Jiří Rosický

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CATEGORIES OF MODELS OF INFINITARY HORN THEORIES

JIŘÍ ROSICKÝ, Brno

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Our aim is to characterize underlying functors of categories of models of infinitary Horn theories. The characterization is, in fact, an infinitary version of one result of O. Kean ([2], Prop. 1.4.1).

Infinitary Horn theories are theories of a language $L_{\infty,\infty}$. The language $L_{\infty,\infty}$ has a set (possibly empty) of n-ary function symbols for each cardinal number $n \geqq 1$, a set (possibly empty) of n-ary relation symbols for each cardinal number $n \geqq 1$ and a set of constant symbols. Further, we have a proper class V of variables. If n is a cardinal number, then the string $(x_i)_{i \in n}$ of variables will be denoted by x and sometimes x will be identified with a map $x: n \to V$. Terms and atomic formulas are defined as usual. Formulas are built up from atomic formulas by means of a negation, conjunctions $\bigwedge_{i \in I} \varphi_i$, where I can be an arbitrary set and quantifiers $\forall x$, where $x: n \to V$ and n is is an arbitrary cardinal number. Remark that no genuine occurence of a quantifier will appear in our considerations because all formulas will be universal. Concerning infinitary logic consult [1].

An infinitary Horn theory H is a theory of $L_{\infty,\infty}$ whose axioms are all of the form (where we will assume that the following formulas all have their free variables universally quantified in front):

(1) φ where φ is an atomic formula

(2) $\bigwedge_{i \in I} \varphi_i \to \Theta$ where φ_i, $i \in I$ and Θ are atomic formulas.

Let $\mathscr{A}_H$ be the category of all models of a given infinitary Horn theory H (morphisms are homomorphisms, i.e. maps which preserve atomic formulas). Let $U_H : \mathscr{A}_H \to Set$ be the forgetful functor. Our permission of a class of function and relation symbols can cause two inconveniences. The functor U_H need not have a left adjoint and U_H need not be fibre-small (i.e. there can be a proper class of models on the same underlying set). The first inconvenience can be easily excluded syntactically by the assumption that there is only a set of n-ary terms in H for each cardinal number n. Namely, then the algebraic reduct of $\mathscr{A}_H$ (if we consider operations only) is varietal in the sense of [3] and if we endow the free algebra over a set X by the weakest relational structure we get the free $\mathscr{A}_H$-object over X (see [2], 1.6.). The syntactical counterpart of the second inconvenience is not clear and so we adopt the following convention.

Definition: *A fibre-small functor $U : \mathscr{A} \to Set$ will be called a Horn functor if there is an infinitary Horn theory H such that for each cardinal number n there is only a set of n-ary terms and an equivalence $M : \mathscr{A} \to \mathscr{A}_H$ such that $U_H \,.\, M = U$.*

We are going to give a characterization of Horn functors analogous to the characterization of varietal functors from [3]. We say that pushouts preserve onto morphisms if in a pushout

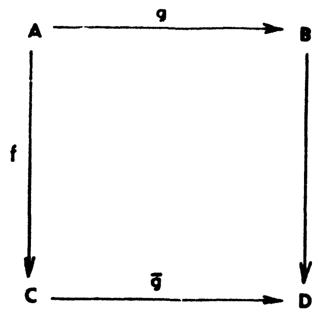

Fig. 1

Uf onto infers that $U\bar{f}$ is onto. This condition implies that U carries coequalizers on epics for

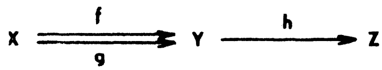

Fig. 2

is a coequalizer iff the following diagram is a pushout

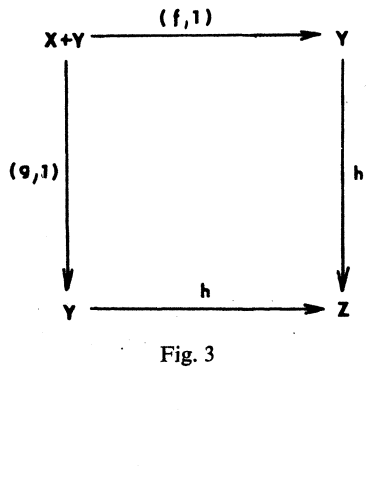

Fig. 3

Theorem: *$U : \mathscr{A} \to Set$ is a Horn functor iff $\mathscr{A}$ is cocomplete and co-well-powered, U is faithful, has a left adjoint and the following conditions hold*
(i) *Pushouts preserve onto morphisms*

(ii) *If $Uf_i : UA_i \to UB_i$ are onto, then $U \sum_i f_i : U \sum_i A_i \to U \sum_i B_i$ is onto.*

Proof: Necessity is a matter of a direct verification. Let U fulfil the mentioned properties. Denote by F a left adjoint of U, by $\varphi = \varphi_{n,A} : \mathscr{A}(Fn, A) \to Set(n, UA)$ the adjunction isomorphism, by $\eta : 1 \to UF$ the unit and by $\varepsilon : FU \to 1$ the counit of the adjunction. Consider the language $L_{\infty,\infty}$ which has morphisms $f : F1 \to Fn$ as n-ary function symbols (constants will be treated as 0-ary function symbols) and morphisms $p : Fn \to X$ such that Up is onto as n-ary relation symbols. If $g : Fn \to Fm$ and $i : 1 \to n$ maps the unique element of 1 on $i \in n$, then the composition $g \, . \, Fi$ will be denoted by g_i. Consider the Horn theory H with the following axioms:

(A1) (a) If $F1 \xrightarrow{f} Fm \xrightarrow{g} Fn$, then

$$(gf)(x) = f(g_1(x), g_2(x), \ldots)$$

(b) If $i : 1 \to n$, then $(Fi)(x) = x_i$.

(A2) If

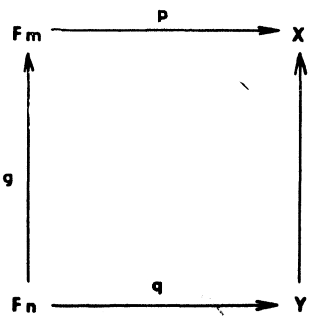

Fig. 4

commutes and Up, Uq are onto, then

(a) $p(x) \to q(xg)$

Moreover, if the square is a pushout, then

(b) $p(x) \leftrightarrow q(xg)$

(A3) If $Fn \overset{f}{\underset{g}{\rightrightarrows}} Fm \xrightarrow{p} X$ is a coequalizer, then

$$p(x) \leftrightarrow \bigwedge_{i \in n} f_i(x) = g_i(x)$$

(A4) If

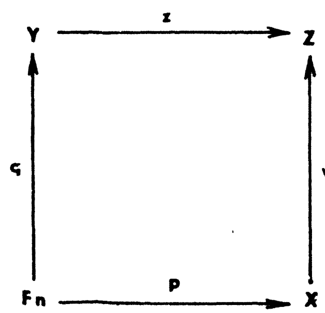

Fig. 5

is a pushout, Up, Uq are onto and $r = w \,.\, p$, then

$$r(x) \leftrightarrow p(x) \wedge q(x).$$

(A5) If I is a set and $p_i : Fn_i \to X_i$, Up_i onto for any $i \in I$ and $u_i : n_i \to \sum_{i \in I} n_i$ are injections, then

$$\left(\sum_{i \in I} p_i\right)(x) \leftrightarrow \bigwedge_{i \in I} p_i(x \,.\, u_i).$$

Here, (i) was used in (A3), (A4) and (ii) in (A5).

Define the functor $M : \mathscr{A} \to \mathscr{A}_H$ as follows. Consider $A \in \mathscr{A}$. Let $M(A)$ have UA as the underlying set, interpret $f : F1 \to Fn$ as

$$f^A : (UA)^n \xrightarrow{\varphi^{-1}} \mathscr{A}(Fn, A) \xrightarrow{\mathscr{A}(f, A)} \mathscr{A}(F1, A) \xrightarrow{\varphi} UA$$

and interpret $p : Fn \to X$, Up onto as the n-ary relation p^A on UA such that $p^A = \{a : n \to UA / \text{there is } g : X \to A \text{ such that } \varphi(g \,.\, p) = a\}$. It is easy to verify that $M(A)$ is a model of H. Clearly $Uh : M(A) \to M(B)$ carries a homomorphism of models for any $h : A \to B$. Thus M is a functor. M is faithful and we will show that it is full. Consider a homomorphism $h : M(A) \to M(B)$ of models. Since $\varepsilon_A^{MA}(1_{UA})$ holds, we get $\varepsilon_A^{MB}(U_H h)$. Thus there is $g : A \to B$ such that $U_H h = \varphi(g \,.\, \varepsilon_A) = Ug$. Hence $h = M(g)$ and M is full. It remains to show that M is an equivalence, i.e. that any $C \in \mathscr{A}_H$ is isomorphic to $M(A)$ for some $A \in \mathscr{A}$.

Let $C \in \mathscr{A}_H$ and denote by p^C the interpretation of in C for each relation symbol p. The map $U\varepsilon_A$ is onto for each $A \in \mathscr{A}$ because $U\varepsilon \,.\, \eta U = 1$ (see [4] p. 80) and thus we may put $\bar{C}(A) = (\varepsilon_A)^C$. By (A2) (a) applied to the square $\varepsilon_B \,.\, FUf = f \,.\, \varepsilon_A$ we get that $Set(Uf, U_H C)$ induces a map $\bar{C}(f) : \bar{C}(B) \to \bar{C}(A)$ for any $f : A \to B$ in $\mathscr{A}$. Hence we get a functor $\bar{C} : \mathscr{A}^{\mathrm{op}} \to Set$.

Let $A = \sum_{i \in I} A_i$ in $\mathscr{A}$, $t_i : A_i \to A$ be injections and denote by $k : \sum_i UA_i \to UA$ the canonical map. Let $e : FUA \to E$ be the coequalizer of $FUF\Sigma UA_i \underset{\varepsilon_{FUA} . FUFK}{\overset{FU\varepsilon_A . FUFk}{\rightrightarrows}}$ FUA. Since ε_A equalizes $FU\varepsilon_A$, ε_{FUA}, there is a unique morphism $v : E \to A$ such that

$v . e = \varepsilon_A$. Hence $v . e . Fk = \varepsilon_A . Fk = \Sigma\varepsilon_{A_i}$. Let the left square in the following diagram be a pushout and $\bar{u}$ be the unique morphism such that $\bar{u} . \bar{v} = v . e$ and $\bar{u} . u = 1$.

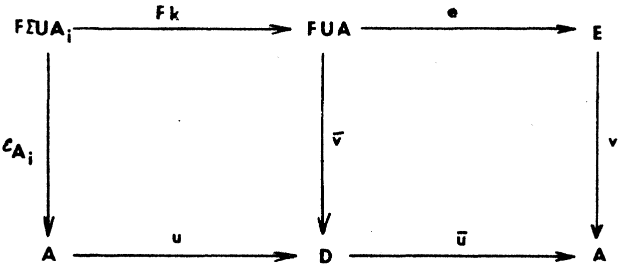

Fig. 6

Then the outer rectangle is a pushout. Namely, we have to prove that $r . e . Fk = s . \Sigma\varepsilon_{A_i}$ implies $s . v = r$. But it follows from $r . e . FU\Sigma\varepsilon_{A_i} = r . e . FU\varepsilon_A . FUFk = r . e . \varepsilon_{FUA} . FUFk = r . e . Fk . \varepsilon_{F\Sigma UA_i} = s . \Sigma\varepsilon_{A_i} . \varepsilon_{F\Sigma UA_i} = s . v . e . Fk . \varepsilon_{F\Sigma UA_i} = s . v . e . \varepsilon_{FUA} . FUFk = s . v . e . FU\varepsilon_A . FUFk = s . v . e . FU\Sigma\varepsilon_{A_i}$ because $FU\Sigma\varepsilon_{A_i}$ is epi by (ii). Hence the right square is a pushout. Following (A2) (b), (A5), (A4) and (A3) we have that $\bar{v}(x) \leftrightarrow (\Sigma\varepsilon_{A_i})(xk) \leftrightarrow \bigwedge_{i\in I} \varepsilon_{A_i}(x . Ut_i)$, $\varepsilon_A(x) \leftrightarrow e(x) \wedge \bar{v}(x)$ and $e(x) \leftrightarrow \bigwedge_{j\in UF\Sigma UA_i} x_{U\Sigma\varepsilon_{A_i}}(j) = \varphi^{-1}(j)(k . x)$ because $(FU(\varepsilon_A . Fk))_j(x) = x_{U\Sigma\varepsilon_{A_i}}(j)$ and $(\varepsilon_{FUA} . FUFk)_j(x) = (Fk . \varepsilon_{F\Sigma UA_i})_j(x) = (\varepsilon_{F\Sigma UA_i})_j(x . k) = \varphi^{-1}(j)(x . k)$. Hence

$$(1) \qquad \varepsilon_A(x) \leftrightarrow (\bigwedge_{i\in I} \varepsilon_{A_i}(x . Ut_i) \wedge \bigwedge_{j\in UF\Sigma UA_i} x_{U\Sigma\varepsilon_{A_i}}(j) = \varphi^{-1}(j)(x . k))$$

Consider the canonical map $t : \bar{C}(A) \to \prod_{i\in I} \bar{C}(A_i)$ which is given by $t(c) = \langle c . Ut_i\rangle_{i\in I}$ for any $c : UA \to U_HC$ from $\bar{C}(A)$. Since $U\Sigma\varepsilon_{A_i}$ is onto, $r = U\Sigma\varepsilon_{A_i}(j)$ for any $r \in UA$. Following (1) $c_r = \varphi^{-1}(j)(c . k)$ for any $c \in \bar{C}(A)$. Hence t is injective. Let $\langle c^i\rangle_i \in \prod_i \bar{C}(A_i)$ and let $\bar{c} : \Sigma UA_i \to U_HC$ be determined by c^i. By (A5) $(\Sigma\varepsilon_{A_i})^C(\bar{c})$ holds. Let $j_1, j_2 \in UF\Sigma UA_i$ and $U\Sigma\varepsilon_{A_i}(j_1) = U\Sigma\varepsilon_{A_i}(j_2)$. Then $(\Sigma\varepsilon_{A_i}) . \varphi^{-1}(j_1) = (\Sigma\varepsilon_{A_i}) . \varepsilon_{F\Sigma UA_i} . Fj_1 = \varepsilon_A . FU\Sigma\varepsilon_{A_i} . Fj_1 = (\Sigma\varepsilon_{A_i}) . \varphi^{-1}(j_2)$. Let p be a coequalizer of $\varphi^{-1}(j_1), \varphi^{-1}(j_2)$. Since $\Sigma\varepsilon_{A_i}$ can be factorized through e, $p^C(\bar{c})$ holds by (A2) (a) and $\varphi^{-1}(j_1)(\bar{c}) = \varphi^{-1}(j_2)(\bar{c})$ by (A3). Hence $c_r = \varphi^{-1}(j)(\bar{c})$, where $r = U\Sigma\varepsilon_{A_i}(j)$ defines $c : UA \to U_HC$ and $c \in \bar{C}(A)$ by (1). Thus t is bijective and C preserves products.

Let $A \overset{f}{\underset{g}{\rightrightarrows}} B \overset{e}{\to} D$ be a coequalizer diagram in $\mathscr{A}$. Since Ue is epi, the canonical map t from $\bar{C}(D)$ into an equalizer of $\bar{C}(B) \overset{\bar{C}(f)}{\underset{\bar{C}(g)}{\rightrightarrows}} \bar{C}(A)$ is injective. We will prove that it is onto. Let $y : UB \to U_HC \in \bar{C}(B)$ and $y . Uf = y . Ug$. Let $h : UB \to E$ be an equalizer of Uf, Ug and $k : E \to UD$ be the unique map such that $k \,.\, h = Ue$. We are going to show that the following square is a pushout

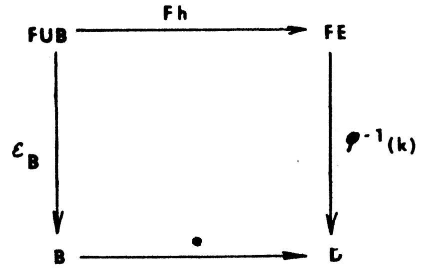

Fig. 7

Consider $u : B \to X$ and $v : FE \to X$ with $u \,.\, \varepsilon_B = v \,.\, Fh$. It holds $u \,.\, f \,.\, \varepsilon_A = u \,.\, \varepsilon_B \,.\, FUf = v \,.\, Fh \,.\, FUf = v \,.\, Fh \,.\, FUg = u \,.\, g \,.\, \varepsilon_A$ and thus $u \,.\, f = u \,.\, g$. There is a unique $r : D \to X$ such that $r \,.\, e = u$. Further $r \,.\, \varphi^{-1}(k) \,.\, Fh = r \,.\, \varepsilon_D \,.\, F(k \,.\, h) = = r \,.\, \varepsilon_D \,.\, FUe = r \,.\, e \,.\, \varepsilon_B = u \,.\, \varepsilon_B = v \,.\, Fh$ and thus $r \,.\, \varphi^{-1}(k) = v$ because Fh is epi. By (A4)

(2) $$(e \,.\, \varepsilon_B)\,(y) \leftrightarrow (\varepsilon_B(y) \wedge (Fh)\,(y))$$

Since Fh is a coequalizer of FUf, FUg, (A3) implies that

(3) $$F(h)\,(y) \leftrightarrow \bigwedge_{i \in UA} y_{Uf(i)} = y_{Ug(i)}$$

Since we have supposed that $(\varepsilon_B)^C\,(y)$ and $y \,.\, Uf = y \,.\, Ug$, we get by (2) and (3) that $(e \,.\, \varepsilon_B)^C\,(y)$ and hence $(FUe)^C\,(y)$ holds following (A2) (a). Further, Ue is a coequalizer of its kernel pair $r, s : Z \to UB$. Thus FUe is a coequalizer of Fr, Fs and by (A3)

$$(FUe)\,(y) \leftrightarrow \bigwedge_{i \in Z} y_{r(i)} = y_{s(i)}$$

Hence $y \,.\, r = y \,.\, s$ and there is a unique $x : UD \to U_H C$ such that $x \,.\, Ue = y$. The following rectangle is a pushout because we have proved that the left square is a pushout and the right square is a pushout for Fk is

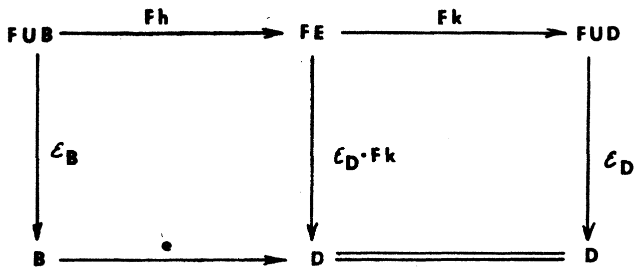

Fig. 8

epi. By (A2) (b) $(\varepsilon_B)^C\,(x \,.\, Ue) \leftrightarrow (\varepsilon_D)^C\,(x)$. Hence $y = t(x)$.

We have proved that $\bar{C}$ preserves limits. Since $\mathscr{A}^{op}$ is complete, well-powered and $F1$ is its cogenerator, $\bar{C}$ is representable by the Freyd's theorem (see [4], p. 126).

Denote by $N(C)$ a representing object and by $\zeta : \mathscr{A}(-, N(C)) \to \bar{C}$ a representing isomorphism. We will prove that $C \cong MN(C)$. Namely, we will show that the following mapping carries the isomorphism of models $MN(C) \to C$.

$$\alpha : UN(C) \xrightarrow{\varphi^{-1}} \mathscr{A}(F1, N(C)) \xrightarrow{\zeta} \bar{C}(F1) \xrightarrow{(U_H C)^{\eta_1}} U_H C$$

Clearly ε_{Fn} is a coequalizer of 1_{FUFn}, $F\eta_n \cdot \varepsilon_{Fn}$ for any set n. By (A3)

$$\varepsilon_{Fn}(x) \leftrightarrow \bigwedge_{i \in UFn} x_i = \varphi^{-1}(i)\,(x \cdot \eta_n) \tag{4}$$

for $(F\eta_n \cdot \varepsilon_{Fn})_i(x) = (F\eta_n \cdot \varphi^{-1}(\text{i}))\,(x) = \varphi^{-1}(i)\,(x \cdot \eta_n)$. Hence $(U_H C)^{\eta_1} : \bar{C}(F1) \to$ $\to U_H C$ is bijective and therefore α is bijective. Let $f : F1 \to Fn$ be an n-ary function symbol. We denote by f^D the interpretation of f in a model D of H. The diagram

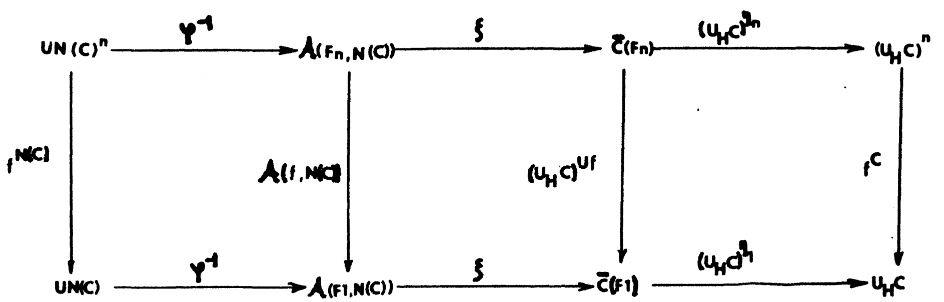

Fig. 9

commutes by the definition of $f^{N(C)}$, the naturality of ζ and by (4) because $f^C(c \cdot \eta_n) =$ $= c_{\varphi(f)} = c_{Uf \cdot \eta_1}$ for any $c : UFn \to U_H C \in \bar{C}(Fn)$. Hence α preserves f because for any $x : n \to UN(C)$ and $i \in n$ it holds $\alpha^n(x)\,(i) = \alpha(x \cdot i) = \zeta(\varphi^{-1}(x \cdot i)) \cdot \eta_1 =$ $= \zeta(\varphi^{-1}(x) \cdot Fi) \cdot \eta_1 = \zeta(\varphi^{-1}(x)) \cdot UFi \cdot \eta_1 = (\zeta(\varphi^{-1}(x) \cdot \eta_n))\,(i)$.

Let $p : Fn \to X$ be an n-ary relation symbol and consider $a : n \to UN(C)$. Let $p^{N(C)}(a)$ hold. Then there is $g : X \to N(C)$ such that $\varphi(g \cdot p) = a$. Further, $\alpha^n(a) =$ $= \zeta(\varphi^{-1}(a)) \cdot \eta_n = \zeta(g \cdot p) \cdot \eta_n = \zeta(g) \cdot Up \cdot \eta_n$. Since $\varepsilon_X \cdot F(Up \cdot \eta_n) = p$, following (A2) (a) $\xi_X(x) \to p(x \cdot Up \cdot \eta_n)$. Since $(\varepsilon_x)^C\,(\zeta(g))$, we have $p^C(\alpha^n(a))$.

Let the both squares in the following diagram be pushouts

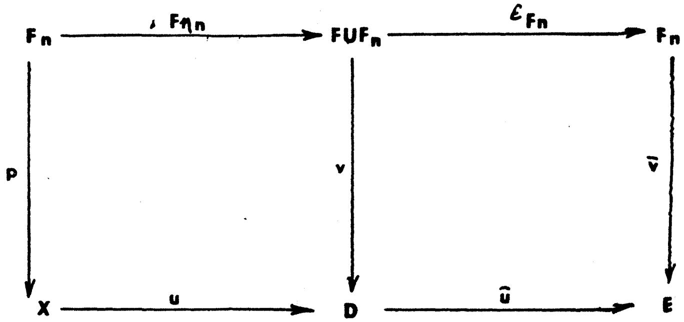

Fig. 10

Then the outer rectangle is a pushout and since the top row is equal to 1_{Fn}, one gets that $\bar{v} = p$. Hence $(p(x \,.\, \eta_n) \wedge \varepsilon_{Fn}(x)) \leftrightarrow v(x) \wedge \varepsilon_{Fn}(x) \leftrightarrow (p \,.\, \varepsilon_{Fn})\,(x) \rightarrow (FUp)\,(x)$.

Let $p^C(\alpha^n(a))$. Then $p^C((\zeta(\varphi^{-1}(a)) \,.\, \eta_n)$ and $\varepsilon^C_{Fn}(\zeta(\varphi^{-1}(a)))$. Therefore $(FUp)^C$ $(\zeta(\varphi^{-1}(a))$. In the same way as in the proof that $\bar{C}$ preserves equalizers it can be shown that there is $b : UX \rightarrow U_H C$ such that $b \,.\, Up = \zeta(\varphi^{-1}(a))$. Now,

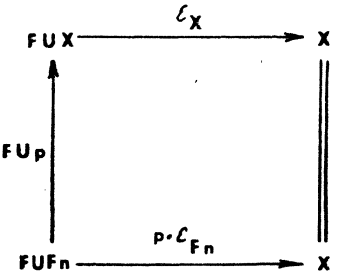

Fig. 11

is a pushout because $u \,.\, p \,.\, \varepsilon_{Fn} = v \,.\, FUp$ implies $u \,.\, \varepsilon_X \,.\, FUp = u \,.\, p \,.\, \varepsilon_{Fn} = v \,.\, FUp$. Hence $\varepsilon_X(x) \leftrightarrow (p \,.\, \varepsilon_{Fn})\,(x \,.\, Up)$. All these facts together yield $\varepsilon^C_X(b)$. Finally, $\varphi^{-1}(a) = \zeta^{-1}(b \,.\, Up) = \zeta^{-1}(b) \,.\, p$ and $p^{N(C)}(a)$ is true.

We have proved that α carries an isomorphism and thus M is an equivalence.

To compare the just proved Theorem with Prop. 1.4.1 of [2] we remark that $\mathscr{A}^{\mathrm{op}}$ plays a role of Kean's abstract Horn theory with $F1$ as its M and onto morphism as its monics. We gave a complete proof of the Theorem for the proof is only sketched in [2]. The associated Horn theory H in our paper differs slightly from that one of the paper [2]. The reason for this change is the fact that the author was unable to succeed with the Kean's original H_τ.

Our Theorem shows that topological spaces are given by an infinitary Horn theory It would be useful to find a convenient presentation of it.

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J. Rosický
662 95 Brno, Janáčkovo nám. 2a
Czechoslovakia