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Jiří Barot

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A MULTIPLICATIVE FUNCTIONAL ON THE COMMUTATIVE TOPOLOGICAL FIELD

JIŘÍ BAROT, Brno

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The aim of this paper is to determine some conditions for the commutative topological field in order the functional F on T to be multiplicative.

Definition. A set T of elements $x, y, z, \dots$ is called a topological field, if:

1) T is a complex linear space with the identity element e, in which the operation xy is defined such that

$$x(y + z) = xy + xz; \qquad (xy)\,z = x(yz); \qquad xe = ex = x.$$

2) T is a topological field, where

a) the mapping $(\lambda, x) \to \lambda x$ of the product $C \times T$ into T is continuous, C is the set of complex numbers

b) the mapping $(x, y) \to x + y$ of the product $T \times T$ into T is continuous

c) the mapping $(x, y) \to xy$ of the product $T \times T$ into T is continuous

d) the mapping $x \to x^{-1}$ of the space $T \backslash \{0\}$ into T is continuous.

Lemma. [1]. *Let P be a complex topological linear space, $x_0 \in P$, $x_0 \neq 0$. Then there exists on P a continuous linear functional f, $f(x_0) \neq 0$ if and only if $x_0 \in P$ has a convex neighbourhoud not containing the zero element.*

Theorem. *Let T be a commutative topological field, $x_0 \in T$, $x_0 \neq 0$ has a convex neighbourhood not containing the zero element of T. Then there exists a multiplicative continuous linear functional F such that $F(x_0) \neq 0$.*

Proof. With respect to lemma, there is defined on T a continuous linear functional $f: T \to C$ such that $f(x_0) \neq 0$. Let us consider the inverse element $x_0^{-1} \in T$ and $x_0^{-1} - \lambda e \in T$, $\lambda \in C$. Suppose that $x_0^{-1} - \lambda e \neq 0$ for every $\lambda \in C$. Then for every λ there exists $(x_0^{-1} - \lambda e)^{-1}$. Let us show that the function $G: C \to C$, $G(\lambda) = f[u(\lambda)] = f[(x_0^{-1} - \lambda e)^{-1}]$ satisfies the following two conditions.

1°. The function $G: C \to C$ is analytic in the whole Gaussian plane.

Indeed, $\mathrm{d}G(\lambda)/\mathrm{d}\lambda = \lim_{h \to 0} [G(\lambda + h) - G(\lambda)]/h = \lim_{h \to 0} \{f[u(\lambda + h)] - f[u(\lambda)]\}/h =$

$= \lim_{h\to 0} f\{[u(\lambda + h) - u(\lambda)]/h\} = \lim_{h\to 0} f\{[u(\lambda + h)\, u(\lambda)]\, [u^{-1}(\lambda) - u^{-1}(\lambda + h)]/h\} =$
$= \lim_{h\to 0} f\{[(x_0^{-1} - (\lambda + h)\, e]^{-1}\, [x_0^{-1} - \lambda e]^{-1}\, (he)/h\} = f[(x_0^{-1} - \lambda e)^{-2}]$ and
$\lim_{\lambda\to\infty} f[(x_0^{-1} - \lambda e)^{-2}] = \lim_{\lambda\to\infty} (1/\lambda^2) f[(x_0^{-1}/\lambda - e)^{-2}] = 0.$

2°. The function $G : C \to C$ is bounded in the whole Gaussian plane.

Indeed, for every $|\lambda| > N$ there is $|G(\lambda)| = |f[u(\lambda)]| = |f[(x_0^{-1} - \lambda e)^{-1}]| = (1/|\lambda|)\, |f[(x_0^{-1}/\lambda - e)^{-1}]| < (1/N)\, |f[(x_0^{-1}/\lambda - e)^{-1}]| < (1/N)\, |f(-e)|$ and $\lim_{\lambda\to\lambda_0} f[(x_0^{-1} - \lambda e)^{-1}] = f[(x_0^{-1} - \lambda_0 e)^{-1}]$. Thus the function $G : C \to C$, $G(\lambda) = f[(x_0^{-1} - \lambda e)^{-1}]$ is bounded in the neighbourhood of an arbitrary radius of the point λ_0.

In accordance with Liouville's theorem (from the theory of functions of a complex variable) holds the assertion that $f[(x_0^{-1} - \lambda e)^{-1}]$ is a constant for every λ.

This constant is zero, since

$$\lim_{\lambda\to\infty} f[(x_0^{-1} - \lambda e)^{-1}] = \lim_{\lambda\to\infty} (1/\lambda) f[(x_0^{-1}/\lambda - e)^{-1}] = 0.$$

Thus for $\lambda = 0$ we get $f[u(0)] = f(x_0) = 0$, which is the contradiction.

It follows that $\lambda_0^{-1} \in C$ exists such that

$$x_0^{-1} - \lambda_0^{-1} e = 0 \qquad \text{i.e.} \qquad x_0 = \lambda_0 e.$$

From equalities $x = \lambda_0 e$, $y = \mu_0 e$ we get $f(x) = \lambda_0 f(e)$, $f(y) = \mu_0 f(e)$, $f(xy) = (\lambda_0 \mu_0) f(e)$, $f(x) f(y) = (\lambda_0 \mu_0) f^2(e)$ and $f(x) f(y) = f(e) f(xy)$. Finally, denoting $\alpha = 1/f(e)$, $F = \alpha f$ we see that $F(xy) = F(x)\, F(y)$ i.e. the functional $F = \alpha f$ is multiplicative.

Corollary. *Let R be a complex commutative topological ring with the identity element e, $M \subset R$, $M \neq R$* the *maximal ideal. Let $K_0 \subset R$ be a nonempty, open and convex set, which does not contain the elements of the ideal $M \subset R$. Then there exists a homorphic mapping $\varphi : R \to C$.*

Proof. Obviously, R/M is the commutative topological fields and the image of $K_0 \subset R$ in the canonical mapping of the topological ring R onto R/M is a nonempty, open and convex set in R/M which does not contain the zero element of R/M. The assertion follows from Theorem above.

REFERENCES

[1] W. Nef: *Invariante Linearformen,* Mathematische Nachrichten 28, (1964), Satz 28, 123—140.
It is easy to see that the assertion is true for complex topological space, too.

J. Barot
Brno 616 00, Vrázova 33
Czechoslovakia