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ON THE STRUCTURE OF SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS WITH GIVEN CHARACTERISTIC MULTIPLIERS IN THE GENERALIZED FLOQUET THEORY

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1. INTRODUCTION

In [1] and [2] established O. Borůvka the functions X that for every solution u of the both-sided oscillatory equation (q): $y'' = q(t)\,y$, $q \in C^0_{\mathbf{R}}$, $\mathbf{R} = (-\infty, \infty)$ is $\frac{uX(t)}{\sqrt{|X'(t)|}}$ a solution of the same equation (on $\mathbf{R}$) again. M. Laitoch extended in [6] on the above basis the classical Floquet theory (e. g. [7]) also to equations (q), where q is in general no periodic function. By means of the theory of phases and dispersions there are expressed characteristic multipliers of (q) in both the classical ([2]–[5], [8]) and the generalized ([10]) Floquet theory. In [9] there is investigated the structure of equations (q) with π periodic carrier q with given characteristic multipliers. The aim of this paper is to investigate the structure of equations (q) with given characteristic multipliers in the generalized Floquet theory.

2. BASIC CONCEPTS, PROPERTIES AND NOTATION

In what follows we are investigate differential equations of type

$$\text{(q)} \qquad y'' = q(t)\,y, \; q \in C^0_{\mathbf{R}}$$

being both-sided oscillatory on $\mathbf{R}$ (i.e. every nontrivial solution of (q) has infinitely many zeros to the right and to the left of the point $t_0 \in \mathbf{R}$). Occasionaly the function q will be called the carrier of the equation (q). The trivial solution of (q) will be excluded.

Convention. Throughout this article f^{-1} will denote the inverse function (so far such exists) to the function f; f^{σ} will denote the function f for $\sigma = 1$ and the function

f^{-1} for $\sigma = -1$. The composite functions $\alpha[X(t)]$, $\varepsilon[\alpha[\gamma(t)]]$ will be written more briefly $\alpha X(t)$, $\varepsilon\alpha\gamma(t)$.

Let u, v be independent solutions of (q). Following [1] and [2] we say that a function $\alpha : \mathbf{R} \to \mathbf{R}$, $\alpha \in C^0_{\mathbf{R}}$ is a (first) phase of the (ordered) pair of solutions u, v if

$$\operatorname{tg} \alpha(t) = \frac{u(t)}{v(t)}, \qquad \text{for } t \in \mathbf{R} - \{t \in \mathbf{R},\ v(t) = 0\}.$$

If u, v are independent solutions of (q), $uv' - u'v = w$, then there exists a phase α of u, v satisfying $u(t) = \sqrt{|w|}\,\dfrac{\sin \alpha(t)}{\sqrt{|\alpha'(t)|}}$, $v(t) = \sqrt{|w|}\,\dfrac{\cos \alpha(t)}{\sqrt{|\alpha'(t)|}}$, $t \in \mathbf{R}$. We say that α is a (first) phase of (q) if there exist independent solutions u, v of (q) possessing a phase α.

Every phase α of (q) has the following properties:

$$(1) \qquad \alpha \in C^3_{\mathbf{R}}, \qquad \alpha'(t) \neq 0 \qquad \text{for } t \in \mathbf{R},\ \alpha(\mathbf{R}) = \mathbf{R}$$

and if α is a phase of independent solutions u, v of (q) with the Wronskian determinant w $(=uv' - u'v)$, then $\operatorname{sign} \alpha' = -\operatorname{sign} w$. The set of all functions α possessing the properties of (1) form a group $\mathbf{G}$ with respect to composition of functions.

The set of phases of equation $y'' = -y$ is denoted by $\mathbf{E}$. If α is a phase of (q), then $\mathbf{E}\alpha := \{\varepsilon\alpha,\ \varepsilon \in \mathbf{E}\}$ are all phases of this equation. For every $\varepsilon \in \mathbf{E}$ we have: $\varepsilon(t + \pi) = \varepsilon(t) + \pi \,.\, \operatorname{sign} \varepsilon'$. If for some $\varepsilon \in \mathbf{E}$, $t_0 \in \mathbf{R}$ and an integer $k : \varepsilon(t_0) = t_0 + k\pi$, then $t + (k - 1)\pi < \varepsilon(t) < t + (k + 1)\pi$ for $t \in \mathbf{R}$.

Let $t_0 \in \mathbf{R}$ and let u be a solution of (q), $u(t_0) = 0$. Let $\varphi(t_0)$ be the first zero of u lying on the right of t_0. Then the function φ is defined on $\mathbf{R}$ and is called the basic central dispersion (of the first kind) of (q). This function has the following properties:

$$\varphi \in C^3_{\mathbf{R}}, \qquad \varphi(t) > t, \qquad \varphi'(t) > 0 \qquad \text{for } t \in \mathbf{R}.$$

$\varphi_n(t)$ denotes the function $\underbrace{\varphi \ldots \varphi}_{n}(t)$ and $\varphi_{-n}(t)$ denotes the inverse function to $\varphi_n(t)$; $\varphi_0(t) \equiv t$ for $t \in \mathbf{R}$. There holds the Abelian relation $\alpha\varphi_n(t) = \alpha(t) + n\pi \,.\, \operatorname{sign} \alpha'$ between every phase α of (q) and the basic central dispersion φ of (q).

The function $X \in C^3_{\mathbf{R}}$, $X' \neq 0$ is called a dispersion (of the 1st kind) of (q) if and only if it is a solution (on $\mathbf{R}$) of the differential equation

$$\text{(qq)} \qquad \sqrt{|X'|}\left(\frac{1}{\sqrt{|X'|}}\right)'' + X'^2 \,.\, q(X) = q(t).$$

Let α be a phase of (q). Then X is a dispersion of (q) exactly if $X = \alpha^{-1}\varepsilon\alpha$ for an $\varepsilon \in \mathbf{E}$. Therefore $\alpha^{-1}\mathbf{E}\alpha := \{\alpha^{-1}\varepsilon\alpha,\ \varepsilon \in \mathbf{E}\}$ is the set of all dispersions of (q). Every dispersion X maps $\mathbf{R}$ onto $\mathbf{R}$ and for every solution u of (q) $\dfrac{uX(t)}{\sqrt{|X'(t)|}}$ is again a solution of this equation. The above definitions and properties are given in [1] and [2].

Let X be a dispersion of (q) and φ its basic central dispersion. By the generalized Floquet theory ([6, 10]) there exist independent solutions u, v of (q) satisfying either

$$\text{(2)} \qquad \frac{uX(t)}{\sqrt{|X'(t)|}} = \varrho_{-1}u(t), \qquad \frac{vX(t)}{\sqrt{|X'(t)|}} = \varrho_1 v(t), \qquad \varrho_{-1}\cdot\varrho_1 = \pm 1$$

or

$$\text{(3)} \qquad \frac{uX(t)}{\sqrt{|X'(t)|}} = \varrho_{-1}u(t), \qquad \frac{vX(t)}{\sqrt{|X'(t)|}} = u(t) + \varrho_1 v(t), \qquad \varrho_{-1} = \varrho_1 = \pm 1$$

(Generally complex) numbers ϱ_{-1}, ϱ_1 are called the characteristic multipliers of (q) relative to the dispersion X (see [10]).

Remark 1. Let u, v be independent solutions of (q) for which (2) holds. Let $u_1(t) := := -u(t)$ for $t \in \mathbf{R}$. Then u_1, v are again independent solutions of (q) satisfying (2), where we write u_1 instead of u. If α is a phase of u, v and α_1 is a phase of u_1, v, then $\operatorname{sign}\alpha' = -\operatorname{sign}\alpha_1'$.

It has been proved in [10] (Theorems 1–3): Let $\operatorname{sign} X' = 1$ and let for an $x \in \mathbf{R}$ and for an integer n be $X(x) = \varphi_n(x)$. Then $(-1)^n \sqrt{\frac{\varphi_n'(x)}{X'(x)}}$ and $(-1)^n \sqrt{\frac{X'(x)}{\varphi_n'(x)}}$ are the characteristic multipliers of (q) relative to the dispersion X. The characteristic multipliers of (q) relative to the dispersion X are complex and equal to $e^{\pm a\pi i}$ $(0 < < a < 1)$ if and only if $\operatorname{sign} X' = 1$ and if

$$\text{(4)} \qquad \alpha X(t) = \alpha(t) + (a + 2n)\,\pi, \qquad (n \text{ is an integer})$$

for a phase α of (q). If $\operatorname{sign} X' = -1$ and $X(x) = x$, then $-\sqrt{-X'(x)}$ and $\frac{1}{\sqrt{-X'(x)}}$ are the characteristic multipliers of (q) relative to the dispersion X.

Definition 1. *We say that the equation* (q) *relative to the dispersion X is of the category* (1, *n*), *where n is an integer, when* $\operatorname{sign} X' = 1$ *and* $X(x) = \varphi_n(x)$ *for any* $x \in \mathbf{R}$. *Let us say that* (q) *relative to the dispersion X is of the category* (2, *n*) *with n being an integer when* $\operatorname{sign} X' = 1$ *and there exists a number a,* $0 < a < 1$ *and a phase α of* (q) *for which* (4) *holds. Finally say that* (q) *relative to the dispersion X is of the category* (3,0) *when* $\operatorname{sign} X' = -1$.

Remark 2. Every equation (q) relative to the dispersion X is precisely of one of the three categories given in Definition 1 as follows from [10]. The definitions of categories (i, n), $i = 1, 2$, are for $X = t + \pi$ identical with those given in [2].

Lemma 1. *Let X be a dispersion of* (q) *and φ be the basic central dispersion of* (q).

Then

a) *the equation* (q) *has two different characteristic multipliers relative to the dispersion* X *and is of the category* $(1, n)$ *if and only if* sign $X' = 1$ *and the function* $X(t) - \varphi_n(t)$ *changes its* sign *on* $\mathbf{R}$,

b) *the equation* (q) *has two equal (real) characteristic multipliers relative to the dispersion* X, *and is of the category* $(1, n)$ *and there exist independent solutions* u, v *of* (q) *for which* (3) *holds precisely if* sign $X' = 1$, $X(t) \not\equiv \varphi_n(t)$ *for* $t \in \mathbf{R}$ *and* $\min_{t\in\mathbf{R}} \tau \, . \, (X(t) - \varphi_n(t)) = 0$, *where* $\tau = \pm 1$,

c) *the equation* (q) *has two equal (real) characteristic multipliers relative to the dispersion* X, *and is of the category* $(1, n)$ *and there exist independent solutions* u, v *of* (q) *for which* (2) *holds precisely if* $X(t) = \varphi_n(t)$ *for* $t \in \mathbf{R}$,

d) *the equation* (q) *relative to the dispersion* X *is of the category* $(2, n)$ *if and only if either* $\varphi_{2n}(t) < X(t) < \varphi_{2n+1}(t)$ *or* $\varphi_{-2n-1}(t) < X(t) < \varphi_{-2n}(t)$ *for* $t \in \mathbf{R}$.

Proof. Lemma 1 immediately follows from Theorem 4 [10].

Definition 2. *We say that* (q_1) *and* (q_2) *relative to the same dispersion* X *have the same behaviour if* 1° *they have the same characteristic multipliers and* 2° *if they are of the same category and* 3° *if* (3) *holds for an appropriate pair of solutions of one of the equations, then it holds for an appropriate pair of solutions of the other equation, too and the Wronskian determinants of both pairs have the same signs.*

Remark 3. In case $X = t + \pi$, the definition of the same behaviour of (q_1) and (q_2) relative to the same dispersion X is identical with the definition of the same behaviour of (q_1) and (q_2) given in [9].

3. THE MAIN RESULT

Lemma 2. *Let* $X \in \mathbf{G}$. *Then* $\mathscr{S}_X: = \{\alpha \in \mathbf{G};\ \alpha X = X^{\operatorname{sign} \alpha'} \alpha\}$ *is a subgroup of the group* $\mathbf{G}$.

Proof. Let $\alpha_1, \alpha_2 \in \mathscr{S}_X, \alpha_1 X = X^{\operatorname{sign} \alpha_1'} \alpha_1, \alpha_2 X = X^{\operatorname{sign} \alpha_2'} \alpha_2$. Then $\alpha_1 \alpha_2 X = \alpha_1 X^{\operatorname{sign} \alpha_2'} \alpha_2 = X^{\operatorname{sign} \alpha_1' . \operatorname{sign} \alpha_2'} \alpha_1 \alpha_2$, $\alpha_1^{-1} X = X^{\operatorname{sign} \alpha_1'} \alpha_1^{-1}$. Hence $\alpha_1 \alpha_2$ and α_1^{-1} are the elements $\mathscr{S}_X$ and $\mathscr{S}_X$ is a subgroup of the group $\mathbf{G}$.

Remark 4. Let $X = t + \pi$. Then $\alpha \in \mathscr{S}_X$ if and only if $\alpha(t + \pi) = \alpha(t) + \pi \, . \operatorname{sign} \alpha'$. In this case $\mathscr{S}_X$ is called the subgroup of the elementary phases (see [1, 2]).

Theorem. *Let* X *be a dispersion of* (q_1) *and* α_1 *be its phase. The equation* (q_2) *has the dispersion* X *and* (q_1) *and* (q_2) *relative to the same dispersion* X *have the same behaviour if and only if any (and then every) phase* α_2 *of* (q_2) *is satisfying*

$$\alpha_2 = \varepsilon \alpha_1 \gamma$$

for any $\varepsilon \in \mathbf{E}$ *and* $\gamma \in \mathscr{S}_X(: = \{\gamma \in \mathbf{G};\ \gamma X = X^{\operatorname{sign} \gamma'} \gamma)$.

Proof. ($\Rightarrow$) Let (q_2) have the dispersion X and (q_1) and (q_2) relative to the same dispersion X have the same behaviour. Let (q_1) be of the category $(1, n)$ relative to the dispersion X and let ϱ_{-1}, ϱ_1 be its characteristic multipliers. Let (2) hold for independent solutions u, v of (q_1) and let α be a phase of the solutions u, v. Then $\alpha X = \varepsilon\alpha$, where $\varepsilon \in \mathbf{E}$ and $\operatorname{tg} \varepsilon(t) = \frac{\varrho_{-1}}{\varrho_1} \operatorname{tg} t$, sign $\varepsilon' = 1$. According to the properties of 1° it holds for some independent solutions u_2, v_2 of (q_2):

$$\frac{u_2X(t)}{\sqrt{|X'(t)|}} = \varrho_{-1}u_2(t),$$

$$(5) \qquad \frac{v_2X(t)}{\sqrt{|X'(t)|}} = \varrho_1 v_2(t), \qquad t \in \mathbf{R}.$$

Let α_2 be a phase of solutions u_2, v_2. It follows from Remark 1 that u_2, v_2 may always be chosen so that sign α' = sign α_2'. Then $\alpha_2 X = \varepsilon\alpha_2 + k\pi$, where k is an integer. Let φ and $\bar{\varphi}$ be the basic central dispersions of (q_1) and (q_2). According to the property of 2° there exist numbers x_1, x_2: $X(x_1) = \varphi_n(x_1)$, $X(x_2) = \bar{\varphi}_n(x_2)$. Then $\alpha X(x_1) = \alpha\varphi_n(x_1) = \alpha(x_1) + n\pi \,.\, \text{sign}\,\alpha' = \varepsilon\alpha(x_1)$, $\alpha_2 X(x_2) = \alpha_2\bar{\varphi}_n(x_2) = \alpha_2(x_2) + n\pi \,.\, \text{sign}\,\alpha_2' = \varepsilon\alpha_2(x_2) + k\pi$ and therefore $\varepsilon\alpha(x_1) = \alpha(x_1) + n\pi \,.\, \text{sign}\,\alpha'$, $\varepsilon\alpha_2(x_2) = \alpha_2(x_2) + (n - k \,.\, \text{sign}\,\alpha_2)\,\pi \,.\, \text{sign}\,\alpha_2'$. It follows from the first equality $t + (n \,.\, \text{sign}\,\alpha' - 1)\,\pi < \varepsilon(t) < t + (n \,.\, \text{sign}\,\alpha' + 1)\,\pi$ for $t \in \mathbf{R}$. Then $\alpha_2(x_2) + (n \,.\, \text{sign}\,\alpha' - 1)\,\pi < \varepsilon\alpha_2(x_2) = \alpha_2(x_2) + (n - k \,.\, \text{sign}\,\alpha_2')\,\pi \,.\, \text{sign}\,\alpha_2' < \alpha_2(x_2) + (n \,.\, \text{sign}\,\alpha' + 1)\,\pi$. This yields $-\pi < -k\pi < \pi$, hence $k = 0$. From $\alpha X = \varepsilon\alpha$, $\alpha_2 X = \varepsilon\alpha_2$ we obtain $\alpha X = \alpha_2 X \alpha_2^{-1}\alpha$, $\alpha^{-1}\alpha_2 X = X\alpha^{-1}\alpha_2$. For $\gamma := \alpha^{-1}\alpha_2$ we have sign $\gamma' = 1$, $\gamma X = X\gamma$, consequently $\gamma \in \mathcal{S}_X$ and $\alpha_2 = \alpha\gamma$. Further α and α_1 are phases of (q_1) thus $\alpha = \varepsilon\alpha_1$ for any $\varepsilon \in \mathbf{E}$ and we have $\alpha_2 = \varepsilon\alpha_1\gamma$.

Let (q_1) relative to the dispersion X be of the category $(1, n)$ and let (3) hold for independent solutions u, v of (q_1). This yields for a phase α of solutions u, v: $\alpha X = \varepsilon\alpha$, where $\varepsilon \in \mathbf{E}$, $\operatorname{tg} \varepsilon(t) = \dfrac{\varrho \operatorname{tg} t}{\varrho + \operatorname{tg} t}$ ($\varrho = \pm 1$). According to the properties of 2° and 3° there exist independent solutions u_2, v_2 of (q_2) satisfying

$$\frac{u_2X(t)}{\sqrt{|X'(t)|}} = \varrho u_2(t),$$

$$\frac{v_2X(t)}{\sqrt{|X'(t)|}} = u_2(t) + \varrho v_2(t), \qquad t \in \mathbf{R},$$

whereby the solutions u, v and u_2, v_2 have the same signs of the Wronskian determinants (i.e. sign $(uv' - u'v)$ = sign $(u_2v_2' - u_2'v_2)$). Let α_2 be a phase of the solutions u_2, v_2. Then sign α' = sign α_2' and $\alpha_2 X = \varepsilon\alpha_2 + k\pi$, where k is an integer. If we proceed in the same manner as we did in the first part of the proof, we find that $k = 0$ and thus $\alpha_2 = \varepsilon\alpha_1\gamma$ for any $\varepsilon \in \mathbf{E}$ and $\gamma \in \mathcal{S}_X$.

Let (q_1) relative to the dispersion X be of the category $(2, n)$ and let $e^{\pm a\pi i}$, $0 < a < 1$ be its characteristic multipliers. Then there exist a phase α of (q_1) and a phase α_2 of (q_2): $\alpha X = \alpha + (a + 2n)\pi$, $\alpha_2 X = \alpha_2 + (a + 2n)\pi$. From this we get $\operatorname{sign} \alpha' = \operatorname{sign} \alpha'_2$ and $\alpha X \alpha^{-1} = \alpha_2 X \alpha_2^{-1}$, $\alpha^{-1}\alpha_2 X = X\alpha^{-1}\alpha_2$. For $\gamma := \alpha^{-1}\alpha_2$ we obtain $\operatorname{sign} \gamma' = 1$ and $\gamma X = X\gamma$, hence $\gamma \in \mathcal{S}_X$ and $\alpha_2 = \varepsilon\alpha_1\gamma$ for any $\varepsilon \in \mathbf{E}$.

Let (q_1) relative to the dispersion X be of the category $(3, 0)$ and let ϱ_{-1}, ϱ_1 be its characteristic multipliers; $\varrho_{-1} \cdot \varrho_1 = -1$ (see [10]). Then there exist independent solutions u, v and u_2, v_2 of (q_1) and (q_2), respectively, satisfying (2) and (5). Let α and α_2 be phases of the solutions u, v and u_2, v_2. By Remark 1 u_2, v_2 may always be chosen so that $\operatorname{sign} \alpha' = \operatorname{sign} \alpha'_2$. Then $\alpha X = \varepsilon\alpha$, $\alpha_2 X = \varepsilon\alpha_2 + k\pi$ with k being an integer, $\varepsilon \in \mathbf{E}$, $\operatorname{tg} \varepsilon(t) = \dfrac{\varrho_{-1}}{\varrho_1} \operatorname{tg} t$. Let $X(x) = x$ be for $x \in \mathbf{R}$. Then $\alpha(x) = \varepsilon\alpha(x)$, $\alpha_2(x) = \varepsilon\alpha_2(x) + k\pi$. It follows now from the first equality: $t - \pi < \varepsilon(t) < t + \pi$ for $t \in \mathbf{R}$. Then $\alpha_2(x) - \pi < \varepsilon\alpha_2(x) = \alpha_2(x) - k\pi < \alpha_2(x) + \pi$, that we get inserting $\alpha_2(x)$ instead of t into the last inequality. From this we get $-\pi < -k\pi < \pi$ and therefore $k = 0$. Then $\alpha X = \varepsilon\alpha$, $\alpha_2 X = \varepsilon\alpha_2$ and we prove in the same way as before that $\alpha_2 = \varepsilon\alpha_1\gamma$ for any $\varepsilon \in \mathbf{E}$ and $\gamma \in \mathcal{S}_X$.

$(\Leftarrow)$ Let $\varepsilon \in \mathbf{E}$, $\gamma \in \mathcal{S}_X$, $\sigma = \operatorname{sign} \gamma'$, $\alpha_2 := \varepsilon\alpha_1\gamma$ be a phase of (q_2) and $X(= \alpha_1^{-1}\varepsilon_1\alpha_1$, $\varepsilon_1 \in \mathbf{E})$ be a dispersion of (q_1). Then $\alpha_2 X = \varepsilon\alpha_1\gamma X = \varepsilon\alpha_1 X^\sigma\gamma = \varepsilon\alpha_1 X^\sigma \alpha_1^{-1}\varepsilon^{-1}\alpha_2 = \varepsilon\alpha_1\alpha_1^{-1}\varepsilon_1^\sigma\alpha_1\alpha_1^{-1}\varepsilon^{-1}\alpha_2 = \varepsilon\varepsilon_1^\sigma\varepsilon^{-1}\alpha_2 = \varepsilon_2\alpha_2$ for any $\varepsilon_2 (:= \varepsilon\varepsilon_1^\sigma\varepsilon^{-1}) \in \mathbf{E}$. Thus X is also a dispersion of (q_2). Let φ and $\bar{\varphi}$ be the basic central dispersions of (q_1) and (q_2). Then $\alpha_2\bar{\varphi} = \varepsilon\alpha_1\gamma\bar{\varphi} = \alpha_2 + \pi \cdot \operatorname{sign} \alpha'_2 = \varepsilon\alpha_1\gamma + \pi \cdot \operatorname{sign} \alpha'_2 = \varepsilon(\alpha_1\gamma + \pi \cdot \operatorname{sign} \alpha'_2 \cdot \operatorname{sign} \varepsilon')$ and $\alpha_1\gamma\bar{\varphi} = \alpha_1\gamma + \pi\sigma \cdot \operatorname{sign} \alpha'_1$, $\alpha_1\gamma\bar{\varphi}\gamma^{-1} = \alpha_1 + \pi\sigma \cdot \operatorname{sign} \alpha'_1 = \alpha_1\varphi_\sigma$. Therefore $\gamma\bar{\varphi}_\sigma\gamma^{-1} = \varphi$ and $\gamma\bar{\varphi}_{n\sigma}\gamma^{-1} = \varphi_n$.

Let (q_1) be relative to the dispersion X of the category $(1, n)$. We have then for any number x_1: $X(x_1) = \varphi_n(x_1)$ and $\alpha_1 X(x_1) = \alpha_1\varphi_n(x_1) = \alpha_1(x_1) + n\pi \cdot \operatorname{sign} \alpha'_1$. For $x_{-1} := \varphi_n(x_1)$ we get $X^{-1}(x_{-1}) = \varphi_{-n}(x_{-1})$. Let $x_2 := \gamma^{-1}(x_\sigma)$. Then $\alpha_2 X(x_2) = \varepsilon\alpha_1\gamma X(x_2) = \varepsilon\alpha_1 X^\sigma\gamma(x_2) = \varepsilon\alpha_1 X^\sigma(x_\sigma) = \varepsilon\alpha_1\varphi_{n\sigma}(x_\sigma) = \varepsilon\alpha_1(x_\sigma) + \sigma n\pi \cdot \operatorname{sign} \varepsilon' \cdot \operatorname{sign} \alpha'_1 = \varepsilon\alpha_1\gamma(x_2) + n\pi \cdot \operatorname{sign} \alpha'_2 = \alpha_2\bar{\varphi}_n(x_2)$, hence $X(x_2) = \bar{\varphi}_n(x_2)$. Next we have

$$\frac{\bar{\varphi}'_n(x_2)}{X'(x_2)} = \frac{\gamma^{-1\prime}\varphi_{n\sigma}\gamma(x_2) \cdot \varphi'_{n\sigma}\gamma(x_2) \cdot \gamma'(x_2)}{X'\gamma^{-1}(x_\sigma)} = \frac{\gamma^{-1\prime}\varphi_{n\sigma}(x_\sigma) \cdot \varphi'_{n\sigma}(x_\sigma)}{\gamma^{-1\prime}(x_\sigma) \cdot X'\gamma^{-1}(x_\sigma)} =$$

$$= \frac{\gamma^{-1\prime}\varphi_{n\sigma}(x_\sigma) \cdot \varphi'_{n\sigma}(x_\sigma)}{(X\gamma^{-1}(t))'_{t=x_\sigma}} = \frac{\gamma^{-1\prime}\varphi_{n\sigma}(x_\sigma) \cdot \varphi'_{n\sigma}(x_\sigma)}{(\gamma^{-1}X^\sigma(t))'_{t=x_\sigma}} = \frac{\gamma^{-1\prime}\varphi_{n\sigma}(x_\sigma) \cdot \varphi'_{n\sigma}(x_\sigma)}{\gamma^{-1\prime}X^\sigma(x_\sigma) \cdot X^{\sigma\prime}(x_\sigma)} =$$

$$= \frac{\gamma^{-1\prime}X^\sigma(x_\sigma) \cdot (\varphi'_n(x_1))^\sigma}{\gamma^{-1\prime}X^\sigma(x_\sigma) \cdot (X'(x_1))^\sigma} = \left[\frac{\varphi'_n(x_1)}{X'(x_1)}\right]^\sigma$$

and (q_1) and (q_2) relative to the dispersion X are of the same category and have the same characteristic multipliers.

Now let there exist independent solutions u, v of (q_1) for which (3) holds. Then for any phase α of $u, v (\varrho = \pm 1)$:

$$\frac{\sin \alpha X(t)}{\sqrt{|\alpha' X(t) \,.\, X'(t)|}} = \varrho \frac{\sin \alpha(t)}{\sqrt{|\alpha'(t)|}},$$
$$\frac{\cos \alpha X(t)}{\sqrt{|\alpha' X(t) \,.\, X'(t)|}} = \frac{\sin \alpha(t)}{\sqrt{|\alpha'(t)|}} + \varrho \frac{\cos \alpha(t)}{\sqrt{|\alpha'(t)|}}.$$

Putting $X(t)$ in place of t in the last formulas and with some modifications we obtain

$$-\frac{\sin \alpha X^{-1}(t)}{\sqrt{|\alpha' X^{-1}(t) \,.\, X^{-1\prime}(t)|}} = -\varrho \frac{\sin \alpha(t)}{\sqrt{|\alpha'(t)|}},$$
$$\frac{\cos \alpha X^{-1}(t)}{\sqrt{|\alpha' X^{-1}(t) \,.\, X^{-1\prime}(t)|}} = -\frac{\sin \alpha(t)}{\sqrt{|\alpha'(t)|}} + \varrho \frac{\cos \alpha(t)}{\sqrt{|\alpha'(t)|}}.$$

Therefore

$$\sigma \frac{\sin \alpha X^{\sigma}(t)}{\sqrt{|\alpha' X^{\sigma}(t) \,.\, X^{\sigma\prime}(t)|}} = \sigma\varrho \frac{\sin \alpha(t)}{\sqrt{|\alpha'(t)|}},$$
$$\frac{\cos \alpha X^{\sigma}(t)}{\sqrt{|\alpha' X^{\sigma}(t) \,.\, X^{\sigma\prime}(t)|}} = \sigma \frac{\sin \alpha(t)}{\sqrt{|\alpha'(t)|}} + \varrho \frac{\cos \alpha(t)}{\sqrt{|\alpha'(t)|}}$$

and on putting $\gamma(t)$ instead of t we get ($X^{\sigma}\gamma = \gamma X$):

$$\sigma \frac{\sin \alpha\gamma X(t)}{\sqrt{|[\alpha\gamma X(t)]'|}} = \sigma\varrho \frac{\sin \alpha\gamma(t)}{\sqrt{|[\alpha\gamma(t)]'|}},$$
$$\frac{\cos \alpha\gamma X(t)}{\sqrt{|[\alpha\gamma X(t)]'|}} = \sigma \frac{\sin \alpha\gamma(t)}{\sqrt{|[\alpha\gamma(t)]'|}} + \varrho \frac{\cos \alpha\gamma(t)}{\sqrt{|[\alpha\gamma(t)]'|}}.$$

Let $\alpha_3 := \alpha\gamma$, $u_2 := \sigma \frac{\sin \alpha_3}{\sqrt{|\alpha_3'|}}$, $v_2 := \frac{\cos \alpha_3}{\sqrt{|\alpha_3'|}}$. Then u_2, v_2 are independent solutions of (q_2) having the phase $\sigma\alpha_2$ and satisfying (3), where we write u_2, v_2 instead of u, v. Since $\operatorname{sign} \alpha' = \operatorname{sign} \sigma\alpha_2'$, the Wronskian determinants of u, v and u_2, v_2 have the same signs.

Let (q_1) be relative to the dispersion X of the category $(2, n)$. Then there exists a phase α of (q_1): $\alpha X = \alpha + (a + 2n)\pi$, with $0 < a < 1$. From this $\alpha X^{-1} = \alpha - (a + 2n)\pi$, hence $\alpha X^{\sigma} = \alpha + \sigma(a + 2n)\pi$. Since $\alpha_4 := \sigma \,.\, \alpha\gamma$ is a phase of (q_2) and $\alpha_4 X = \sigma \,.\, \alpha\gamma X = \sigma \,.\, \alpha X^{\sigma}\gamma = \sigma \,.\, \alpha\gamma + (a + 2n)\pi = \alpha_4 + (a + 2n)\pi$, the equations ($q_1$) and ($q_2$) relative to the dispersion X have the same behaviour.

If (q_1) relative to the dispersion X is of the category (3,0), then $\operatorname{sign} X' = -1$ and since X is also a dispersion of (q_2), this equation relative to the dispersion X is of the category (3,0) and has the same characteristic multipliers as (q_1). Thus both equations relative to the dispersion X have the same behaviour.

Remark 5. Let $X = t + \pi$. Then $\mathscr{S}_X$ is a subgroup of the elementary phases and from the above Theorem follows the Theorem of [9] as a special case.

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