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REDUCIBILITY THEOREMS FOR DIFFERENTIABLE LIFTINGS IN FIBER BUNDLES

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1. INTRODUCTION

Let $\mathscr{D}_n$ be the category whose objects are n-dimensional, Hausdorff differential manifolds satisfying the second axiom of countability, and whose morphisms are injective immersions. Let $\mathscr{PB}_n$ be the category formed by all principal fiber bundles over the manifolds from $\mathscr{D}_n$, and by the homomorphisms of principal fiber bundles Recall that a homomorphism of a principal G_1-bundle (Y_1, π_1, X_1) into a principal. G_2-bundle (Y_2, π_2, X_2) is by definition a triple (σ, σ_0, ν), where $\nu : G_1 \to G_2$ is a homomorphism of Lie groups and $\sigma : Y_1 \to Y_2$, $\sigma_0 : X_1 \to X_2$ are maps such that $\pi_2\sigma = \sigma_0\pi_1$ and $\sigma(y \,.\, g) = \sigma(y) \,.\, \nu(g)$ for all $y \in Y_1$ and $g \in G_1$. If G is a Lie group then $\mathscr{PB}_n(G)$ will denote the subcategory of $\mathscr{PB}_n$ formed by principal G-bundles and their G-homomorphisms.

This paper is devoted to the theory of liftings in fiber bundles. Our approach is in accordance with *Nijenhuis*' "natural bundles" [11] with only minor modifications consisting in the use of principal fiber bundles. We work with the following

Definition 1. A covariant functor $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ is called a *lifting to the group* G, if it has the following properties:

1. For every $X \in \operatorname{Ob} \mathscr{D}_n$, τX has X for its base space, i.e. $\tau X = (\tau_0 X, \tau_X, X)$, and for every $\alpha \in \operatorname{Mor} \mathscr{D}_n$, $\tau\alpha$ has α for its projection, i.e., $\tau\alpha = (\tau_0\alpha, \alpha, \mathrm{id}_G)$.

2. For every $X \in \operatorname{Ob} \mathscr{D}_n$ and every open submanifold U of X the relations

$$(1) \qquad \tau_0 U = \pi_X^{-1}(U), \qquad \pi_U = \pi_X|_{\tau_0 T}, \qquad \tau_0(\mathrm{id}_X|_U) = \mathrm{id}_{\tau_0 X}|_{\tau_0 U}$$

hold.

Similar definitions are used in the papers by *Salvioli* [13], *Krupka and Trautman* [10], *Krupka* [8], and *Chuu-Lian Terng* [3]. In these papers, the concept of lifting is applied to the theory of geometric objects and their Lie derivatives, the invariant variational problems in fiber bundles, the classification of natural vector bundles and invariant differential operators. In [5], the jet prolongations of the lifting functors associated with the frame lifting are discussed, and in [7] the differential invariants are interpreted as natural transformations of liftings. The ideas of the lifting theory either in a "classical" fashion or in a "modern" one have been used in various branches of applied mathematics – in the theory of invariant variational problems, the field theory, and the general relativity (see, e.g., [1], [5], [6], [8], [10], [12]).

The main results of this paper consist in proving the finite order theorem for differentiable liftings in principal fiber bundles and the reducibility of a differentiable lifting to its covariance group, or, which is the same, to a transitive lifting. Further, we shall show that every lifting in the category of fibre bundles, associated with a differentiable lifting, can be considered as associated with the r-frame lifting $\mathscr{F}^r$, where $r \geqq 0$ is an integer.

All manifolds and maps considered in this paper belong to the category $\mathscr{C}^\infty$.

2. ELEMENTARY PROPERTIES OF A LIFTING

Let us consider the categories $\mathscr{D}_n$ and $\mathscr{PB}_n(G)$. For $X \in \mathrm{Ob}\, \mathscr{D}_n$, let $\mathscr{D}_X$ denote the full subcategory of $\mathscr{D}_n$ whose objects are open submanifolds of X. Denote by R^n the real, n-dimensional Euclidean space. Let $\mathscr{PB}_{R^n \times G}$ be the full subcategory of $\mathscr{PB}_n(G)$ whose objects are restrictions of the trivial principal G-bundle $(R^n \times G, \pi, R^n)$ to open submanifolds of R^n.

Proposition 1. *Let $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ be a lifting. For each $\alpha \in \mathrm{Mor}\, \mathscr{D}_n$, $\alpha : X_1 \to X_2$, and each $U \in \mathrm{Ob}\, \mathscr{D}_{X_1}$,*

$$\tau_0(\alpha|_U) = \tau_0\alpha|_{\tau_0 U}. \tag{2}$$

Proof. (2) is a direct consequence of (1).

Let $\tau : \mathscr{D}_n \to \mathscr{QB}_n(G)$ be a lifting and consider the principal G-bundle $\tau R^n = (\tau_0 R^n, \pi_{R^n}, R^n)$. Choose a point $y \in \pi_{R^n}^{-1}(0)$. To each $y \in \pi_{R^n}^{-1}(0)$ it is related an element $v(y) \in G$ by the formula $y = y_0 \,.\, v(y)$. The arising map $v : \pi_{R^n}^{-1}(0) \to G$ is a diffeomorphism. From now on let t_x denote the translation $x' \to x' - x$ of R^n. It is easily verified that the formula

$$\varepsilon_0(y) = (x, v(\tau_0 t_x(y))), \tag{3}$$

where $x = \pi_{R^n}(y)$, defines an isomorphism $\varepsilon = (\varepsilon_0, \mathrm{id}_{R^n}, \mathrm{id}_G)$ of τR^n onto $(R^n \times G, \pi, R^n)$, i.e., a trivialization of the principal G-bundle τR^n. The inverse isomorphism is defined by $\varepsilon_0^{-1}(x, g) = \tau_0 t_{-x}(y_0 \,.\, g)$. This proves the following

Proposition 2. *The principal G-bundle τR^n is trivial.*

Let $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ be a lifting, $\varepsilon = (\varepsilon_0, \mathrm{id}_{R^n}, \mathrm{id}_G)$ a trivialization of the principal G-bundle τR^n. The correspondence $U \to (U \times G, \pi, U)$, $\alpha \to (\tau_0^\varepsilon \alpha, \alpha, \mathrm{id}_G)$, where

$$\tau_0^\varepsilon \alpha = \varepsilon_0 \circ \tau_0 \alpha \circ \varepsilon_0^{-1} \tag{4}$$

and the restrictions of the maps π, ε_0, and ε_0^{-1} are not denoted, is a covariant functor from the category $\mathscr{D}_{R^n}$ to $\mathscr{PB}_{R^n \times G}$. We call this functor the ε-functor associated with the trivialization ε and denote it by τ^ε. The equality

$$\tau_0^\varepsilon(\mathrm{id}_{R^n}|_U) = \mathrm{id}_{R^n \times G}|_{U \times G}$$

holds for each $U \in \mathrm{Ob}\, \mathscr{D}_{R^n}$.

Consider a principal G-bundle (Y, π, X). To each $x_0 \in X$ there exist a chart (U, φ) on X such that $x_0 \in U$, a diffeomorphism $\Phi : \pi^{-1}(U) \to \varphi(U) \times G$ such that for each $y \in \pi^{-1}(U)$ and $g \in G$, $\Phi(y) = (\varphi\pi(y), \tilde{\Phi}(y))$, $\tilde{\Phi}(y \cdot g) = \Phi(y) \cdot g$. The pair $((U, \varphi), \Phi)$ will be called the fiber chart on (Y, π, X). A system $((U_\iota, \varphi_\iota), \Phi_\iota)$, $\iota \in I$, of fiber charts on (Y, π, X) such that (U_ι, φ_ι), $i \in I$, is an atlas on X, defines in a well-known way the differential structure of the manifold Y. Such a system is called a fiber atlas on (Y, π, X).

Proposition 3. *Let $X \in \mathrm{Ob}\, \mathscr{D}_n$ and let (U_ι, φ_ι), $\iota \in I$, be an atlas on X. Then the system $((U_\iota, \varphi_\iota), \varepsilon_0 \circ \tau_0 \varphi_\iota)$, $\iota \in I$, is a fiber atlas on the principal G-bundle τX.*

Proof. Obviously $\varepsilon_0 \circ \tau_0 \varphi_\iota \circ (\varepsilon \circ \tau_0 \varphi_\varkappa)^{-1} = \tau_0^\varepsilon(\varphi_\iota \varphi_\varkappa^{-1})$, where τ^ε is the ε-functor associated with ε, holds for all $\iota, \varkappa \in I$ such that the expressions on both sides are defined. Our assertion immediately follows from this relation.

The following proposition establishes a method of constructing the liftings by extending the functors from the category $\mathscr{D}_{R^n}$ into the category $\mathscr{PB}_{R^n \times G}$. Its proof is elementary, and we give it in a shortened form because of the formulas needed later.

Proposition 4. *Let $\tilde{\tau} : \mathscr{D}_{R^n} \to \mathscr{PB}_{R^n \times G}$ be a covariant functor assigning to $U \in \mathrm{Ob}\, \mathscr{D}_{R^n}$ the principal G-bundle $\tilde{\tau} U = (\tilde{\tau}_0 U, \tilde{\pi}_U, U)$, where $\tilde{\tau}_0 U = U \times G$ and $\tilde{\pi}_U : U \times G \to U$ is the natural projection on the first factor, and to $\alpha \in \mathrm{Mor}\, \mathscr{D}_{R^n}$ a morphism $\tilde{\tau}\alpha = (\tilde{\tau}_0 \alpha, \alpha, \mathrm{id}_G)$. Assume that for every $U \in \mathrm{Ob}\, \mathscr{D}_{R^n}$*

$$\tilde{\tau}_0(\mathrm{id}_{R^n}|_U) = \mathrm{id}_{R^n \times G}|_{U \times G}. \tag{5}$$

Then there exist a lifting $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ and a trivialization ε of τR^n such that $\tau^\varepsilon = \tilde{\tau}$. For two such liftings τ, ϱ and trivializations ε, v satisfying $\tau^\varepsilon = \varrho^v = \tilde{\tau}$ there exists a natural transformation $X \to N_X$ of the functor τ to ϱ (in the category $\mathscr{PB}_n$) such that for every $X \in \mathrm{Ob}\, \mathscr{D}_n$, N_X is an isomorphism of principal G-bundles.

Proof. With the aid of a general construction [2, p. 62] we define to each $X \in$

$\in \operatorname{Ob} \mathscr{D}_n$ a principal G-bundle $\tau X = (\tau_0 X, \pi_X, X)$. Let $X \in \operatorname{Ob} \mathscr{D}_n$, let (U_ι, φ_ι), $\iota \in I$, be a countable atlas on X. In the set of all triples (x, g, ι), where $x \in U_\iota$, $g \in G$, there is defined an equivalence relation such that the triples (x_1, g_1, ι), $(x_2, g_2, \varkappa)$ are equivalent if one only if $x_1 = x_2$, $(\varphi_\iota(x_1), g_1) = \tilde{\tau}_0(\varphi_\iota \varphi_\varkappa^{-1})(\varphi_\varkappa(x_2), g_2)$. Let $\tau_0 X$ denote the corresponding quotient and $[x, g, \iota]$ the equivalence class of a triple (x, g, ι). For $y \in \tau_0 X$, $y = [x, g, \iota]$ put $\pi_X(y) = x$. We obtain a surjection $\pi_X : \tau_0 X \to X$. The formula

$$\Phi_\iota(y) = (\varphi_\iota(x), g) \tag{6}$$

defines a bijection $\Phi_\iota : \pi_X^{-1}(U_\iota) \to \varphi_\iota(U_\iota) \times G$. There exists one and only one differential structure on $\tau_0 X$ such that all the maps Φ_ι, $\iota \in I$, are diffeomorphisms. Further, put for $y \in \tau_0 X$, $y = [x, g, \iota]$, and $g' \in G$, $y \,.\, g' = [x, g \,.\, g', \iota]$. This formula gives rise to a right action $\tau_0 X \times G \ni (y, g) \to y \,.\, g \in \tau_0 X$ of G on $\tau_0 X$. It is readily checked that the triple $\tau X = (\tau_0 X, \pi_X, X)$ becomes a principal G-bundle.

Let $\alpha \in \operatorname{Mor} \mathscr{D}_n$, $\alpha : X_1 \to X_2$. There exists one and only one map $\tau_0 \alpha : \tau_0 X_1 \to \tau_0 X_2$ satisfying the following condition: For every atlas (U_ι, φ_ι), $\iota \in I$, on X_1 and every atlas $(V_\varkappa, \psi_\varkappa)$, $\varkappa \in K$, on X_2,

$$\tau_0 \alpha \,|_{\pi_{X_1}^{-1}(U_\iota \cap \alpha^{-1}(U_\varkappa))} = \Psi_\varkappa^{-1} \circ \tilde{\tau}_0(\psi_\varkappa \alpha \varphi_\iota^{-1}) \circ \Phi_\iota, \tag{7}$$

where $\iota \in I$, $\varkappa \in K$, and Φ_ι, $\Psi_\varkappa$ are defined by (6). It follows that $\tau\alpha = (\tau_0 \alpha, \alpha, \mathrm{id}_G)$ is an injective homomorphism of τX_1 into τX_2, i.e., $\tau\alpha \in \operatorname{Mor} \mathscr{PB}_n(G)$.

The correspondence $X \to \tau X$, $\alpha \to \tau\alpha$ is a lifting from $\mathscr{D}_n$ to $\mathscr{PB}_n(G)$. Using the canonical trivialization $\varepsilon = (\varepsilon_0, \mathrm{id}_{R^n}, \mathrm{id}_G)$ of τR^n one easily obtains from (7) that for every $\alpha \in \operatorname{Mor} \mathscr{D}_{R^n}$, $\tau_0^\varepsilon \alpha = \varepsilon_0 \circ \tau_0 \alpha \circ \varepsilon_0^{-1} = \tilde{\tau}_0 \alpha$.

Let (U_ι, φ_ι), $\iota \in I$, be an atlas on a manifold $X \in \operatorname{Ob} \mathscr{D}_n$. According to Proposition 3, $((U_\iota, \varphi_\iota), \varepsilon_0 \circ \tau_0 \varphi_\iota)$, $\iota \in I$, is a fiber atlas on τX, and $((U_\iota, \varphi_\iota), \nu_0 \circ \varrho_0 \varphi_\iota)$, $\iota \in I$, is a fiber atlas on ϱX. For each $\iota \in I$ there is defined a map $(\nu_0 \circ \varrho_0 \varphi_\iota)^{-1} \circ \varepsilon_0 \circ \tau_0 \varphi_\iota$ from $\tau_0 U_\iota$ to $\varrho_0 U_\iota$. Assume that $\tau^\varepsilon = \varrho^\nu = \tilde{\tau}$. Then for every $\iota, \varkappa \in I$ such that the considered expressions make sense,

$$\begin{aligned} \nu_0 \circ \varrho_0 \varphi_\iota \circ (\nu_0 \circ \varrho_0 \varphi_\varkappa)^{-1} &= \nu_0 \circ \varrho_0(\varphi_\iota \varphi_\varkappa^{-1}) \circ \nu_0^{-1} = \varrho_0^\nu(\varphi_\iota \varphi_\varkappa^{-1}) = \\ &= \tilde{\tau}_0(\varphi_\iota \varphi_\varkappa^{-1}) = \varepsilon_0 \circ \tau_0 \varphi_\iota \circ (\varepsilon_0 \circ \tau_0 \varphi_\varkappa)^{-1}. \end{aligned}$$

This shows that there exists an isomorphism $N_X = (N_X^{(0)}, \mathrm{id}_X, \mathrm{id}_G)$ of τX onto ϱX such that for every $\iota \in I$,

$$N_X^{(0)} = (\nu_0 \circ \varrho_0 \varphi_\iota)^{-1} \circ \varepsilon_0 \circ \tau_0 \varphi_\iota. \tag{8}$$

To show that the correspondence $X \to N_X$, $X \in \operatorname{Ob} \mathscr{D}_n$, is a natural transformation of functors we should verify that for each $\alpha \in \operatorname{Mor} \mathscr{D}_n$, $\alpha : X_1 \to X_2$, $N_{X_2}^{(0)} \circ \tau_0 \alpha = \varrho_0 \alpha \circ N_{X_1}^{(0)}$. This follows, however, from (7), Proposition 3, and (8). This completes the proof.

Let G be a Lie group, e its identity, and consider the trivial principal G-bundle $(R^n \times G, \pi, R^n)$. Let $(\tilde{\sigma}, \sigma, \mathrm{id}_G)$ be a local automorphism of $(R^n \times G, \pi, R^n)$, $\sigma : U \to R^n$. There is one and only one map $\bar{\sigma} : \sigma(U) \to G$ such that for each $(x, g) \in U \times G$, $\tilde{\sigma}(x, g) = (\sigma(x), \bar{\sigma}\sigma(x) . g)$. If $(\tilde{\sigma}_1, \sigma_1, \mathrm{id}_G)$, $(\tilde{\sigma}_2, \sigma_2, \mathrm{id}_G)$ are two local automorphisms of $(R^n \times G, \pi, R^n)$ such that the composed map $\sigma_1\sigma_2$ is defined then for every x from the domain of definition of σ_2

$$\tilde{\sigma}_1\tilde{\sigma}_2(x, g) = (\sigma_1\sigma_2(x), \bar{\sigma}\sigma_1\sigma_2(x) . \bar{\sigma}_2\sigma_2(x) . g). \tag{9}$$

Consider a lifting $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ and a trivialization ε of τR^n. Let $\alpha \in \mathrm{Mor}\,\mathscr{D}_n$, $\alpha : U \to R^n$. Then $\tau_0^\varepsilon\alpha$ (4) is of the form

$$\tau_0^\varepsilon\alpha(x, g) = (\alpha(x), \bar{\tau}^\varepsilon\alpha(\alpha(x)) . g), \tag{10}$$

where $(x, g) \in U \times G$ and $\bar{\tau}^\varepsilon\alpha$ maps $\alpha(U)$ to G. For $\alpha_1, \alpha_2 \in \mathrm{Mor}\,\mathscr{D}_{R^n}$ such that $\alpha_1\alpha_2$ si defined, (9) gives the identity

$$\bar{\tau}^\varepsilon(\alpha_1\alpha_2)(\alpha_1\alpha_2(x)) = \bar{\tau}^\varepsilon\alpha_1(\alpha_1\alpha_2(x)) . \bar{\tau}^\varepsilon\alpha_2(\alpha_2(x)). \tag{11}$$

Obviously, $\bar{\tau}^\varepsilon\, \mathrm{id}_{R^n}(x) = e$.

As before let t_x denote the translation of R^n sending the point $x \in R^n$ to the origin $0 \in R^n$.

Proposition 5. *Let $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ be a lifting, let ε be a trivilisation of τR^n. For every $x, x' \in R^n$,*

$$\bar{\tau}^\varepsilon t_x(x') = e. \tag{12}$$

Proof. Let $\alpha \in \mathrm{Mor}\,\mathscr{D}_{R^n}$, $y_0 \in \pi_{R^n}^{-1}(0)$, and consider the ε-functor τ^ε defined by the trivialisation (3) of τR^n. Since τ is a covariant functor we obtain for every $g \in G$ and x_0 from the domain of α, $\tau_0^\varepsilon\alpha(x_0, g) = (\alpha(x_0), \nu(\tau_0(t_{\alpha(x_0)}\alpha t_{-x_0})(y_0 . g)))$. Putting $\alpha = t_x$ we get

$$\tau_0^\varepsilon t_x(x_0, g) = (t_x(x_0), \nu(\tau_0(t_{x_0-x}t_x t_{-x_0})(y_0 . g))) = (x_0 - x, \nu(y_0 . g)). \tag{13}$$

But $\nu(y_0 . g)$ satisfies the relation $y_0 . g = y_0 . \nu(y_0 . g)$ which gives $\nu(y_0 . g) = g$. On comparing (10) and (13) we obtain $\bar{\tau}^\varepsilon t_x(x_0 - x) = e$. This shows that (12) holds for the trivialisation (3). Let now $\nu = (\nu_0, \mathrm{id}_{R^n}, \mathrm{id}_G)$ be any trivialisation of τR^n. Since for every $\alpha \in \mathrm{Mor}\,\mathscr{D}_n$

$$\tau_0^\nu\alpha = \nu_0 \circ \tau_0\alpha \circ \nu_0^{-1} = \nu_0\varepsilon_0^{-1} \circ \tau_0^\varepsilon\alpha \circ \varepsilon_0\nu_0^{-1}, \tag{14}$$

the formula (9) immediately leads to the relation $\bar{\tau}^\nu t_x(x') = e$ proving Proposition 5.

Let $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ be a lifting, ε a trivialisation of τR^n, $x_0 \in R^n$ a point. Denote by $\mathscr{A}_{x_0}$ the set of all $\alpha \in \mathrm{Mor}\,\mathscr{D}_{R^n}$ defined at x_0 and leaving x_0 fixed, and by $G_{x_0}^\varepsilon$ the set of all $g \in G$ which can be expressed as $\bar{\tau}^\varepsilon\alpha(x_0)$ for some $\alpha \in \mathscr{A}_{x_0}$. It is easily seen that G_{x_0} is a subgroup of G.

Further let $x_1, x_2 \in R^n$. Every $\beta \in \operatorname{Mor} \mathscr{D}_{R^n}$ sending x_1 to x_2 defines an isomorphism of $G^\varepsilon_{x_1}$ and $G^\varepsilon_{x_2}$ as follows. Let $\alpha \in \mathscr{A}_{x_1}$. Then $\beta\alpha\beta^{-1} \in \mathscr{A}_{x_2}$ and, by (11), $\bar{\tau}^\varepsilon(\beta\alpha\beta^{-1})(x_2) = \bar{\tau}^\varepsilon\beta(x_2) \,.\, \bar{\tau}^\varepsilon\alpha(x_1) \,.\, \bar{\tau}^\varepsilon\beta^{-1}(x_1)$, $\bar{\tau}^\varepsilon\beta^{-1}(x_1) = (\bar{\tau}^\varepsilon\beta(x_2))^{-1}$, and we have $\bar{\tau}^\varepsilon(\beta\alpha\beta^{-1})(x_2) = \bar{\tau}^\varepsilon\beta(x_2) \,.\, \bar{\tau}^\varepsilon\alpha(x_1) \,.\, (\bar{\tau}^\varepsilon\beta(x_2))^{-1}$. The desired isomorphism is established as the map

$$G^\varepsilon_{x_1} \ni g \to \bar{\tau}^\varepsilon\beta(x_2) \,.\, g \,.\, (\bar{\tau}^\varepsilon\beta(x_2))^{-1} \in G^\varepsilon_{x_2}. \tag{15}$$

Proposition 6. *Let $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ be a lifting, ε a trivialisation of τR^n. The following assertions hold:*

1. *There exists a subgroup G^ε of G such that $G^\varepsilon_x = G^\varepsilon$ for every $x \in R^n$.*
2. *For every $\alpha \in \operatorname{Mor} \mathscr{D}_{R^n}$, $\alpha : U \to R^n$, the map $\bar{\tau}^\varepsilon\alpha : \alpha(U) \to G$ takes values in G^ε.*
3. *If ν is another trivialisation of τR^n then the groups G^ε and G^ν are similar.*

Proof. Let $x_1, x_2 \in R^n$. From (12) we conclude that for $\beta = t_{x_1 - x_2}$ the map (15) becomes the identity map which proves the first assertion. Let $\alpha \in \operatorname{Mor} \mathscr{D}_{R^n}$, $\alpha(x_1) = x_2$. Then $t_{x_2 - x_1} \circ \alpha \in \mathscr{A}_{x_1}$ so that $\bar{\tau}^\varepsilon(t_{x_2 - x_1} \circ \alpha)(x_1) \in G^\varepsilon_{x_1}$. Now (11) and (12) give

$$\bar{\tau}^\varepsilon(t_{x_2 - x_1} \circ \alpha)(x_1) = \bar{\tau}^\varepsilon t_{x_2 - x_1}(x_1) \,.\, \bar{\tau}^\varepsilon\alpha(x_2) = \bar{\tau}^\varepsilon\alpha(x_2)$$

which shows that $\bar{\tau}^\varepsilon\alpha(x_2) \in G^\varepsilon_{x_1}$. This proves the second assertion. The third statement follows from (14).

Accordingly, we define:

Definition 2. Let $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ be a lifting. Each group G^ε, where ε is a trivialisation of the principal G – bundle τR^n, is called the *covariance group* of the lifting τ.

The liftings which we now introduce are of primary importance.

Definition 3. We say that a lifting $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ is *transitive* if for any $X \in \operatorname{Ob} \mathscr{D}_n$ and $y_1, y_2 \in \tau_0 X$ there exists $\alpha \in \operatorname{Mor} \mathscr{D}_X$ such that $\tau_0\alpha(y_1) = y_2$.

Clearly a lifting $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ is transitive if and only if it is transitive on $\pi^{-1}_{R^n}(0)$, where π_{R^n} is the projection map of τR^n. This leads to the following consequence.

Theorem 1. *A necessary and sufficient condition for a lifting $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ to be transitive is that its covariance group is equal to G.*

Proof. Let ε be a trivialisation of τR^n. It follows from (10) that τ is transitive on $\pi^{-1}_{R^n}(0)$ if and only if to every $g_1, g_2 \in G$ one can find $\alpha \in \mathscr{A}_0$, $0 \in R^n$, such that $\bar{\tau}^\varepsilon\alpha(0) \,.\, g_1 = g_2$. This is, however, equivalent to the condition $G^\varepsilon_0 = G$.

Let G and G_0 be Lie groups, $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ and $\varrho : \mathscr{D}_n \to \mathscr{PB}_n(G_0)$ liftings. Assume that G_0 is a Lie subgroup of G.

Definition 4. We say that τ is *reducible* to ϱ, or that τ is *reducible to the subgroup* G_0 of G, if there is a natural transformation N of the functor ϱ to τ such that for every $X \in \mathrm{Ob}\, \mathscr{D}_n$, $N_X : \varrho X \to \tau X$ is a reduction of the principal G-bundle τX to the principal G_0-bundle ϱX. N is called a *reduction* of the lifting τ to ϱ or a *reduction of* τ *to the subgroup* G_0 of G.

Proposition 7. *A sufficient condition for a lifting* $\tau : \mathscr{D}_n \to \mathscr{P}\mathscr{B}_n(G)$ *to be reducible to a lifting* $\varrho : \mathscr{D}_n \to \mathscr{P}\mathscr{B}_n(G_0)$ *is that there exist a trivialization* ε *of* τR^n, *a trivialization* ν *of* ϱR^n *and a natural transformation* $\tilde{N}$ *of the functor* ϱ^ν *to* τ^ε *such that* $\tilde{N}_U : \varrho^\nu U \to \tau^\varepsilon U$, $U \in \mathrm{Ob}\, \mathscr{D}_{R^n}$ *is a reduction of the principal G-bundle* $\tau^\varepsilon U$ *to the principal* G_0*-bundle* $\varrho^\nu U$.

Proof. Assume that ε, ν, and $\tilde{N}$ satisfy the conditions of Proposition 7. We shall construct a natural transformation $N : \varrho \to \tau$ such that for each $X \in \mathrm{Ob}\, \mathscr{D}_n$, $N_X : \varrho X \to \tau X$ is a reduction of the principal G-bundle τX to the principal G_0-bundle ϱX.

Let $X \in \mathrm{Ob}\, \mathscr{D}_n$, let (U_ι, φ_ι), $\iota \in I$, be an atlas on X. According to Proposition 3, $((U_\iota, \varphi_\iota), \varepsilon_0 \circ \tau_0\varphi_\iota)$, $\iota \in I$, is a fiber atlas on τX and $((U_\iota, \varphi_\iota), \nu_0 \circ \varrho_0\varphi_\iota)$, $\iota \in I$, is a fiber atlas on ϱX. For every $\iota \in I$ we have a map

(16) $$\varrho_0 U_\iota \ni y \to N_\iota(y) = (\varepsilon_0 \circ \tau_0\varphi_\iota)^{-1} \circ \tilde{N}_{\varphi_\iota(U_\iota)} \circ \nu_0 \circ \varrho_0\varphi_\iota(y) \in \tau_0 U_\iota .$$

There exists one and only one map $N_X^{(0)} : \varrho_0 X \to \tau_0 X$ such that

(17) $$N_X^{(0)} |_{\varrho_0 U_\iota} = N_\iota .$$

It follows from the definition of N_ι that the triple $N_X = (N_X^{(0)}, \mathrm{id}_X, \lambda)$, where $\lambda : G_0 \to G$ is the natural injection, is an injective homomorphism of the principal G_0-bundle ϱX to the principal G-bundle τX, i.e., a reduction of τX to ϱX.

In order to show that the correspondence $X \to N_X$, $X \in \mathrm{Ob}\, \mathscr{D}_n$, is a natural transformation of functors we shall check that for every $\alpha \in \mathrm{Mor}\, \mathscr{D}_n$, $\alpha : X_1 \to X_2$,

(18) $$\tau_0\alpha \circ N_{X_1}^{(0)} = N_{X_2}^{(0)} \circ \varrho_0\alpha .$$

Let (U_ι, φ_ι), $\iota \in I$, be an atlas on X_1, and let $(V_\varkappa, \psi_\varkappa)$, $\varkappa \in K$, be an atlas on X_2. According to (16) and the properties of the natural transformation $\tilde{N} = (\tilde{N}^{(0)}, \mathrm{id}_{R^n}, \mathrm{id}_G)$, for each $\iota \in I$, $\varkappa \in K$ such that the considered expressions make sense,

$$\begin{aligned} &\varepsilon_0 \circ \tau_0\psi_\varkappa \circ \tau_0\alpha \circ (\varepsilon_0 \circ \tau_0\varphi_\iota)^{-1} \circ \tilde{N}^{(0)}_{\varphi_\iota(U_\iota)} \circ \nu_0 \circ \varrho_0\varphi_\iota = \\ &= \tau_0^\varepsilon(\psi_\varkappa\alpha\varphi_\iota^{-1}) \circ \tilde{N}^{(0)}_{\varphi_\iota(U_\iota)} \circ \nu_0 \circ \varrho_0\varphi_\iota = \tilde{N}^{(0)}_{\psi_\varkappa\alpha(U_\iota)} \circ \varrho_0^\nu(\psi_\varkappa\alpha\varphi_\iota^{-1}) \circ \nu_0 \circ \varrho_0\varphi_\iota = \\ &= \tilde{N}^{(0)}_{\psi_\varkappa\alpha(U_\iota)} \circ \nu_0 \circ \varrho_0\psi_\varkappa \circ \varrho_0\alpha, \end{aligned}$$

which gives

$$\tau_0\alpha \circ (\varepsilon_0 \circ \tau_0\varphi_\iota)^{-1} \circ \tilde{N}^{(0)}_{\varphi_\iota(U_\iota)} \circ \nu_0 \circ \varrho_0\varphi_\iota = (\varepsilon_0 \circ \tau_0\psi_\varkappa)^{-1} \circ \tilde{N}^{(0)}_{\psi_\varkappa\alpha(U_\iota)} \circ \nu_0 \circ \varrho_0\psi_\varkappa \circ \varrho_0\alpha.$$

This proves (18).

3. DIFFERENTIABLE LIFTINGS

We shall now formulate a condition ensuring that the covariance group of a lifting $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ is a Lie subgroup of G.

Definition 5. A lifting $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ is said to be *differentiable* if it has the following property:

For each $X \in \operatorname{Ob} \mathscr{D}_n$, each open interval I, open submanifold U in X, and differentiable map $\alpha : I \times U \to X$ such that for each $t \in I$ the map α_t defined by $\alpha_t(x) = \alpha(t, x)$ is a morphism of the category $\mathscr{D}_X$, the map $I \times \tau_0 U \ni (t, y) \to \tau_0 \alpha_t(y) \in \tau_0 X$ is differentiable.

Let $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ be a lifting, ε a trivialization of τR^n.

Proposition 8. *For τ to be differentiable it suffices that the following condition is satisfied:*

For every open submanifold U of R^n, every open interval I and differentiable map $\alpha : I \times U \to R^n$ such that $\alpha_t \in \operatorname{Mor} \mathscr{D}_{R^n}$, $t \in I$, the map $I \times U \ni (t, x) \to \bar{\tau}^{\varepsilon}\alpha_t(\alpha_t(x)) \in G$ is differentiable.

Proof. The statement follows from the local representation of $\tau_0 \alpha_t$ by means of fiber charts $((U, \varphi), \varepsilon_0 \circ \tau_0 \varphi)$, $((V, \psi), \varepsilon_0 \circ \tau_0 \psi)$ and from (10).

Let us now introduce some notation. We shall denote by $j_x^r f$ the r-jet of a map f at a point x, $r = 1, 2, \ldots, \infty$, and by $*$ the composition of jets. L_n^r will denote the group of all invertible r-jets with source and target at the origin $0 \in R^n$. For finite r, we shall consider this group with the natural structure of a Lie group. For $r = \infty$ the group L_n^∞ will be considered with its algebraic structure.

Let (Y_i, π_i, X_i) be a principal G_i-bundle, $i = 1, 2$, (σ, σ_0, ν) a homomorphism of (Y_1, π_1, X_1) into (Y_2, π_2, X_2). The restriction of the map σ to $\pi_1^{-1}(x)$, $x \in X_1$, will be denoted by $\sigma |_x$. For $X_1, X_2 \in \operatorname{Ob} \mathscr{D}_n$, denote by $\mathscr{J}^\infty(X_1, X_2)$ the set of all invertible ∞-jets with source in X_1 and target in X_2.

Proposition 9. *Let $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ be a lifting, $\alpha \in \operatorname{Mor} \mathscr{D}_n$, $\alpha : X_1 \to X_2$. Then the map $\tau_0 \alpha |_x$, $x \in X_1$, depends only on $j_x^\infty \alpha$.*

Proof. See [3].

As a consequence of Proposition 10 we obtain

Proposition 10. *Let $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ be a lifting, ε a trivialization of τR^n. The relation*

$$\tilde{\varepsilon}(j_x^\infty \alpha) = \bar{\tau}^{\varepsilon}\alpha(\alpha(x)) \tag{19}$$

defines a map $\tilde{\varepsilon} : \mathscr{J}^\infty(R^n, R^n) \to G$ *which has the following property: For each* $j^\infty_x\alpha \in$ $\in \mathscr{J}^\infty(R^n, R^n)$

$$\tilde{\varepsilon}(j^\infty_x\alpha) = \tilde{\varepsilon}(j^\infty_0(t_{\alpha(x)}\alpha t_{-x})), \tag{20}$$

and for each $s_1, s_2 \in \mathscr{J}^\infty(R^n, R^n)$ *such that* $s_1 * s_2$ *is defined,*

$$\tilde{\varepsilon}(s_1 * s_2) = \tilde{\varepsilon}(s_1) \cdot \tilde{\varepsilon}(s_2). \tag{21}$$

Proof. It follows from (10) and Proposition 10 that $\tau^\varepsilon\alpha(\alpha(x))$ depends only on $j^\infty_x\alpha$. From (11) and Proposition 5 we deduce that $\tau^\varepsilon\alpha(\alpha(x)) = \tau^\varepsilon(t_{\alpha(x)}\alpha t_{-x})(0)$ which proves (20). (21) follows from (11).

Let $\tilde{\varepsilon}_0$ be the restriction of $\tilde{\varepsilon}$ (19) to the subset L^∞_n of $\mathscr{J}(R^n, R^n)$,

$$\tilde{\varepsilon}_0 = \tilde{\varepsilon}\,|_{L^\infty_n}. \tag{22}$$

It follows from (21) that $\tilde{\varepsilon}_0$ is a homomorphism of groups. In establishing a more precise result than that one of Proposition 9 we shall use a lemma about normal subgroup structure of L^∞_n. Let Ker ϱ_r denote the kernel of the natural group homomorphism $\varrho_r : L^\infty_n \to L^r_n$.

Lemma. If N is a nontrivial normal subgroup of L^∞_n then there is an integer $k \geqslant 0$ such that Ker $\varrho_k \subset N$.

Proof. See [3].

Theorem 2. Let $\tau : \mathscr{D}_n \to \mathscr{P}\mathscr{B}_n(G)$ be a differentiable lifting, ε a trivialization of τR^n. There exist an integer $r \geqslant 0$ and a homomorphism $\tilde{\varepsilon}_r : L^r_n \to G$ of Lie groups such that

$$\tilde{\varepsilon}_0 = \tilde{\varepsilon}_r \circ \varrho_r. \tag{23}$$

Proof. 1. Let $k \geqslant 0$ be an integer, and denote by $\iota_k : L^k_n \to L^\infty_n$ the natural injection of sets assigning to a k-jet $(a^i_j, a^i_{j_1j_2}, \ldots, a^i_{j_1\ldots j_k})$ the ∞-jet $(a^i_j, a^i_{j_1j_2}, \ldots, a^i_{j_1\ldots j_k}, 0, 0, \ldots)$. Let $a \in L^k_n$ be any point, J an open interval containing the origin $0 \in R$, and $J \ni t \to \psi(t) \in L^k_n$ any differentiable curve passing through a, i.e., such that $\psi(0) = a$. There is a neighbourhood U of $0 \in R^n$ and a differentiable map $J \times U \ni (t, x) \to$ $\to \alpha(t, x) = \alpha_t(x) \in R^n$ such that (1) for each t the map α_t belongs to the class Mor $\mathscr{D}_{R^n}$, (2) $\alpha_t(0) = 0$, (3) $\psi(t) = j^k_0\alpha_t$, and (4) $j^\infty_0\alpha_t = \iota_k(j^k_0\alpha_t)$. This map is easily constructed by means of polynomials whose coefficients depend on t. Let ε be a trivialization of τR^n. Evidently, $\tilde{\varepsilon}_0\iota_k\psi(t) = \tau^\varepsilon_t(0)$. Since the lifting τ is by assumption differentiable we see that the curve $t \to \tilde{\varepsilon}_0\iota_k\psi(t)$ in G must be differentiable at the point $0 \in R$. The curve ψ being arbitrary, the map $\tilde{\varepsilon}_0\iota_k$ is by a well-known theorem differentiable at the point $a = \psi(0)$. We have thus proved that the map $\tilde{\varepsilon}_0 \iota_k : L_n^k \to G$ is differentiable.

2. Let us consider the group homomorphism $\tilde{\varepsilon}_0 : L_n^\infty \to G$, and assume that $\tilde{\varepsilon}_0$ is injective. Then for every k, $\tilde{\varepsilon}_0 \iota_k : L_n^k \to G$ is an injection and, by the first part of this proof, an immersion of differential manifolds. Using the arbitrariness of k and the dimensional arguments one obtains a contradiction showing that $\tilde{\varepsilon}_0$ cannot be an injection. We conclude that the kernel $\operatorname{Ker} \tilde{\varepsilon}_0$ of the group homomorphism $\tilde{\varepsilon}_0$ is nontrivial.

3. The kernel $\operatorname{Ker} \tilde{\varepsilon}_0$ is a normal subgroup of L_n^∞. By virtue of the above Lemma there is an integer $r \geqslant 0$ such that $\operatorname{Ker} \varrho_r \subset \operatorname{Ker} \tilde{\varepsilon}_0$. The quotient $L_n^\infty / \operatorname{Ker} \varrho_r = L_n^r$ has a natural structure of a Lie group, and the equality $\tilde{\varepsilon}_0 = \tilde{\varepsilon}_r \circ \varrho_r$ defines a group homomorphism $\tilde{\varepsilon}_r : L_n^r \to G$. The differentiability of $\tilde{\varepsilon}_r$ follows from the equality $\tilde{\varepsilon}_r = \tilde{\varepsilon}_0 \circ \iota_r$ and from the first part of this proof. This shows that $\tilde{\varepsilon}_r$ is a homomorphism of Lie groups, and the proof is complete.

Let us pass to a description of the covariance group of a differentiable lifting.

Theorem 3. *The covariance group of a differentiable lifting $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ is a Lie subgroup of G. If ε is a trivialization of τR^n then the covariance group G^ε is equal to $\tilde{\varepsilon}_0(L_n^\infty)$.*

Proof. Let ε be a trivialization of τR^n, and define $\tilde{\varepsilon}_0$ by (22). By virtue of Proposition 6, the covariance group of a lifting $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ is defined by $G^\varepsilon = \{g \in G \mid g = \tau^\varepsilon \alpha(0), \alpha \in \mathscr{A}_0\}$. According to (19) and (22), $G^\varepsilon = \{g \in G \mid g = \tilde{\varepsilon}_0(s), s \in L_n^\infty\} = \tilde{\varepsilon}_0(L_n^\infty)$.

It thus remains to show that $\tilde{\varepsilon}_0(L_n^\infty)$ is a Lie subgroup of G. According to Theorem 2 there is an integer $r \geqslant 0$ and a homomorphism $\tilde{\varepsilon}_r : L_n^r \to G$ of Lie groups such that $\tilde{\varepsilon}_0 = \tilde{\varepsilon}_r \circ \varrho_r$. For this r, let $L_n^{r(+)}$ denote the maximal connected subgroup of L_n^r. Since $L_n^{r(+)}$ is linearly connected, to each $s \in L_n^{r(+)}$ one can find a curve $[0, 1] \ni t \to s_t \in L_n^{r(+)}$ such that $s_0 = j_0^r \operatorname{id}_{R^n}$ and $s_1 = s$. The differentiability of τ implies that the curve $t \to \tilde{\varepsilon}_0 \iota_r(s_t) = \tilde{\varepsilon}_r(s_t)$ is differentiable. Further, $\tilde{\varepsilon}_r(j_0^r \operatorname{id}_{R^n}) = e$, where e is the identity element of G, and we see that the element $\tilde{\varepsilon}_r(s) \in G$ can be joined to the identity e by a curve lying in $\tilde{\varepsilon}_r(L_n^{r(+)})$. This implies, however, that the algebraic subgroup $\tilde{\varepsilon}_r(L_n^{r(+)})$ is a Lie subgroup of G [4, p. 275]. Let $L_n^{r(-)}$ be the complement of $L_n^{r(+)}$ in L_n^r, $s_0 \in L_n^{r(-)}$. The map $s \to s_0 * s$ defines a diffeomorphism of $L_r^{r(+)}$ and $L_n^{r(-)}$ which shows that $\tilde{\varepsilon}_r(L_n^{r(-)})$ and hence $\tilde{\varepsilon}_r(L_n^r)$ is a submanifold of G. This means that $\tilde{\varepsilon}_0(L_n^\infty)$ is a Lie subgroup of G.

Let $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ be a differentiable lifting, ε a trivialization of τR^n. Using the notation of Theorem 2 we define:

Definition 6. The smallest number r such that there is a homomorphism $\tilde{\varepsilon}_r : L_n^r \to G$ of Lie groups satisfying $\tilde{\varepsilon}_0 = \tilde{\varepsilon}_r \circ \varrho_r$ is called the *order* of the lifting τ.

Clearly the order of a lifting τ is defined independently of the choice of the trivialization ε.

Let us return to the dependence of $\tau_0\alpha|_x$ on $j_x^\infty\alpha$ (Proposition 9). The differentiability condition leads to the following result:

Theorem 4. *Let $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ be a differentiable lifting, $\alpha \in \operatorname{Mor}\mathscr{D}_n$, $\alpha : X_1 \to \to X_2$. Then the map $\tau_0\alpha|_x$, $x \in X_1$, depends only on $j_x^r\alpha$, where r is the order of the lifting τ.*

Proof. Choose a trivialization ε of τR^n and consider a fiber atlas $((U_\iota, \varphi_\iota), \varepsilon_0 \circ \tau_0\varphi_\iota)$, $\iota \in I$, on τX_1 and a fiber atlas $((V_\varkappa, \psi_\varkappa), \varepsilon_0 \circ \tau_0\psi_\varkappa)$, $\varkappa \in K$, on τX_2. For each $\iota \in I$, $\varkappa \in K$ such that the considered expressions make sense, $\varepsilon_0 \circ \tau_0\psi_\varkappa \circ \tau_0\alpha \circ (\varepsilon_0 \circ \tau_0\varphi_\iota)^{-1} = = \tau_0^\varepsilon(\psi_\varkappa\alpha\varphi_\iota^{-1})$. According to (10) and Proposition 10

$$\tau_0^\varepsilon(\psi_\varkappa\alpha\varphi_\iota^{-1})(x', g) = (\psi_\varkappa\alpha\varphi_\iota^{-1}(x'), \tilde{\varepsilon}_0(j_0^\infty(t_{\psi_\varkappa\alpha\varphi_\iota^{-1}(x')}\psi_\varkappa\alpha\varphi_\iota^{-1}t_{-x'})) \cdot g).$$

Theorem 2 and Definition 6 imply that

$$\tilde{\varepsilon}_0(j_0^\infty(t_{\psi_\varkappa\alpha\varphi_\iota^{-1}(x')}\psi_\varkappa\alpha\varphi_\iota^{-1}t_{-x'})) = \tilde{\varepsilon}_r(j_0^r(t_{\psi_\varkappa\alpha\varphi_\iota^{-1}(x')}\psi_\varkappa\alpha\varphi_\iota^{-1}t_{-x'})),$$

where r is the order of the lifting τ. This relation shows that $\tau_0\alpha|_x$ is a function of $j_x^r\alpha$. By virtue of the identity $\tau_0\alpha = \tau_0\psi_\varkappa^{-1} \circ \tau_0(\psi_\varkappa\alpha\varphi_\iota^{-1}) \circ \tau_0\varphi_\iota$, this function is independent of the trivialization ε. This finishes the proof.

Theorems 2–4 can be called the *finite order theorems* for differentiable liftings in principal fiber bundles.

We shall end this section by proving a theorem concerning the reducibility of a differentiable lifting.

Theorem 5. *Every differentiable lifting is reducible to its covariance group. The reduction is unique up to a natural transformation.*

Proof. Let $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ be a differentiable lifting, ε a trivialization of τR^n, $\tau^\varepsilon : \mathscr{D}_{R^n} \to \mathscr{PB}_{R^n \times G}$ the corresponding ε-functor (Section 2), G^ε the covariance group of τ, and $\iota_\varepsilon : G^\varepsilon \to G$ the natural injection. In order to show that τ is reducible to G^ε it suffices, according to Propositions 7 and 4, to find a functor $\tilde{\varrho} : \mathscr{D}_{R^n} \to \mathscr{PB}_{R^n \times G^\varepsilon}$ and a natural transformation $\tilde{N}$ of $\tilde{\varrho}$ to τ^ε such that $\tilde{N}_U : \tilde{\varrho}U \to \tau^\varepsilon U$ is a reduction of $\tau^\varepsilon U$ to $\tilde{\varrho}U$ for each $U \in \operatorname{Ob}\mathscr{D}_{R^n}$. Let us define such a functor satisfying (5). For $U \in \operatorname{Ob}\mathscr{D}_{R^n}$ we set $\tilde{\varrho}U = (U \times G^\varepsilon, \pi_U^\varepsilon, U)$, where $\pi_U^\varepsilon : U \times G^\varepsilon \to U$ is the natural projection. According to Proposition 6, for each $\alpha \in \operatorname{Mor}\mathscr{D}_{R^n}$, $\alpha : U \to V$, the map $\tau^\varepsilon\alpha : \alpha(U) \to G$ takes values in G^ε. Consequently, the equality

$$(\mathrm{id}_V \times \iota_\varepsilon) \circ \tilde{\varrho}_0\alpha = \tau_0^\varepsilon\alpha \circ (\mathrm{id}_U \times \iota_\varepsilon) \tag{24}$$

defines a morphism $\tilde{\varrho}\alpha = (\tilde{\varrho}_0\alpha, \alpha, \mathrm{id}_{G^\varepsilon})$ from $\tilde{\varrho}U$ into $\tilde{\varrho}V$. It is directly seen that the correspondence $U \to \tilde{\varrho}U$, $\alpha \to \tilde{\varrho}\alpha$ is the desired functor.

According to Proposition 4, there exist a lifting $\varrho : \mathscr{D}_n \to \mathscr{PB}_n(G^\varepsilon)$ and a trivialization ν of ϱR^n such that $\varrho^\nu = \tilde{\varrho}$. We shall show that τ is reducible to ϱ. By Proposition 7, it suffices to find a natural transformation $\tilde{N}$ of ϱ^ν to τ^ε such that $\tilde{N}_U : \varrho^\nu U \to \tau^\varepsilon U$, $U \in \mathrm{Ob}\,\mathscr{D}_{R^n}$, is a reduction. Let $U \in \mathrm{Ob}\,\mathscr{D}_{R^n}$, $(x, g) \in U \times G^\varepsilon$. We set $\tilde{N}_U^{(0)}(x, g) = (x, \iota_\varepsilon(g))$. For each $\alpha \in \mathrm{Mor}\,\mathscr{D}_{R^n}$, $\alpha : U \to V$, and $(x, g) \in U \times G^\varepsilon$, the equality $\varrho^\nu = \tilde{\varrho}$ together with (24) imply $\iota_\varepsilon(\bar{\varrho}^\nu\alpha(\alpha(x))) = \bar{\tau}^\varepsilon\alpha(\alpha(x))$ which gives $\tilde{N}_V^{(0)} \circ \tilde{\varrho}_0^\nu\alpha(x, g) = (\alpha(x), \bar{\tau}^\varepsilon\alpha(\alpha(x)) \,.\, \iota_\varepsilon(g))$. Since $\tau_0^\varepsilon\alpha \circ \tilde{N}_U^{(0)}(x, g) = (\alpha(x), \bar{\tau}^\varepsilon\alpha(\alpha(x)) \,.\, \iota_\varepsilon(g))$ we see that the relation $\tau_0^\varepsilon\alpha \circ \tilde{N}_U^{(0)} = \tilde{N}_V^{(0)} \circ \tilde{\varrho}_0^\nu\alpha$ must hold. This proves that the correspondence $U \to \tilde{N}_U = (\tilde{N}_U^{(0)}, \mathrm{id}_U, \iota_\varepsilon)$ is a natural transformation of ϱ^ν to τ^ε. Applying Proposition 7 we see that ϱ is reducible to τ.

The second part of our assertion follows from Proposition 4.

4. ASSOCIATED LIFTINGS

We begin this section by introducing some categories of fiber bundles.

$\mathscr{FB}_n$ will denote the category whose objects are fiber bundles associated with the principal fiber bundles from the category $\mathscr{PB}_n$, and whose morphisms are homomorphisms of fiber bundles over the morphisms from the category $\mathscr{D}_n$. Let (Y_i, π_i, X_i) be a principal G_i-bundle, and let Q_i be a left G_i-space, $i = 1, 2$. Denote by $(Y_i \times_{G_i} Q_i, {}_i, \bar{\pi}_i X_i)$ the fiber bundle with fiber Q_i associated with (Y_i, π_i, X_i). Recall that a collection $((\sigma, \sigma_0, \nu), \bar{\sigma}, F)$ is called a homomorphism of $(Y_1 \times_{G_1} Q_1, \bar{\pi}_1, X_1)$ into $(Y_2 \times_{G_2} Q_2, \bar{\pi}_2, X_2)$ if (σ, σ_0, ν) is a homomorphism of the principal G_1-bundle (Y_1, π_1, X_1) into the principal G_2-bundle (Y_2, π_2, X_2), $F : Q_1 \to Q_2$ is a map such that for each $q \in Q_1$ and $g \in G_1$, $F(g \,.\, q) = \nu(g) \,.\, F(q)$, and $\bar{\sigma} : Y_1 \times_{G_1} Q_1 \to Y_2 \times_{G_2} Q_2$ is a map such that for each $z \in Y_1 \times_{G_1} Q_1$ represented (as an equivalence class) by a pair $(y, q) \in Y_1 \times Q_1$, $\bar{\sigma}(z) = [\sigma(y), F(q)]$ (compare with [14]).

Let G be a Lie group. The subcategory of $\mathscr{FB}_n$ formed by all fiber bundles associated with the principal G-bundles and by their G-homomorphisms, will be denoted by $\mathscr{FB}_n(G)$.

Let $\tau : \mathscr{D}_n \to \mathscr{PB}_n(G)$ be a lifting and Q a left G-space. For $X \in \mathrm{Ob}\,\mathscr{D}_n$, write $\tau_Q X = (\tau_0 X \times_G Q, \bar{\pi}_X, X)$ for the fiber bundle associated with the principal G-bundle $\tau X = (\tau_0 X, \pi_X, X)$. Let $\alpha \in \mathrm{Mor}\,\mathscr{D}_n$, $\alpha : X_1 \to X_2$, $\tau\alpha = (\tau_0\alpha, \alpha, \mathrm{id}_G)$. If $z \in \tau_0 X \times_G Q$, $z = [y, q]$, then the point $\bar{\tau}_Q\alpha(z) = [\tau_0\alpha(y), q] \in \tau_0 X \times_G Q$ is independent of the choice of the pair (y, q) representing the equivalence class z. The collection $\tau_Q\alpha = ((\tau_0\alpha, \alpha, \mathrm{id}_G), \bar{\tau}_Q\alpha, \mathrm{id}_Q)$ is a morphism of the category $\mathscr{FB}_n(G)$. The correspondence $X \to \tau_Q X$, $\alpha \to \tau_Q\alpha$ has the properties of a covariant functor from $\mathscr{D}_n$ to $\mathscr{FB}_n(G)$. We denote this functor by τ_Q.

Definition 7. τ_Q is called the Q-lifting, associated with the lifting τ.

Let $\mathscr{F}^r$ be the r-frame functor. $\mathscr{F}^r$ is a differentiable lifting from the category $\mathscr{D}_n$ to $\mathscr{P}\mathscr{B}_n(L_n^r)$. For $X \in \operatorname{Ob} \mathscr{D}_n$ and $\alpha \in \operatorname{Mor} \mathscr{D}_n$ we write $\mathscr{F}^r X = (\mathscr{F}_0^r X, \varrho_X^r, X)$, $\mathscr{F}_\alpha^r = (\mathscr{F}_0^r \alpha, \alpha, \mathrm{id}_{L_n^r})$. The following theorem describes the class of Q-liftings associated with the differentiable liftings in principal fiber bundles.

Theorem 6. *Every Q-lifting $\tau_Q : \mathscr{D}_n \to \mathscr{F}\mathscr{B}_n(G)$ associated with a differentiable lifting $\tau : \mathscr{D}_n \to \mathscr{P}\mathscr{B}_n(G)$ is associated with the r-frame lifting $\mathscr{F}^r$, where r is the order of τ. More precisely, there is a lifting $\mathscr{F}_Q^r : \mathscr{D}_n \to \mathscr{F}\mathscr{B}_n(L_n^r)$ associated with $\mathscr{F}^r$, and a natural transformation $N : \mathscr{F}_Q^r \to \tau_Q$ of functors such that for every $X \in \operatorname{Ob} \mathscr{D}_n$, N_X is of the form $N_X = ((N_X^{(0)}, \mathrm{id}_X, v), \bar{N}_X, \mathrm{id}_Q)$, where $\bar{N}_X : \mathscr{F}_0^r X \times_{L_n^r} Q \to \tau_0 X \times_G Q$ is a diffeomorphism.*

Proof. Choose a trivialization $\varepsilon = (\varepsilon_0, \mathrm{id}_{R^n}, \mathrm{id}_G)$ of τR^n. The map $(s, q) \to \tilde{\varepsilon}_r(s) \,.\, q$, where r is the order of τ, defines a left action of L_n^r on Q (Theorem 2). This action gives rise to a lifting $\mathscr{F}_Q^r : \mathscr{D}_n \to \mathscr{F}\mathscr{B}_n(L_n^r)$ associated with $\mathscr{F}^r$.

Let $X \in \operatorname{Ob} \mathscr{D}_n$, and let e denote the identity of G. According to Theorem 4, to each $y \in \mathscr{F}_0^r X$, $y = j_0^r \varphi$, there is associated an element $\varepsilon_X^{(0)}(y) = \tau_0 \varphi \circ \varepsilon_0^{-1}(0, e) \in \tau_0 X$. For each $s \in L_n^r$, $s = j_0^r \alpha$ the relations (4), (10), (19), (22), and (23) give

$$\varepsilon_X^{(0)}(y * s) = \tau_0(\varphi\alpha) \circ \varepsilon_0^{-1}(0, e) = \tau_0 \varphi \circ \varepsilon_0^{-1} \circ \tau_0^{\varepsilon}\alpha(0, e) = \tag{25}$$
$$= \tau_0 \varphi \circ \varepsilon_0^{-1}(0, \tilde{\varepsilon}_r(s)) = \tau_0 \varphi \circ \varepsilon_0^{-1}(0, e) \,.\, \tilde{\varepsilon}_r(s) = \varepsilon_X^{(0)}(y) \,.\, \tilde{\varepsilon}_r(s),$$

which shows that the triple $(\varepsilon_X^{(0)}, \mathrm{id}_X, \tilde{\varepsilon}_r)$ is a morphism of the category $\mathscr{P}\mathscr{B}_n$. This morphism gives rise to a map $\bar{\varepsilon}_X : \mathscr{F}_0^r X \times_{L_n^r} Q \to \tau_0 X \times_G Q$ as follows. For $z \in \mathscr{F}_0^r X \times_{L_n^r} Q$, $z = [y, q]$, we set $\bar{\varepsilon}_X(z) = [\varepsilon_X^{(0)}(y), q]$. It follows from (25) that the element $\bar{\varepsilon}_X(z)$ is well defined. Obviously, $((\varepsilon_X^{(0)}, \mathrm{id}_X, \tilde{\varepsilon}_r), \bar{\varepsilon}_X, \mathrm{id}_Q)$ is a morphism in the category $\mathscr{F}\mathscr{B}_n$. We shall verify that $\bar{\varepsilon}_X$ is a bijection. Firstly, we shall show that it is an injection. Choose $z_i \in \mathscr{F}_0^r X \times_{L_n^r} Q$, $z_i = [y_i, q_i]$, $y_i = j_0^r \varphi_i$, $i = 1, 2$, and assume that $\bar{\varepsilon}_X(z_1) = \bar{\varepsilon}_X(z_2)$. Then there is an element $g \in G$ such that $\tau_0 \varphi_2 \circ \varepsilon_0^{-1}(0, e) = \tau_0 \varphi_1 \circ \varepsilon_0^{-1}(0, e) \,.\, g$, $q_2 = g^{-1} \,.\, q_1$. The first equality leads to the relation $\tau_0^{\varepsilon}(\varphi_1^{-1}\varphi_2)(0, e) = (0, g)$ or, equivalently, $\bar{\tau}^{\varepsilon}(\varphi_1^{-1}\varphi_2)(0) = \tilde{\varepsilon}_r(j_0^r(\varphi_1^{-1}\varphi_2)) = g$. Using the second equality we obtain for $s \in L_n^r$, $s = j_0^r(\varphi^{-1}\varphi_2)$,

$$z_2 = [y_2, q_2] = [j_0^r \varphi_1 * s, q_2] = [y_1, \tilde{\varepsilon}_r(s) \,.\, q_2] = [y_1, g \,.\, q_2] = z_1$$

proving that $\bar{\varepsilon}_X$ is a bijection. Secondly, let us verify that $\bar{\varepsilon}_X$ is a surjection. Choose $\bar{z} \in \tau_0 X \times_G Q$, $\bar{z} = [\bar{y}, q]$, and any element $y \in \mathscr{F}_0^r X$, $y = j_0^r \varphi$, such that $\varrho_X^r(y) = \bar{\pi}_X(\bar{z})$. Then $\bar{y} = \tau_0 \varphi \circ \varepsilon_0^{-1}(0, e) \in \tau_0 X$, and $\bar{z}$ has a representative of the form $(\bar{y}, q)$ for some $q \in Q$. Obviously, for $z = [y, q]$ we have $\bar{\varepsilon}_X(z) = \bar{z}$ proving that $\bar{\varepsilon}_X$ is a surjection. This means that $\bar{\varepsilon}_X$ is a bijection, hence a diffeomorphism.

To complete the proof it remains to verify that for each $\alpha \in \operatorname{Mor} \mathscr{D}_n$, $\alpha : X_1 \to X_2$, the following two diagrams are commutative:

$$\begin{array}{ccc} \mathscr{F}^r_0 X_1 & \xrightarrow{\varepsilon^{(0)}_{X_1}} & \tau_0 X_1 \\ {\scriptstyle \mathscr{F}^r_0\alpha}\downarrow & & \downarrow{\scriptstyle \tau_0\alpha} \\ \mathscr{F}^r_0 X_2 & \xrightarrow{\varepsilon^{(0)}_{X_2}} & \tau_0 X_2 \end{array} \qquad \begin{array}{ccc} \mathscr{F}^r_0 X_1 \times_{L^r_n} Q & \xrightarrow{\varepsilon_{X_1}} & \tau_0 X_1 \times_G Q \\ {\scriptstyle \overline{\mathscr{F}}^r_Q\alpha}\downarrow & & \downarrow{\scriptstyle \tau_Q\alpha} \\ \mathscr{F}^r_0 X_2 \times_{L^r_n} Q & \xrightarrow{\varepsilon_{X_2}} & \tau_0 X_2 \times_G Q \end{array}$$

This is, however, a direct consequence of the definitions.

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Added in proof. Recently, the finite order theorem was proved by R. S. Palais and Chun-Lian Terng for smooth locally trivial fiber bundles whose structure groups are not a priori specified (Topology, 16 (1977), 271-277).

REFERENCES

[1] G. W. Horndeski, D. Lovelock, *Scalar-tensor field theories,* Tensor, 24 (1972), 79—92.

[2] D. Husemoller, *Fibre Bundles,* McGraw-Hill, New York, 1966.

[3] Chuu-Lian Terng, *Natural vector bundles and natural differential operators.* A dissertation, Brandeis University, Massachusetts, 1976.

[4] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry,* Vol. 1, Interscience, New York, 1963.

[5] D. Krupka, *A setting for generally invariant Lagrangian structures in tensor bundles,* Bull. Acad. Polon. Sci, Sér. Math. Astronom. Phys., 22 (1974), 967—972.

[6] D. Krupka, *A theory of generally invariant Lagrangians for the metric fields.* II., Intern. J. Theor. Phys. 15 (1976), 949—959; I., Intern. J. Theor. Phys. 17 (1978), 359-368

[7] D. Krupka, *Elementary theory of differential invariants,* Arch. Math. (Brno), 14 (1978), 207-214

[8] D. Krupka, *Finite order liftings in principal fibre bundles,* Beiträge zur Algebra und Geometrie (Halle), to appear.

[9] D. Krupka, *Lagrangians and topology,* Scripta Fac. Sci. Nat. UJEP Brunensis, Physica 3—4 (1975), 265—270.

[10] D. Krupka, A. Trautman, *General invariance of Lagrangian structures,* Bull. Acad. Polon. Sci, Sér. Math. Astronom. Phys., 22 (1974), 207—211.

[11] A. Nijenhuis, *Natural bundles and their general properties,* Differential geometry, in honour of K. Yano, Kinokunyia, Tokyo, 1972, 317—334.

[12] H. Rund, *Invariant theory of variational problems for geometric objects,* Tensor, 18 (1967), 240—257.

[13] S. E. Salvioli, *On the theory of geometric objects,* J. Diff. Geometry 7 (1972), 257—278.

[14] R. Sulanke, P. Wintgen, *Differentialgeometrie und Faserbündel,* Berlin, 1972.

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