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NOTE ON THE OSCILLATORY BEHAVIOR OF BOUNDED SOLUTIONS OF HIGHER ORDER DIFFERENTIAL EQUATION WITH RETARDED ARGUMENT

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This paper contains the theorem which gives the sufficient conditions for bounded solutions of nth order differential equation with retarded argument to be oscillatory. The assertion of this theorem is not true for the corresponding ordinary differential equation.

Some theorems which have specific character for differential equations with retarded argument are given in [3]–[5]. In [1] D. L. Lovelady studied the asymptotic behavior of bounded solutions of ordinary differential equations

$$(p_{n-1}(\dots p_2(p_1u')'\dots)')' + (-1)^{n+1} qu = 0,$$
$$(p_{n-1}(t)(\dots p_2(t)(p_1(t)u'(t))'\dots)')' + (-1)^{n+1} F(t, u) = 0$$

and the oscillatory behavior of bounded solutions of ordinary differential equations

$$(p_{n-1}(\dots p_2(p_1u')'\dots)')' + (-1)^{n} qu = 0,$$
$$(p_{n-1}(t)(\dots p_2(t)(p_1(t)u'(t))'\dots)')' + (-1)^{n} F(t, u) = 0.$$

In [2] authors T. Kusano and H. Onose studied the asymptotic behavior of bounded nonoscillatory solutions of functional differential equation

$$(r_{n-1}(t)(r_{n-2}(t)(\dots(r_2(t)(r_1(t)y'(t))')'\dots)')')' + a(t)f(y(g(t))) = b(t).$$

We consider the nth order differential equation with retarded argument

$$(1)\qquad (p_{n-1}(t)(\dots p_2(t)(p_1(t)y'(t))'\dots)')' + (-1)^{n+1} q(t)y(g(t)) = 0,$$

where

$$(2)\qquad p_k \in C[[0, \infty), (0, \infty)], \qquad k = 1, \dots, n-1,$$

$$(3)\qquad q \in C[[0, \infty), [0, \infty)],$$

$$(4)\qquad g \in C[[0, \infty), R], \qquad g(t) < t, \qquad g'(t) \geqq 0, \qquad \lim_{t\to\infty} g(t) = \infty.$$

Let $v_1 = y(t)$, $v_2 = p_1 v_1'$, ..., $v_n = p_{n-1} v_{n-1}'$ on $[0, \infty)$. Now the system

(5)
$$\begin{aligned} v_1' &= \frac{v_2}{p_1} \\ v_2' &= \frac{v_3}{p_2} \\ &\vdots \\ v_{n-1}' &= \frac{v_n}{p_{n-1}} \\ v_n' &= -(-1)^{n+1} q y(g) \end{aligned}$$

is satisfied.

We put

(6) $$P_0(s, t) = 1, \qquad P_k(s, t) = \int_t^s \frac{P_{k-1}(z, t)}{p_k(z)} \mathrm{d}z, \qquad \text{for } s \geqq t \geqq 0,\ k = 1, \ldots, n-1.$$

Lemma. Let functions $v_k \in C[[0, \infty), \boldsymbol{R}]$ ($k = 1, \ldots n$) satisfy the system (5) and be of constant sign in the interval $[t_0, \infty)$, $t_0 \in [0, \infty)$ and

(7) $$(-1)^{k+1} v_k(t) v_1(t) \geqq 0, \qquad \text{for } t \geqq t_0, \qquad k = 1, \ldots, n.$$

Then

(8) $$| v_1(t) | \geqq P_{n-1}(s, t) | v_n(s) | \qquad \text{for } s \geqq t \geqq t_0.$$

Proof. An induction argument shows that if $s \geq t \geq t_0$ and $0 \leq i \leq n-1$, then with regard to (5) and (6)

$$v_1(t) = \sum_{k=0}^{i} (-1)^k P_k(s, t) v_{k+1}(s) + (-1)^{i+1} \int_t^s P_i(z, t) v_{i+1}'(z) \mathrm{d}z.$$

In view of (7) and for $i = n - 1$ we get

$$| v_1(t) | = \sum_{k=0}^{n-1} P_k(s, t) | v_{k+1}(s) | + \int_t^s P_{n-1}(z, t) | v_n'(z) | \mathrm{d}z.$$

From this (8) already follows.

A function $y \in C[[0, \infty), \boldsymbol{R}]$ satisfying the initial conditions $y(t) = \Phi(t)$, $t \leqq 0$, $\Phi \in C[\boldsymbol{E}_0, \boldsymbol{R}]$ ($\boldsymbol{E}_0$ is the initial set), $y^{(k)}(0) = y_0^{(k)}$, $k = 1, \ldots, n-1$, is called a solution of (1) if and only if $y, p_1 y', p_2(p_1 y')', \ldots, p_{n-1}(\ldots p_2(p_1 y')' \ldots)'$ are differentiable, and (1) is true.

A solution $y(t)$ of the equation (1) is called oscillatory if the set of zeros of $y(t)$ is not bounded from the right. A solution $y(t)$ of the equation (1) is called nonoscillatory if it is eventually of constant sign.

We restrict our consideration to those solutions $y(t)$ of (1) which satisfy

$$\sup \{| y(t) | : t_0 \leqq t < \infty\} > 0$$

for any $t_0 \in [0, \infty)$.

Theorem. Let the following conditions hold:

$$\int_0^\infty \frac{1}{p_k(s)}\,\mathrm{d}s = \infty, \qquad k = 1, \dots, n-1, \tag{9}$$

$$\limsup_{t\to\infty} \int_{g(t)}^{t} q(s)\, P_{n-1}(g(t), g(s))\,\mathrm{d}s > 1. \tag{10}$$

Then all bounded solutions of (1) are oscillatory.

Proof. We shall use the methods used in [1] and [3]. Let $y(t)$ be a bounded nonoscillatory solution of (1). We may suppose without loss of generality that $y(t) > 0$, for $t \geqq t_0$, $t_0 \in [0, \infty)$ (the case $y(t) < 0$ is treated similarly). By (4) there exists $t_1 \geqq t_0$ such that $g(t) \geqq t_0$ for $t \geqq t_1$. Thus $y(g(t)) > 0$ for $t \geqq t_1$. By (5), v_n' is one-signed on $[t_1, \infty)$, so v_n is eventually one-signed. Thus v_{n-1}' is eventually one-signed, so v_{n-1} is eventually one-signed. Continuing this, we see that there is t_2 in (t_1, ∞) such that each v_k, $1 \leqq k \leqq n$, is one-signed on $[t_2, \infty)$. Now we shall prove that if $k \geqq 2$, then $v_k v_k' \leqq 0$ in $[t_2, \infty)$. If $k \geqq 2$ and $t \geqq t_2$ then

$$v_{k-1}(t) = v_{k-1}(t_2) + \int_{t_2}^{t} \frac{v_k(s)}{p_{k-1}(s)}\,\mathrm{d}s. \tag{11}$$

Suppose that $k \geqq 2$ and $v_k v_k' \leqq 0$ fails on $[t_2, \infty)$. Since v_k and v_k' are both one-signed on $[t_2, \infty)$, we see that $v_k v_k' > 0$ on $[t_2, \infty)$ for some $k \geqq 2$. Thus v_k is either eventually positive and nondecreasing or eventually negative and nonincreasing. In either case, (11) and (9) say that v_{k-1} is unbounded and has the same eventual sign as v_k. Repeating this procedure $k - 1$ times, we see that $y(t)$ is unbounded, a contradiction, so we conclude that $v_k v_k' \leqq 0$ on $[t_2, \infty)$ whenever $k \geqq 2$. In view of (5) and $v_k v_k' \leqq 0$ for $k \geqq 2$ there is $v_k \leqq 0$ on $[t_2, \infty)$ if k is even, and $v_k \geqq 0$ on $[t_2, \infty)$ if k is odd. Then by Lemma (8) holds.

Let n be odd, then by Lemma

$$y(t) \geqq P_{n-1}(s, t)\, v_n(s), \qquad s \geqq t \geqq t_2.$$

With regard to (4) we have

$$y(g(t)) \geqq P_{n-1}(g(s), g(t))\, v_n(g(s)) \qquad \text{for } s \geqq t \geqq t_3,$$

where a $t_3 \geqq t_2$ is so large that $g(t) \geqq t_2$ for $t \geqq t_3$.

From the last equation of (5) we get

$$v_n'(t) \leqq -q(t)\, P_{n-1}(g(s), g(t))\, v_n(g(s)), \qquad s \geqq t \geqq t_3.$$

Integrating the last inequality with respect to t from $g(s)$ to s, for s sufficiently large, we obtain

$$v_n(s) \leqq v_n(g(s)) \left[1 - \int_{g(s)}^{s} q(t)\, P_{n-1}(g(s), g(t))\,\mathrm{d}t\right].$$

In view of (10) and (7) the left hand side is positive while the right one is negative, for sufficiently large s, which is a contradiction. If n is even the proof is similar.

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