n+j-1}(\Phi_j) | \leqq | q_j q_{j+1} \dots q_{n+j-1}(| \Phi_j |) | \leqq$$
$$\leqq \sup_{s \in J} \varphi_j(s) \, | q_j q_{j+1} \dots q_{n+j-1}(1) | \leqq \sup_{s \in J} \varphi_j(s) \sup_{s \in J} \gamma_j(s),$$

which is the inequality of (5,7) for $r = 1$.

Now suppose that (5,7) holds. Then according to (2,8)

$$| \mathscr{C}^{r+1}_j(\Phi_j) | = | \mathscr{C}_j \mathscr{C}^r_j(\Phi_j) | \leqq | q_j q_{j+1} \dots q_{n+j-1}(| \mathscr{C}^r_j(\Phi_j) |) | \leqq$$
$$\leqq \sup_{s \in J} \varphi_j(s) (\sup_{s \in J} \gamma_j(s))^r \, | q_j q_{j+1} \dots q_{n+j-1}(1) | \leqq$$
$$\leqq \sup_{s \in J} \varphi_j(s) (\sup_{s \in J} \gamma_j(s))^r \sup_{s \in J} \gamma_j(s) =$$
$$= \sup_{s \in J} \varphi_j(s) (\sup_{s \in J} \gamma_j(s))^{r+1}.$$

The proof is thus completed.

Theorem 5.7. *Suppose that 2° holds. Let us assume that for some s, $0 \leqq s \leqq n - 1$, $a_{s+1} = \infty$, and that there exists at least one i, $0 \leqq i \leqq n$ so that $a_i = a$. Suppose that it holds*

$$(5,8) \qquad \gamma_{s+1}(t) < \infty, \qquad \varphi_{s+1}(t) < \infty.$$

If $a \geqq t_0$ is such that $\gamma_{s+1}(a) < 1$, then the series (4,3) *converges uniformly on the interval $\langle a, \infty)$.*

Proof. If the assumptions of (5,8) are fulfilled, the functions $\gamma_{s+1}(t)$ and $\varphi_{s+1}(t)$ are finite, continuous, and decreasing. Hence there exists a number $a \geqq t_0$ such that

$$\gamma_{s+1}(t) \leqq \gamma_{s+1}(a) < 1$$

for $t \in \langle a, \infty)$. According to Lemma 5.6., for $t \geqq a$ it is

$$(5,9) \qquad | \mathscr{C}^r_{s+1}(\Phi_{s+1}) | \leqq \varphi_{s+1}(a) \, \gamma^r_{s+1}(a).$$

Since $\gamma_{s+1}(a) < 1$, the geometric series $\sum_{r=0}^{\infty} \varphi_{s+1}(a) \, . \, \gamma^r_{s+1}(a)$ is a convergent majorant to the series (4,3); therefore (4,3) converges uniformly on the interval $\langle a, \infty)$.

The proof is completed.

Theorem 5.8. *If the assumptions of Theorem 5.7 are fulfilled, then for $t \in \langle a, \infty)$ the following estimate holds:*

$$(5,10) \qquad \left| L_s y_1 - \sum_{r=0}^{n} \mathscr{C}_{s+1}^r(\Phi_{s+1}) \right| \leqq \gamma_{s+1}^{n+1}(a) \frac{\varphi_{s+1}(a)}{1 - \gamma_{s+1}(a)}.$$

Proof. Denote

$$R_{n+1}(t) = \sum_{r=n+1}^{\infty} \mathscr{C}_{s+1}^r(\Phi_{s+1}).$$

According to (5,9)

$$| R_{n+1}(t) | \leqq \varphi_{s+1}(a) \sum_{r=n+1}^{\infty} \gamma_{s+1}^r(a) =$$
$$= \varphi_{s+1}(a) [\gamma_{s+1}^{n+1}(a) + \gamma_{s+1}^{n+2}(a) + \gamma_{s+1}^{n+3}(a) + \gamma_{s+1}^{n+4}(a) + \dots] =$$
$$= \varphi_{s+1}(a) \gamma_{s+1}^{n+1}(a) [1 + \gamma_{s+1}(a) + \gamma_{s+1}^2(a) + \dots] =$$
$$= \gamma_{s+1}^{n+1}(a) \frac{\varphi_{s+1}(a)}{1 - \gamma_{s+1}(a)}.$$

The proof is completed.

Theorem 5.9. *Suppose that 2° holds. Let $a_{s+1} = a$ and there exists i, $0 \leqq i \leqq n$ so that $a_i = \infty$. Let a_l be the first number in the cycle of $a_{s+1}, a_{s+2}, \dots, a_{n+s}$ such that $a_l = \infty$. Suppose that*

$$(5,11) \qquad \gamma_l(t) < \infty, \qquad \varphi_l(t) < \infty,$$

hold and there exists a number $a \geqq t_0$ such that $\gamma_l(a) < 1$. Then the series (4,3) converges uniformly on the interval $\langle a, b \rangle$, where b is an arbitrary number such that $b > a$.

Proof. If the conditions of (5,11) are satisfied, the functions $\gamma_l(t)$ and $\varphi_l(t)$ are continuous, finite, and decreasing. Accordingly, there exists a number $a \geqq t_0$ such that $\gamma_l(t) \leqq \gamma_l(a) < 1$ for all $t \geqq a$. Taking into account the statement of Lemma 5.6, we can easily prove that

1. if $s + 1 \leqq l$, then for $t \in \langle a, b \rangle$ it holds

$$(5,12) \qquad | \mathscr{C}_{s+1}^r(\Phi_{s+1}) | \leqq \varkappa_{s+1,l}(b) \gamma_l^r(a) \varphi_l(a), \qquad r = 0, 1, \dots$$

2. if $s + 1 > l$, then for $t \in \langle a, b \rangle$ it holds'

$$(5,13) \qquad | \mathscr{C}_{s+1}^r(\Phi_{s+1}) | \leqq \varkappa_{s+1,l}(b) \gamma_l^{r-1}(a) \varphi_l(a), \qquad r = 1, 2, \dots$$

On the assumption that $\gamma_l(a) < 1$, there exists in either above case a convergent geometric series to the series (4,3), which is its majorant; thus the series (4,3) converges uniformly on the interval $\langle a, b\rangle$.

Theorem 5.10. *Suppose the conditions of* (5,9) *are satisfied. Then for* $t \in \langle a, b\rangle$ *it holds*

$$(5,14) \qquad \left| L_s y_1 - \sum_{r=0}^{n} \mathscr{C}^r_{s+1}(\Phi_{s+1}) \right| \leqq \varkappa_{s+1,l}(b)\, \gamma_l^{s+1}(a) \frac{\varphi_l(a)}{1 - \gamma_l(a)}$$

for $s + 1 \leqq l$,

$$(5,15) \qquad \left| L_s y_1 - \Phi_{s+1} - \sum_{r=1}^{n} \mathscr{C}^r_{s+1}(\Phi_{s+1}) \right| \leqq \varkappa_{s+1,l}(b)\, \gamma_l^{n}(a) \frac{\varphi_l(a)}{1 - \gamma_l(a)},$$

for $s + 1 > l$.

Since the proof of this Theorem is analogous to that of 5.8, it may be omitted.

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