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Pavol Šoltés

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OSCILLATION OF SOLUTIONS OF A NON-LINEAR DELAY DIFFERENTIAL EQUATION OF THE FOURTH ORDER

PAVEL ŠOLTÉS, Košice

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The papers [3], [4] and [5] investigate the properties of solutions of third and fourth order differential equations without argument delay. Certain results of these papers were extended and generalized in [1] and [2]; the latter papers are concerned with investigating the properties of solutions of the non-linear differential equation

$$(1) \qquad (\varrho(x)\, y''')' + p(x)\, y'' + q(x)\, y' + r(x)\, y + y(x) \sum_{i=1}^{n} Q_i(x)\, F_i(y(h_i(x))) = g(x)$$

with $\varrho(x) \equiv 1$.

The present paper contains results which are an extension and generalization of certain results of these papers, especially [2] and [6].

We shall assume throughout that the functions $\varrho(x) > 0$, $\varrho'(x) \geqq 0$, $\varrho''(x) \leqq 0$, $p(x)$, $q(x) \geqq 0$, $r(x)$, $g(x)$, $Q_i(x)$ and $h_i(x)$ $(i = 1, 2, \ldots, n)$ are from $C_0(J)$ where $J = \langle x_0, \infty)$, n is a natural number.

Suppose further that

$$\inf_{x \in J} [x - h_i(x)] \geqq d > 0, \qquad h_i(x) \to +\infty \quad \text{as} \quad x \to \infty,$$
$$F_i(z) \in C_0(-\infty, \infty), \qquad F_i(z) \geqq 0 \qquad \text{for } i = 1, 2, \ldots, n.$$

We can now define the initial problem: let $\Phi(x)$ be defined and continuous on the initial set

$$E_{x_0} = \bigcup_{i=1}^{n} E^i_{x_0}, \qquad E^i_{x_0} = \langle \inf h_i(x), x_0 \rangle$$

and let $y_0^{(k)}$, $k = 1, 2, 3$ be arbitrary real numbers. We want to find a solution $y(x)$ of (1) defined on J satisfying the initial conditions:

$$(2) \qquad y(x_0) = \Phi(x_0) = y_0, \qquad y^{(k)}(x_0 + 0) = y_0^{(k)}, \qquad k = 1, 2, 3$$
$$y(x) = \Phi(x) \qquad \text{for } x \in E_{x_0}.$$

We have the following theorem:

Theorem 1. *Suppose that $q(x) \in C_1(J)$ and that for every $x \in J$*

$$2\varrho(x) - |p(x)| \geqq 0, \qquad 2r(x) - |p(x)| - q'(x) - |g(x)| \geqq 0,$$
$$Q_i(x) \geqq 0, \qquad i = 1, 2, \ldots, n.$$

If

$$\int_{x_0}^{\infty} \frac{q(s)}{\varrho(s)} \, ds = +\infty, \tag{3}$$

then any solution $y(x)$ satisfying (2) and such that

$$H(y(x_0)) + \frac{1}{2} \int_{x_0}^{\infty} |g(s)| \, ds \leqq K_0 \leqq 0, \tag{4}$$

where $H(y(x)) = \varrho(x)\, y(x)\, y'''(x) - \varrho(x)\, y'(x)\, y''(x) + \frac{1}{2} \varrho'(x)\, y'^2(x) + \frac{1}{2} q(x)\, y^2(x)$ *is oscillatory on J.*

Proof. Let $y(x)$ be a solution of (1) and (2) satisfying (4) which is not oscillatory. This means that e.g. $y(x) > 0$ for all $x \geqq x_1 \geqq x_0$.

Multiplying (1) by $y(x)$ and integrating from x_0 to $x \geqq x_0$, we get after some manipulations

$$\begin{aligned} H(y(x)) &+ \int_{x_0}^{x} \left[\varrho(s) - \frac{1}{2} |p(s)|\right] y''^2(s) \, ds + \\ &+ \int_{x_0}^{x} \left[r(s) - \frac{1}{2} |p(s)| - \frac{1}{2} q'(s)\right] y^2(s) \, ds - \\ &- \frac{1}{2} \int_{x_0}^{x} \varrho''(s)\, y'^2(s) \, ds + \sum_{i=1}^{n} \int_{x_0}^{x} y^2(s)\, Q_i(s)\, F_i(y(h_i(s))) \, ds \leqq \\ &\leqq H(y(x_0)) + \int_{x_0}^{x} |g(s)| \, |y(s)| \, ds \end{aligned} \tag{5}$$

and thus

$$\begin{aligned} H(y(x)) &+ \int_{x_0}^{x} \left[r(s) - \frac{1}{2} |p(s)| - \frac{1}{2} q'(s) - \frac{1}{2} |g(s)|\right] y^2(s) \, ds \leqq \\ &\leqq H(y(x_0)) + \frac{1}{2} \int_{x_0}^{x} |g(s)| \, ds \leqq K_0 \leqq 0. \end{aligned} \tag{6}$$

For $x \geqq x_1$, from (6) we have

$$\frac{d}{dx}\left[\frac{y''(x)}{y(x)}\right] \leqq -\frac{1}{2} \frac{q(x)}{\varrho(x)} \tag{7}$$

and therefore, owing to (3), $\frac{y''(x)}{y(x)} \to -\infty$ as $x \to \infty$ so that there exists a number $x_2 \geqq x_1$ such that for every $x \geqq x_2$ is $y''(x) < 0$ and therefore $y'(x)$ decreases on $\langle x_2, \infty)$. Therefore one of the following statements must hold:

1. $y'(x) \geqq 0$ for all $x \geqq x_2$.

2. There exists $x_3 \geqq x_2$ such that $y'(x) < 0$ for all $x \geqq x_3$.

Evidently 2 contradicts the assumption that $y(x) \geqq 0$ for $x > x_1$.

Suppose therefore that $y(x) > 0$, $y'(x) > 0$, $y''(x) < 0$. Then, owing to (7)

$$\frac{y''(x)}{y(x_2)} \leqq \frac{y''(x)}{y(x)} \leqq \frac{y''(x_2)}{y(x_2)} - \frac{1}{2}\int_{x_2}^{x} \frac{q(s)}{\varrho(s)}\, ds$$

and thus $y''(x) \to -\infty$ as $x \to \infty$ which contradicts the assumption $y'(x) \geqq 0$.

If we assume $y(x) < 0$, the proof is analogous.

Remark 1. Theorem 1 is a generalization of Theorem 3 of [2] and Theorem 3 of [6]. It is evident from the proof of Theorem 1 of [2] that it is not enough to assume that $F_i(z)$, $i = 1, 2, \ldots, n$ are increasing functions. Some additional hypothesis is needed, e.g. that for all $i = 1, 2, \ldots, n$

(8) $F_i(z)$ decreases on $(-\infty, 0)$ and increases on $(0, \infty)$ or for all $i = 1, 2, \ldots, n$,

(9) $$\inf_{\delta < |z| < \infty} F_i(z) = F_{i\delta} > 0 \qquad \text{for every } \delta > 0.$$

In that case it is possible to generalize Theorem 1 of [2]. We obtain the following two theorems:

Theorem 2. *The hypotheses of this theorem are the same as those of Theorem 1 with* (3) *replaced by the condition*

(10) $$\int_{x_0}^{\infty} \frac{1}{\varrho(s)}\, ds = +\infty, \qquad \int_{x_0}^{\infty} Q_i(s)\, ds = +\infty$$

holds true at least for one $i = 1, 2, \ldots, n$.

In addition, suppose that for $x \in J$ $p(x) \geqq 0$, $r(x) \geqq 0$ *and that* $F_i(z)$ *satisfy the condition* (8). *Then every solution* $y(x)$ *of* (1) *and* (2) *satisfying* (4) *with* $K_0 < 0$ *is oscillatory on* J.

Proof. Suppose that a solution $y(x)$ of (1) satisfies (2) and (4) and that e.g. $y(x) > 0$ for every $x \geqq x_1 \geqq x_0$. From (7) we see that $\dfrac{y''(x)}{y(x)}$ is nonincreasing on $\langle x_1, \infty)$ and therefore one of the following statements must hold:

1. $y''(x) > 0$ for all $x > x_1$.

2. There exists $x_2 \geqq x_1$ such that for every $x \geqq x_2$ holds $y''(x) < 0$.

Suppose that 1. holds. Then $y'(x)$ is nondecreasing and therefore one of the following statements must hold:

α) $y'(x) \leqq 0$ for all $x \geqq x_1$.

β) There exists $x_2 \geqq x_1$ such that for all $x \geqq x_2$ $y'(x) > 0$.

For the case α) we can prove from (6) that

$$\varrho(x)\,(y(x)\,y'''(x) - y'(x)\,y''(x)) \leqq K_0 < 0$$

and therefore

$$y'''(x) \leqq \frac{K_0}{\varrho(x)\, y(x)} \leqq \frac{K_0}{y(x_1)\, \varrho(x)}$$

so that $y''(x) \to -\infty$ as $x \to \infty$ which is a contradiction. Therefore $y(x) > 0$, $y'(x) > 0$, $y''(x) \geqq 0$ for each $x \geqq x_2$. Considering the hypotheses, we derive from (1)

$$(\varrho(x)\, y'''(x))' + y(x_2) \sum_{i=1}^{n} F_i(y(x_2))\, Q_i(x) \leqq g(x) \qquad \text{for } x \geqq \bar{x}_2,$$

where $\bar{x}_2 \geqq x_2$ is large enough, and thus $\varrho(x)\, y'''(x) \to \infty$ as $x \to -\infty$, again a contradiction.

Suppose now that statement 2° holds. Then necessarily $y(x) > 0$, $y'(x) \geqq 0$, $y''(x) < 0$. Owing to (7), we have therefore

$$\frac{y''(x)}{y(x_2)} \leqq \frac{y''(x)}{y(x)} \leqq \frac{y''(x_2)}{y(x_2)}$$

and therefore $y'(x) \to -\infty$ as $x \to \infty$, a contradiction.

The proof is analogous if we assume that $y(x) < 0$.

Analogously we can prove

Theorem 3. *The hypotheses are the same as those of Theorem* 2 *with* (8) *replaced by* (9). *Then every solution $y(x)$ of* (1) *satisfying* (4) *with $K_0 < 0$ is oscillatory on J.*

Theorem 4. *The hypotheses are the same as those of Theorem* 1 *with* (3) *replaced by*

$$\int_{x_0}^{\infty} \frac{ds}{\varrho(s)} = \int_{x_0}^{\infty} r(s)\, ds = +\infty. \tag{11}$$

Suppose in addition that $p(x) \geqq 0$ and $r(x) \geqq 0$ on J. Then any solution $y(x)$ of (1) *satisfying* (2) *and* (4) *with $K_0 < 0$ is oscillatory on J.*

Proof. Suppose that $y(x)$ satisfies (1), (2) and (4) and is not oscillatory. It is evident from the proofs of previous theorems that it is sufficient to investigate the case

$$y(x) > 0, \qquad y'(x) > 0, \qquad y''(x) \geqq 0.$$

From (1) we get for $x \geqq x_2$

$$(\varrho(x)\, y'''(x))' + y(x_2)\, r(x) \leqq g(x)$$

and from this we derive a contradiction analogously as in the proof of Theorem 2,

The following theorem is evident

Theorem 5. *Suppose that the hypotheses of Theorem* 1 *hold except for* (3) *which is replaced by*

$$\int_{x_0}^{\infty} \frac{ds}{\varrho(s)} = \int_{x_0}^{\infty} [r(s) - q'(s)]\, ds = +\infty \tag{12}$$

and that $p(x) \geqq 0$, $r(x) - q'(x) \geqq 0$ *for all* $x \in J$. *Then every solution* $y(x)$ *of* (1) *satisfying* (2) *and* (4) *with* $K_0 < 0$ *is oscillatory on* J.

Remark 2. This theorem is a generalization of Theorem 2 in [2].
Evidently for any real number $a > 0$, b and any real x

$$ax^2 + bx \geqq -\frac{b^2}{4a}.$$

Under the assumptions of Theorem 1 we get from (5)

$$H(y(x)) \leqq H(y(x_0)) + \frac{1}{2}\int_{x_0}^{x} \frac{g^2(s)}{2r(s) - |p(s)| - q'(s)}\, ds$$

provided

$$2r(x) - |p(x)| - q'(x) > 0, \qquad x \in J.$$

If instead of assuming the convergence of the integral $\int_{x_0}^{\infty} |g(s)|\, ds$ which is evident from (4) we assume the convergence of the integral

$$\int_{x_0}^{\infty} \frac{g^2(s)}{2r(s) - |p(s)| - q'(s)}\, ds,$$

where $2r(x) - |p(x)| - q'(x) > 0$ for all $x \in J$, we can easily formulate the following theorems:

Theorem 6. *Suppose that* $q \in C_1(J)$ *and that for all* $x \in J$

$$2\varrho(x) - |p(x)| \geqq 0, \qquad 2r(x) - |p(x)| - q'(x) > 0, \qquad Q_i(x) \geqq 0,$$
$$i = 1, 2, \ldots, n.$$

If (3) *holds, then any solution* $y(x)$ *of* (1) *which satisfies* (2) *and such that*

$$H(y(x_0)) + \frac{1}{2}\int_{x_0}^{\infty} \frac{g^2(s)}{2r(s) - |p(s)| - q'(s)}\, ds \leqq K_0^* \leqq 0 \tag{4'}$$

is oscillatory on J.

Theorem 7. *Suppose that the hypotheses of Theorem* 6 *hold except for* (3) *which is replaced by* (10). *Suppose further that* $p(x) > 0$, $r(x) > 0$ *and that* $F_i(z)$ *satisfy* (8) *or* (9). *If*

$$\int_{x_0}^{\infty} |g(s)|\, ds < \infty, \tag{13}$$

then any solution $y(x)$ *of* (1) *satisfying* (2) *and* (4') *with* $K_0^* < 0$ *is oscillatory on* J.

Theorem 8. *Suppose that the hypotheses of Theorem* 6 *hold except for* (3) *which is replaced by* (11). *Suppose further that* $p(x) \geqq 0$, $r(x) \geqq 0$.

If (13) *holds, then every solution* $y(x)$ *of* (1) *satisfying* (2) *and* (4') *with* $K^* < 0$ *is oscillatory on* J.

Theorem 9. *Suppose that the hypotheses of Theorem* 6 *hold except for* (3) *which is replaced by* (12) *and that* $p(x) \geqq 0$, $r(x) - q'(x) \geqq 0$ *for all* $x \in J$. *If* (13) *holds, then any solution* $y(x)$ *of* (1) *satisfying* (2) *and* (4') *with* $K_0^* < 0$ *is oscillatory on* J.

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P. Šoltés
041 54 Košice, Komenského 14
Czechoslovakia