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HOMOMORPHISMS OF MACHINES
(Part II)

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2. COROLLARIES FOR THE EXISTENCE OF m-HOMOMORPHISMS

Now, we want to study the category of all c-machines and the category of all machines. In all this paragraph, we denote by $\mathcal{M}^c$ the category of c-machines and by $\mathcal{M}$ the category of all machines.

2.0. Notation. Let $\mathcal{A}$ be a category. Then we denote by ob $\mathcal{A}$ the class of all objects of $\mathcal{A}$ and, for arbitrary, $P, Q \in$ ob $\mathcal{A}$, by $[P, Q]_{\mathcal{A}}$ the set of all morphisms from P to Q. In most cases we shall write shortly $\mathcal{A}$ instead of ob $\mathcal{A}$. Further, $\bigcup [P, Q]_{\mathcal{A}}$ means the class of all morphisms of $\mathcal{A}$. The sign $\cong$ means an isomorphism of categories and $\subseteqq$ a full subcategory.

If $\mathcal{A}$ is a category such that, for each, $P, Q \in \mathcal{A}$, $[P, Q]_{\mathcal{A}} \neq \emptyset$, $[Q, P]_{\mathcal{A}} \neq \emptyset$ implies $P = Q$ then $\mathcal{A}$ is called *antisymmetric*. If $\mathcal{A}$ is a category such that, for each $P, Q \in \mathcal{A}$, $|[P, Q]_{\mathcal{A}}| \leqq 1$ then $\mathcal{A}$ is called a *thin* category. An antisymmetric and thin category is called an *ordered class*. An ordered class $\mathcal{A}$ is called an *antichain* if $[P, Q]_{\mathcal{A}} = \emptyset$ for each $P, Q \in \mathcal{A}$, $P \neq Q$ and it is called a *chain* if $[P, Q]_{\mathcal{A}} \neq \emptyset$ or $[Q, P]_{\mathcal{A}} \neq \emptyset$ for each $P, Q \in \mathcal{A}$.

In the paper [6], various arithmetic operations for categories are introduced. We want to use the operation of the lexicographic sum Σ^l that is defined for a system $\{\mathcal{A}_G; G \in \mathcal{G}\}$ of categories where $\mathcal{G}$ is an antisymmetric category and the operation of the cardinal product for two categories. If $\{\mathcal{A}_G; G \in \mathcal{G}\}$ is a system such that $\mathcal{G}$ is an antichain then Σ^l is called the cardinal sum (and we write Σ) and if $\mathcal{G}$ is a chain then Σ^l is called the ordinal sum (and we write Σ°).

The *lexicographic sum* $\sum_{G \in \mathcal{G}}^l \mathcal{A}_G$ of $\{\mathcal{A}_G; G \in \mathcal{G}\}$ is the class $\bigcup_{G \in \mathcal{G}} \bigcup_{P \in \mathcal{A}_G} (G, P)$ of objects and the class $\bigcup[(G_1, P), (G_2, Q)]_{\Sigma^l}$ of morphisms where

$$[(G_1, P), (G_2, Q)]_{\Sigma l} = \begin{cases} [G_1, G_2]_{\mathscr{G}} \times [P, Q]_{\mathscr{A}_{G_1}} & \text{if} \quad G_1 = G_2, \\ [G_1, G_2]_{\mathscr{G}}, & \text{if} \quad G_1 \neq G_2. \end{cases}$$

Especially, if we suppose that all $\mathscr{A}_G$ and $\mathscr{G}$ are thin categories then the definition of composition of morphisms is evident. Further, if $\mathscr{G} = \{1, 2, \ldots, n\}$ is an antichain (a chain with natural order resp.) then we put $\sum\limits_{G \in \mathscr{G}}^{l} \mathscr{A}_G = \sum\limits_{G \in \mathscr{G}} \mathscr{A}_G = \mathscr{A}_1 + \mathscr{A}_2 + \ldots + \mathscr{A}_n$ ($\sum\limits_{G \in \mathscr{G}}^{l} \mathscr{A}_G = \sum\limits_{G \in \mathscr{G}}^{\circ} \mathscr{A}_G = \mathscr{A}_1 \oplus \mathscr{A}_2 \oplus \ldots \oplus \mathscr{A}_n$ resp.).

We see that for the lexicographic sum $\mathscr{A} = \sum\limits_{G \in \mathscr{G}}^{l} \mathscr{A}_G$ with disjoint summands we can put $(G, P) = P$ for each $(G, P) \in \sum\limits_{G \in \mathscr{G}}^{l} \mathscr{A}_G$ and that the sum has the following property: if $G_1, G_2 \in \mathscr{G}$, $G_1 \neq G_2$ then $[G_1, G_2]_{\mathscr{G}} \neq \emptyset$ implies $[P, Q]_{\mathscr{A}} \neq \emptyset$ and $[G_1, G_2]_{\mathscr{G}} = \emptyset$ implies $[P, Q]_{\mathscr{A}} = \emptyset$ for each $P \in \mathscr{A}_{G_1}$, $Q \in \mathscr{A}_{G_2}$. Thus, if, for example, $\mathscr{C} = \mathscr{A} \oplus \mathscr{B}$ and $\mathscr{A} \cap \mathscr{B} = \emptyset$ then, for each $P \in \mathscr{A}$, $Q \in \mathscr{B}$, $[P, Q]_{\mathscr{C}} \neq \emptyset$, $[Q, P]_{\mathscr{C}} = \emptyset$ and if $\mathscr{C} = \mathscr{A} + \mathscr{B}$ and $\mathscr{A} \cap \mathscr{B} = \emptyset$ then, for each $P \in \mathscr{A}$, $Q \in \mathscr{B}$, $[P, Q]_{\mathscr{C}} = \emptyset$, $[Q, P]_{\mathscr{C}} = \emptyset$. (See [6], 1.1, 1.3, 1.4.)

Further, if $\mathscr{A}$, $\mathscr{B}$ are categories then the *cardinal product* $\mathscr{A} . \mathscr{B}$ is the class $\text{ob}\, \mathscr{A} \times \text{ob}\, \mathscr{B}$ of objects and the class $\bigcup[(P, Q), (P', Q')]_{\mathscr{A}.\mathscr{B}}$ of morphisms where $[(P, Q), (P', Q')]_{\mathscr{A}.\mathscr{B}} = [P, P']_{\mathscr{A}} \times [Q, Q']_{\mathscr{B}}$ for each $P, P' \in \mathscr{A}$, $Q, Q' \in \mathscr{B}$ with the composition $(p, q) . (p', q') = (pp', qq')$ for each $(p, q) \in [(P, Q), (P', Q')]_{\mathscr{A}.\mathscr{B}}$, $(p', q') \in [(P', Q'), (P'', Q'')]_{\mathscr{A}.\mathscr{B}}$. (See [6], 2.1, 2.2.)

2.1. Definition.] (a) Let $\mathscr{A}$ be a category. A thin category $\mathscr{A}(b)$ such that $\text{ob}\, \mathscr{A}(b) = \text{ob}\, \mathscr{A}$ and $[P, Q]_{\mathscr{A}(b)} \neq \emptyset$ iff $[P, Q]_{\mathscr{A}} \neq \emptyset$ for each $P, Q \in \mathscr{A}(b)$ is called a *basic category* for $\mathscr{A}$.

(b) A category $\mathscr{A}$ is called a *category with non empty homs* if, for each $P, Q \in \mathscr{A}$, $[P, Q]_{\mathscr{A}} \neq \emptyset$.

A basic category $\mathscr{A}(b)$ for $\mathscr{A}$ is a thin category with the same objects and the same existence of morphisms. Thus, $\mathscr{A}(b)$ gives us a characteristic of the existence of morphisms for the category $\mathscr{A}$. Clearly, for $\boldsymbol{A}, \boldsymbol{B} \in \mathscr{M}$, there is an m-homomorphism $\boldsymbol{A} \to \boldsymbol{B}$ iff $[\boldsymbol{A}, \boldsymbol{B}]_{\mathscr{M}} \neq \emptyset$. Thus, for the categories $\mathscr{M}$ and $\mathscr{M}^c$ of machines, we want to describe by means of $\mathscr{M}(b)$ and $\mathscr{M}^c(b)$ a clasification of machines which is mentioned in [1], [2] and [3].

2.2. Theorem. *Let $\boldsymbol{A}, \boldsymbol{B} \in \mathscr{M}^c$ be arbitrary. Then $[\boldsymbol{A}, \boldsymbol{B}]_{\mathscr{M}^c} \neq \emptyset$ if and only if $\boldsymbol{B}$ is m-admissible for $\boldsymbol{A}$.*

Proof. The assertion follows directly from 1.24 and 1.23.

The following assertions most of which can be found in a different form in [3] follow from this theorem as consequences.

2.3. Definition. We put

$$o - \mathcal{M}^c = \{A \in \mathcal{M}^c;\ RA = 0,\ A^{\infty_1} = \emptyset\},$$
$$\infty_1 - \mathcal{M}^c = \{A \in \mathcal{M}^c;\ RA = 0,\ A^{\infty_1} \neq \emptyset\},$$
$$\infty_2 - \mathcal{M}^c = \{A \in \mathcal{M}^c;\ RA \neq 0\},$$
$$f - \mathcal{M}^c = \{A \in \mathcal{M}^c;\ DA \neq \emptyset\},$$
$$\bar{f} - \mathcal{M}^c = \{A \in \mathcal{M}^c;\ DA = \emptyset\}.$$

2.4. Remark. By (iii), we have $f - \mathcal{M}^c \subseteqq o - \mathcal{M}^c \cup \infty_1 - \mathcal{M}^c$. Clearly, $\mathcal{M}^c = o - \mathcal{M}^c \cup \infty_1 - \mathcal{M}^c \cup \infty_2 - \mathcal{M}^c$ with disjoint summands and $\mathcal{M}^c = f - \mathcal{M}^c \cup \bar{f} - \mathcal{M}^c$ with disjoint summands.

2.5. Definition. We put

$$(f, o) - \mathcal{M}^c = f - \mathcal{M}^c \cap o - \mathcal{M}^c, \qquad (\bar{f}, o) - \mathcal{M}^c = \bar{f} - \mathcal{M}^c \cap o - \mathcal{M}^c,$$
$$(f, \infty_1) - \mathcal{M}^c = f - \mathcal{M}^c \cap \infty_1 - \mathcal{M}^c, \qquad (\bar{f}, \infty_1) - \mathcal{M}^c = \bar{f} - \mathcal{M}^c \cap \infty_1 - \mathcal{M}^c,$$
$$(\bar{f}, \infty_2) - \mathcal{M}^c = \bar{f} - \mathcal{M}^c \cap \infty_2 - \mathcal{M}^c.$$

2.6. Remark. By 2.4, $\mathcal{M}^c = (f, o) - \mathcal{M}^c \cup (f, \infty_1) - \mathcal{M}^c \cup (\bar{f}, o) - \mathcal{M}^c \cup (\bar{f}, \infty_1) - \mathcal{M}^c \cup (\bar{f}, \infty_2) - \mathcal{M}^c$ with disjoint summands.

2.7. Corollary. *Let $A, B \in \mathcal{M}^c$ be arbitrary. Then the following assertions hold*:

(a) *Let $B \in (f, \infty_1) - \mathcal{M}^c$. Then $[A, B]_{\mathcal{M}^c} \neq \emptyset$ iff $A \in f - \mathcal{M}^c$.*

(b) *Let $B \in (\bar{f}, \infty_1) - \mathcal{M}^c$. Then $[A, B]_{\mathcal{M}^c} \neq \emptyset$ iff $A \in (\bar{f}, o) - \mathcal{M}^c \cup (\bar{f}, \infty_1) - \mathcal{M}^c$.*

(c) *Let $B \in (\bar{f}, \infty_2) - \mathcal{M}^c$, $A \notin (\bar{f}, \infty_2) - \mathcal{M}^c$. Then $[A, B]_{\mathcal{M}^c} \neq \emptyset$ iff $A \in (f, o) - \mathcal{M}^c \cup (\bar{f}, \infty_1) - \mathcal{M}^c$.*

Proof of (a). If $B \in (f, \infty_1) - \mathcal{M}^c$ then $DB \neq \emptyset$, $B^{\infty_1} \neq \emptyset$. It follows $RB = 0$ by (iii). Then B is m-admissible for A iff $H(A, B) \neq \emptyset$ which is equivalent to $DA \neq \emptyset$ and the existence of a pair of h-elements of A and B by 1.18 and 1.19 (b). The latest holds iff $DA \neq \emptyset$, i.e. $A \in f - \mathcal{M}^c$ because $SB(\mathrm{d}B) = \infty_1$ and $SA(dA) \in \mathrm{Ord} \cup \{\infty_1\}$ by (vii) and (vi) and thus, dA, dB is a pair of h-elements of A and B by 1.15.

Proof of (b). If $B \in (f, \infty_1) - \mathcal{M}^c$ then $DB = \emptyset$, $RB = 0$ and $B^{\infty_1} \neq \emptyset$. Then B is m-admissible for A iff $H(A, B) \neq \emptyset$ by 1.18 which is equivalent to $DA = \emptyset$ and the existence of a pair of h-elements of A and B by 1.19 (b). But the latest is equivalent to $DA = \emptyset$, i.e. $A \in (\bar{f}, o) - \mathcal{M}^c \cup (\bar{f}, \infty_1) - \mathcal{M}^c$ because, in this case, $RA = 0$ by (iii) and, for arbitrary $x \in A$, $x' \in B^{\infty_1}$, x, x' is a pair of h-elements of A and B.

Proof of (c). If $B \in (\bar{f}, \infty_2) - \mathcal{M}^c$ then $DB = \emptyset$ and $RB \neq 0$. Thus, B is for $A \in (\bar{f}, \infty_2) - \mathcal{M}^c$ m-admissible iff $DA \neq \emptyset$ by 1.18 because, in this case, $RA = 0$ which implies $RB \mid RA$.

2.8. Corollary. *Let $A \in \mathcal{M}^c$ be arbitrary. Then the following assertions hold*:
(a) *Let $B \in (f, o) - \mathcal{M}^c$. Then $[A, B]_{\mathcal{M}^c} \neq \emptyset$ iff $A \in (f, o) - \mathcal{M}^c$ and $\vartheta A \leqq \vartheta B$.*

(b) *Let* $\boldsymbol{B} \in (\bar{f}, o) - \mathscr{M}^c$. *Then* $[A, \boldsymbol{B}]_{\mathscr{M}^c} \neq \emptyset$ *iff* $A \in (\bar{f}, o) - \mathscr{M}^c$ *and* $H(A, \boldsymbol{B}) \neq \emptyset$.

(c) *Let* $A, \boldsymbol{B} \in (\bar{f}, \infty_2) - \mathscr{M}^c$. *Then* $[A, \boldsymbol{B}]_{\mathscr{M}^c} \neq \emptyset$ *iff* $\boldsymbol{RB} \mid \boldsymbol{RA}$.

Proof of (a). Let $\boldsymbol{B} \in (f, o) - \mathscr{M}^c$. Then we have $\boldsymbol{DB} \neq \emptyset$, $\boldsymbol{RB} = 0$ and $\boldsymbol{B}^{\infty_1} = \emptyset$ by (iii). Further, $\vartheta\boldsymbol{B}$ is isolated and $\boldsymbol{SB}(\boldsymbol{dB}) = \vartheta\boldsymbol{B} - 1$ by (vi).

If, now, $A \in (f, o) - \mathscr{M}^c$ and $\vartheta A \leqq \vartheta\boldsymbol{B}$ then $DA \neq \emptyset$, $RA = 0$ and $A^{\infty_1} = \emptyset$. Thus, ϑA is isolated and $SA(dA) = \vartheta A - 1$. It implies that $(dA, \boldsymbol{dB}) \in H(A, \boldsymbol{B})$ by 1.15 and thus, $\boldsymbol{B}$ be m-admissible for A.

Let, on the other hand, $\boldsymbol{B}$ is m-admissible for A. Then $|DA| = |\boldsymbol{DB}| = 1$ by 1.19(c) which implies $\boldsymbol{RB} = 0$ by (iii) and $H(A, \boldsymbol{B}) \neq \emptyset$ by 1.18. If $(x, x') \in H(A, \boldsymbol{B})$ are arbitrary then, for each $n \in N$, the conditions $x \in \operatorname{dom} f^n$, $x' \in \operatorname{dom} g^n$ are equivalent and $SA(f^n(x)) \leqq \boldsymbol{SB}(g^n(x'))$. Thus, for $m \in N$, $x \in \operatorname{dom} f^m$, $f^m(x) = dA$ implies $x' \in \operatorname{dom} g^m$, $g^m(x') = \boldsymbol{dB}$. (See (ii).) We obtain $SA(dA) = SA(f^m(x)) \leqq \boldsymbol{SB}(g^m(x')) = \boldsymbol{SB}(\boldsymbol{dB})$. It follows $SA(dA) \leqq \boldsymbol{SB}(\boldsymbol{dB}) \in \mathrm{Ord}$ which implies $A^{\infty_1} = \emptyset$ by (vii); thus, we have $A \in (f, o) - \mathscr{M}^c$. Further, by (vi), $\vartheta A \leqq \vartheta\boldsymbol{B}$.

Proof of (b). If $\boldsymbol{B} \in (f, o) - \mathscr{M}^c$ then $\boldsymbol{DB} = \emptyset$, $\boldsymbol{RB} = 0$, $\boldsymbol{B}^{\infty_1} = \emptyset$. Thus, by 1.18, $\boldsymbol{B}$ is m-admissible for an $A \in \mathscr{M}^c$ iff $H(A, \boldsymbol{B}) \neq \emptyset$ and $DA = \emptyset$ by 1.19(b) which means $RA = 0$, $A^{\infty_1} = \emptyset$ by 1.14 and 1.15.

Proof of (c). If $A, \boldsymbol{B} \in (\bar{f}, \infty_2) - M^c$ then $\boldsymbol{RB} \neq 0$. Thus, $\boldsymbol{B}$ is m-admissible for A iff $\boldsymbol{RB} \mid RA$ and $DA = \emptyset$ by 1.18.

2.9. Theorem. *Let* $A, B \in \mathscr{M}^c$ *be arbitrary. Then* $[A, \boldsymbol{B}]_{\mathscr{M}^c} \neq \emptyset$ *if and only if precisely one of these following cases occurs*:

(1) $A, \boldsymbol{B} \in (f, o) - \mathscr{M}^c$ *and* $\vartheta A \leqq \vartheta\boldsymbol{B}$.

(2) $A \in (f, o) - \mathscr{M}^c$, $\boldsymbol{B} \in (f, \infty_1) - \mathscr{M}^c$.

(3) $A, \boldsymbol{B} \in (f, \infty_1) - \mathscr{M}^c$.

(4) $A, \boldsymbol{B} \in (\bar{f}, o) - \mathscr{M}^c$ *and* $H(A, \boldsymbol{B}) \neq \emptyset$.

(5) $A \in (\bar{f}, o) - \mathscr{M}^c$, $\boldsymbol{B} \in (\bar{f}, \infty_1) - \mathscr{M}^c$.

(6) $A, \boldsymbol{B} \in (\bar{f}, \infty_1) - \mathscr{M}^c$.

(7) $A \in (\bar{f}, o) - \mathscr{M}^c$, $\boldsymbol{B} \in (\bar{f}, \infty_2) - \mathscr{M}^c$.

(8) $A \in (\bar{f}, \infty_1) - \mathscr{M}^c$, $\boldsymbol{B} \in (\bar{f}, \infty_2) - \mathscr{M}^c$.

(9) $A, \boldsymbol{B} \in (\bar{f}, \infty_2) - \mathscr{M}^c$ *and* $\boldsymbol{RB} \mid RA$.

Proof. Let $\boldsymbol{B} \in \mathscr{M}^c$ be arbitrary. If $\boldsymbol{B} \in (f, o) - \mathscr{M}^c$ then we have (1) by 2.8 (a); if $\boldsymbol{B} \in (f, \infty_1) - \mathscr{M}^c$ then we have (2) or (3) by 2.7 (a). Further, if $\boldsymbol{B} \in (\bar{f}, o) - \mathscr{M}^c$ then we obtain (4) by 2.8 (b), if $\boldsymbol{B} \in (\bar{f}, \infty_1) - \mathscr{M}^c$ then we have (5) or (6) by 2.7 (b) then we obtain (4) by 2.8 (b), if $\boldsymbol{B} \in (\bar{f}, \infty_1)$ new $- \mathscr{M}^c$ the have (5) or (6) by 2.7 (b) and if $\boldsymbol{B} \in (\bar{f}, \infty_2) - \mathscr{M}^c$ then we obtain either (7) or (8) by 2.7 (c) or (9) by 2.8 (c). By 2.6, the proof is finished.

2.10. Remark. If we want to find out the existence of homomorphisms for a given $A \in \mathscr{M}^c$ and arbitrary $\boldsymbol{B} \in \mathscr{M}^c$ we can choose some of the assertions (1)–(9). For example, if $A \in (f, o) - \mathscr{M}^c$ then $[A, \boldsymbol{B}]_{\mathscr{M}}{}^c \neq \emptyset$ for $\boldsymbol{B} \in \mathscr{M}^c$ iff (1) or (2) hold, if $A \in$

$\in (f, \infty_1) - \mathcal{M}^c$ then $[A, B]_{\mathcal{M}^c} \neq \emptyset$ for $B \in \mathcal{M}^c$ iff (3) holds, if $A \in (\bar{f}, o) - \mathcal{M}^c$ then $[A, B]_{\mathcal{M}^c} \neq \emptyset$ for $B \in \mathcal{M}^c$ iff precisely one of (4), (5), (7) holds etc.

Let, now, for each category $\mathcal{A}$, $\mathcal{A}(b)$ means a basic category for $\mathcal{A}$.

2.11. Lemma. *The following assertions hold*:

(a) $\mathcal{M}^c(b) \cong f - \mathcal{M}^c(b) + \bar{f} - \mathcal{M}^c(b)$.

(b) $f - \mathcal{M}^c(b) \cong (f, o) - \mathcal{M}^c(b) \oplus (f, \infty_1) - \mathcal{M}^c(b)$.

(c) $\bar{f} - \mathcal{M}^c(b) \cong (\bar{f}, o) - \mathcal{M}^c(b) \oplus (\bar{f}, \infty_1) - \mathcal{M}^c(b) \oplus (\bar{f}, \infty_2) - \mathcal{M}^c(b)$.

Proof. The assetions are consequences of 2.9.

2.12. Definition. Let $\lambda \in \mathrm{Ord} - \{0\}$, $n \in N - \{0\}$ be arbitrary. Then we put

$$(f, \lambda) - \mathcal{M}^c = \{A \in (f, o) - \mathcal{M}^c;\ \vartheta A = \lambda\},$$
$$(\bar{f}, n) - \mathcal{M}^c = \{A \in (\bar{f}, \infty_2) - \mathcal{M}^c;\ RA = n\}.$$

We denote by $\mathcal{O}$ the thin category such that ob $\mathcal{O} = \mathrm{Ord} - \{0\}$ and, for each $\alpha, \beta \in \mathrm{Ord} - \{0\}$, $[\alpha, \beta]_{\mathcal{O}} \neq \emptyset$ iff $\alpha \leqq \beta$. Further, we denote by $\mathcal{N}$ the thin category such that ob $\mathcal{N} = N - \{0\}$ and, for each $m, n \in N - \{0\}$, $[m, n]_{\mathcal{N}} \neq \emptyset$ iff $n \mid m$. Finally, if $[o]$, $[\infty_1]$ are thin categories defined on $\{o\}$ and $\{\infty_1\}$ then we put $\mathcal{O}^* = \mathcal{O} \oplus [\infty_1]$, $\mathcal{N}^* = [o] \oplus [\infty_1] \oplus \mathcal{N}$.

2.13. Lemma. *The following assertions hold*:

(a) *Let $\mathcal{O}$ be the category defined in* 2.12. *Then* $(f, o) - \mathcal{M}^c(b) \cong \sum^{o}_{\lambda \in \mathcal{O}} (f, \lambda) - \mathcal{M}^c(b)$.

(b) *Let $\mathcal{N}$ be the category defined in* 2.12. *Then* $(\bar{f}, \infty_2) - \mathcal{M}^c(b) \cong \sum^{l}_{n \in \mathcal{N}} (\bar{f}, n) - \mathcal{M}^c(b)$.

(c) *Let $\mathcal{O}^*$ be the category defined in* 2.12. *Then* $f - \mathcal{M}^c(b) \cong \sum^{o}_{\lambda \in \mathcal{O}^*} (f, \lambda) - \mathcal{M}^c(b)$.

(d) *Let $\mathcal{N}^*$ be the category defined in* 2.12. *Then* $\bar{f} - \mathcal{M}^c(b) \cong \sum^{l}_{n \in \mathcal{N}^*} (\bar{f}, n) - \mathcal{M}^c(b)$.

Proof. Indeed, the assertions (a), (b) follows from 2.9 (the cases (1) and (9)), (c) follows from (a) and 2.9 (the case (2)) and (d) follows from (b) and 2.9 the cases (5), (7), (8)).

2.14. Lemma. *The following assertions hold*:

(a) *Let $\lambda \in \mathcal{O}$, $n \in \mathcal{N}$ be arbitrary. Then $(f, \lambda) - \mathcal{M}^c(b)$, $(\bar{f}, n) - \mathcal{M}^c(b)$ are the categories with non empty homs.*

(b) *Let $\mathcal{H}_{\mathcal{M}}$ be the thin category such that* ob $\mathcal{H}_{\mathcal{M}}$ = ob $(\bar{f}, o) - \mathcal{M}^c(b)$ *and, for each $A, B \in$* ob $\mathcal{H}_{\mathcal{M}}$, $[A, B]_{\mathcal{H}_{\mathcal{M}}} \neq \emptyset$ *iff $H(A, B) \neq \emptyset$. Then* $(\bar{f}, o) - \mathcal{M}^c(b) \cong \mathcal{H}_{\mathcal{M}}$.

Proof. The assertions follows from 2.9 (the cases (1), (9) and (4)).

2.15. Theorem. *Let $\mathcal{O}^*$, $\mathcal{N}^*$ be the categories defined in* 2.12 *and let $\mathcal{O}'$, $\mathcal{N}'$ be thin categories such that* ob $\mathcal{O}' = \{(f, \lambda); \lambda \in \mathcal{O}^*\}$, $\mathcal{O}' \cong \mathcal{O}^*$ *and* ob $\mathcal{N}' = \{(\bar{f}, n); n \in \mathcal{N}^*\}$, $\mathcal{N}' \cong \mathcal{N}^*$. *We put $\mathcal{K} = \mathcal{O}' + \mathcal{N}'$. Then $\mathcal{M}^c(b) \cong \sum^{l}_{k \in \mathcal{K}} k - \mathcal{M}(b)$ where $k - \mathcal{M}^c$ is a category with non empty homs for each $k \in \mathcal{K} - \{(\bar{f}, o)\}$ and $(\bar{f}, o) - \mathcal{M}^c(b) \cong \mathcal{H}_{\mathcal{M}}$.*

Indeed, the assertion is a consequence of 2.13 (c), (d), 2.11 (a), 2.9 (the cases (3) and (6)) and 2.14 (a), (b).

2.16. Remark. The theorem 2.15 gives a full simple description of the existence of homomorphisms for the whole category $\mathcal{M}^c$ except the existence of the "internal" homomorphisms of the category $(\bar{f}, o) - \mathcal{M}^c$.

Similarly, we want still to find out a description of the category $\mathcal{M}$ of all machines. If $A \in \mathcal{M}$ is arbitrary, then, by 1.9, $(T, f \mid T) \in \mathcal{M}^c$ for each $T \in A/\varrho A$.

2.17. Definition. Let $A \in \mathcal{M}$ be arbitrary. Then we put $\Theta A = \{\boldsymbol{T} \in \mathcal{M}^c;\ \boldsymbol{T} = (T, f \mid T),\ T \in A/\varrho A\}$. ($\boldsymbol{T} \in \Theta A$ is called a *connected component of* A.)

2.18. Lemma. *Let* $A, \boldsymbol{B} \in \mathcal{M}$ *be arbitrary. Then* $[A, \boldsymbol{B}]_{\mathcal{M}} \neq \emptyset$ *iff, for each* $\boldsymbol{T} \in \Theta A$, *there is* $\boldsymbol{T}' \in \Theta \boldsymbol{B}$ *such that* $[\boldsymbol{T}, \boldsymbol{T}']_{\mathcal{M}^c} \neq \emptyset$.

Indeed, the assertion follows directly from 1.4 and 1.9.

2.19. Definition. We put

$$f - \mathcal{M} = \{A \in \mathcal{M};\ \Theta A \subseteq f - \mathcal{M}^c\},$$
$$\bar{f} - \mathcal{M} = \{A \in \mathcal{M};\ \Theta A \subseteq \bar{f} - \mathcal{M}^c\}.$$

2.20. Theorem. *Let* $f - \emptyset, \bar{f} - \emptyset$ *be two copies of the empty set. Let* $[f - \emptyset], [\bar{f} - \emptyset]$ *be thin categories defined on* $\{f - \emptyset\}$, $\{\bar{f} - \emptyset\}$ *and* $f - \mathcal{M}^* = f - \mathcal{M} + [f - \emptyset]$, $\bar{f} - \mathcal{M}^* = \bar{f} - \mathcal{M} + [\bar{f} - \emptyset]$ *(the cardinal sums). Then* $\mathcal{M} \cong f - \mathcal{M}^* . \bar{f} - \mathcal{M}^*$ *(the cardinal product).*

Proof. Let $A \in \mathcal{M}$ be arbitrary. Putting

$$f - \Theta A = \begin{cases} \Theta A \cap f - \mathcal{M}^c & \text{if} \quad \Theta A \cap f - \mathcal{M}^c \neq \emptyset, \\ f - \emptyset & \text{otherwise} \end{cases}$$

$$\bar{f} - \Theta A = \begin{cases} \Theta A \cap \bar{f} - \mathcal{M}^c & \text{if} \quad \Theta A \cap \bar{f} - \mathcal{M}^c \neq \emptyset \\ \bar{f} - \emptyset & \text{otherwise} \end{cases}$$ let us put $\varphi(A) = (f - \Theta A, \bar{f} - \Theta A)$.

Further, we denote by $1_{f-\emptyset}$ ($1_{\bar{f}-\emptyset}$ resp.) the only morphism on $[f - \emptyset]$ ($[\bar{f} - \emptyset]$ resp.). Then, if $A, \boldsymbol{B} \in \mathcal{M}$ and $F \in [A, \boldsymbol{B}]_{\mathcal{M}}$ are arbitrary then we put

$$f - F = \begin{cases} F \mid \bigcup_{T \in f - \Theta A} T & \text{if } f - \Theta A \neq \emptyset \\ 1_{f-\emptyset} & \text{otherwise} \end{cases}, \quad \bar{f} - F = \begin{cases} F \mid \bigcup_{T \in f - \Theta A} T & \text{if } \bar{f} - \Theta A \neq \emptyset. \\ 1_{\bar{f}-\emptyset} & \text{otherwise} \end{cases}$$

Putting $\varphi(F) = (f - F, \bar{f} - F)$ we can easily prove that φ is an isomorphism by 2.19 and 2.9.

Thus, by 2.20. we see that we can study on only the categories $f - \mathcal{M}$ and $\bar{f} - \mathcal{M}$ and separately.

2.21. Definition. We put

$(f, \infty_1) - \mathcal{M} = \{A \in f - \mathcal{M};$ there is $\boldsymbol{T} \in \Theta A$ such that $\boldsymbol{T} \in (f, \infty_1) - \mathcal{M}^c\}$,

$(f, o) - \mathcal{M} = f - \mathcal{M} - (f, \infty_1) - \mathcal{M}$.

Further, we put

$(\bar{f}, \infty_2) - \mathcal{M} = \{A \in \bar{f} - \mathcal{M};$ there is $T \in \Theta A$ such that $T \in (\bar{f}, \infty_2) - \mathcal{M}^c\}$,

$(\bar{f}, \infty_1) - \mathcal{M} = \{A \in \bar{f} - \mathcal{M} - (\bar{f}, \infty_2) - \mathcal{M};$ there is $T \in \Theta A$ such that $T \in \in (\bar{f}, \infty_1) - \mathcal{M}^c\}$,

$(\bar{f}, o) - \mathcal{M} = \bar{f} - \mathcal{M} - ((\bar{f}, \infty_1) - \mathcal{M} \cup (\bar{f}, \infty_2) - \mathcal{M})$.

2.22. Lemma. *Let $A, B \in \mathcal{M}$ be arbitrary. Then the following assertions hold:*

(a) *If $A \in f - \mathcal{M}$, $B \in (f, \infty_1) - \mathcal{M}$ then $[A, B]_{\mathcal{M}} \neq \emptyset$.*

(b) *Let $A \in \bar{f} - \mathcal{M}$, $B \in (\bar{f}, \infty_1) - \mathcal{M}$. Then $[A, B]_{\mathcal{M}} \neq \emptyset$ iff $A \in (\bar{f}, o) - \mathcal{M} \cup \cup (\bar{f}, \infty_1) - \mathcal{M}$.*

(c) *If $A \in (\bar{f}, o) - \mathcal{M} \cup (\bar{f}, \infty_1) - \mathcal{M}$, $B \in (\bar{f}, \infty_2) - \mathcal{M}$ then $[A, B]_{\mathcal{M}} \neq \emptyset$.*

Proof. Indeed, if $A \in f - \mathcal{M}$, $B \in (f, \infty_1) - \mathcal{M}$ then, for each $T \in \Theta A$, $T \in f - \mathcal{M}^c$ and there is $T' \in \Theta B$ such that $T' \in (f, \infty_1) - \mathcal{M}^c$. Thus, for each $T \in \Theta A$, we have $[T, T']_{\mathcal{M}^c} \neq \emptyset$ by 2.7 (a). It follows $[A, B]_{\mathcal{M}} \neq \emptyset$ by 2.18.

Similarly, we can easily prove the assertion (b) by 2.7 (b), 2.18 and the assertion (c) by 2.7 (c) and 2.18.

2.23. Lemma. *Let $A \in f - \mathcal{M}$, $B \in (f, o) - \mathcal{M}$ be arbitrary. Then $[A, B]_{\mathcal{M}} \neq \emptyset$ iff $A \in (f, o) - \mathcal{M}$ and* $\sup \{\vartheta T \in \mathcal{O};\ T \in \Theta A\} \leqq \sup \{\vartheta T \in \mathcal{O};\ T \in \Theta B\}$.

Proof. Indeed, if $[A, B]_{\mathcal{M}} \neq \emptyset$ then $A \in (f, o) - \mathcal{M}$ by 2.18 and 2.8 (a). Further, $[A, B]_{\mathcal{M}} \neq \emptyset$ iff, for each $T \in \Theta A$, there is $T' \in \Theta B$ such that $\vartheta T \leqq \vartheta T'$ by 2.18 and 2.8 (a) which is equivalent to $\sup \{\vartheta T \in \mathcal{O};\ T \in \Theta A\} \leqq \sup \{\vartheta T \in \Theta B\}$.

We shall formulate a lemma for $(\bar{f}, \infty_2) - \mathcal{M}$ that is analogous to 2.23. If $\mathcal{C}$ is an antisymmetric category and $\mathcal{D} \subseteqq \mathcal{C}$ a full subcategory then we put $\max \mathcal{D} = \{P \in \mathcal{D};$ for each $Q \in \mathcal{D}$, $[P, Q]_{\mathcal{C}} \neq \emptyset$ implies $Q = P\}$.

2.24. Lemma. *Let $A, B \in (\bar{f}, \infty_2) - \mathcal{M}$ and let $\mathcal{N}$ be the category defined in 2.12. Then $[A, B]_{\mathcal{M}} \neq \emptyset$ iff, for each $m \in \max \{RT \in \mathcal{N};\ T \in \Theta A \cap (\bar{f}, \infty_2) - \mathcal{M}^c\}$, there is $n \in \max \{RT \in \mathcal{N};\ T \in \Theta B \cap (f, \infty_2) - \mathcal{M}^c\}$ such that $[m, n]_{\mathcal{N}} \neq \emptyset$.*

Proof. By 2.18, $[A, B]_{\mathcal{M}} \neq \emptyset$ iff, for each $T \in \Theta A$ there is $T' \in \Theta B$ such that $[T, T']_{\mathcal{M}^c} \neq \emptyset$. If $T \in \Theta A \cap ((\bar{f}, o) - \mathcal{M}^c \cup (\bar{f}, \infty_1) - \mathcal{M}^c)$ then, for $T'' \in \Theta B \cap \cap (\bar{f}, \infty_2) - \mathcal{M}^c$, we have $[T, T'']_{\mathcal{M}^c} \neq \emptyset$ by 2.7 (c). Thus, $[A, B]_{\mathcal{M}} \neq \emptyset$ iff, for each $T \in \Theta A \cap (\bar{f}, \infty_2) - \mathcal{M}^c$, there is $T' \in \Theta B \cap (\bar{f}, \infty_2) - \mathcal{M}^c$ such that $RT' \mid RT$ by 2.8 (c). But the latest is equivalent to the condition that, for each $m \in \in \max \{RT;\ T \in \Theta A \cap (\bar{f}, \infty_2) - \mathcal{M}^c\}$, there is $n \in \max \{RT;\ T \in \Theta B \cap (\bar{f}, \infty_2) - - \mathcal{M}^c\}$ such that $n \mid m$.

2.25. Lemma. *Let $A \in \bar{f} - \mathcal{M}$, $B \in (\bar{f}, o) - \mathcal{M}$. Then $[A, B]_{\mathcal{M}} \neq \emptyset$ iff $A \in (\bar{f}, o) - \mathcal{M}$ and, for each $T \in \Theta A$, there is $T' \in \Theta B$ such that $H(T, T') \neq \emptyset$.*

Indeed, the assertion follows from 2.8 (b) and 2.18.

2.26. Theorem. *The following assertions hold:*

(a) *Let $A, B \in f - \mathscr{M}$. Then $[A, B]_{\mathscr{M}} \neq \emptyset$ if and only if precisely one of the following three cases occurs:*

(1) *$A, B \in (f, o) - \mathscr{M}$ and* $\sup\{\vartheta T \in \mathscr{O}; T \in \Theta A\} \leqq \sup\{\vartheta T \in \mathscr{O}; T \in \Theta B\}$.

(2) *$A \in (f, o) - \mathscr{M}$, $B \in (f, \infty_1) - \mathscr{M}$.*

(3) *$A, B \in (f, \infty_1) - \mathscr{M}$.*

(b) *Let $A, B \in \bar{f} - \mathscr{M}$. Then $[A, B]_{\mathscr{M}} \neq \emptyset$ if and only if precisely one of the following six cases occurs:*

(1) *$A, B \in (\bar{f}, o) - \mathscr{M}$ and the conditions of* 2.25 *hold.*

(2) *$A \in (\bar{f}, o) - \mathscr{M}$, $B \in (\bar{f}, \infty_1) - \mathscr{M}$.*

(3) *$A, B \in (\bar{f}, \infty_1) - \mathscr{M}$.*

(4) *$A \in (\bar{f}, o) - \mathscr{M}$, $B \in (\bar{f}, \infty_2) - \mathscr{M}$.*

(5) *$A \in (\bar{f}, \infty_1) - \mathscr{M}$, $B \in (\bar{f}, \infty_2) - \mathscr{M}$.*

(6) *$A, B \in (\bar{f}, \infty_2) - \mathscr{M}$ and the condition of* 2.24 *holds.*

Proof of (a). Let $A \in f - \mathscr{M}$ be arbitrary. If $B \in (f, o) - \mathscr{M}$ then we have (1) by 2.23. If $B \in (f, \infty_1) - \mathscr{M}$ then (2) or (3) holds by 2.22 (a).

Proof of (b). Let $B \in \bar{f} - \mathscr{M}$ be arbitrary. If $B \in (\bar{f}, o) - \mathscr{M}$ then we obtain (1) by 2.25, if $B \in (\bar{f}, \infty_1) - \mathscr{M}$ then (2) or (3) hold by 2.22 (b) and, finally, if $B \in$ $\in (\bar{f}, \infty_2) - \mathscr{M}$ then one of (4), (5), (6) holds by 2.22 (c) and 2.24.

Again, we shall formulate this theorem for corresponding basic categories.

2.27. Lemma. *The following assertions hold:*

(a) $f - \mathscr{M}(b) \cong (f, o) - \mathscr{M}(b) \oplus (f, \infty_1) - \mathscr{M}(b)$.

(b) $\bar{f} - \mathscr{M}(b) \cong (\bar{f}, o) - \mathscr{M}(b) \oplus (\bar{f}, \infty_1) - \mathscr{M}(b) \oplus (\bar{f}, \infty_2) - \mathscr{M}(b)$.

Indeed, the assertions follow from 2.26.

2.28. Lemma. *Let $\mathscr{N}$ be the category defined in* 2.12. *Then*

(a) *$\mathscr{P} \neq \emptyset$ iff* $\max \mathscr{P} \neq \emptyset$ *for each $\mathscr{P} \subseteqq \mathscr{N}$,*

(b) *for each $\mathscr{P} \subseteqq \mathscr{N}$, there is $\mathscr{R} \subseteqq \mathscr{N}$ such that* $\max \mathscr{R} = \mathscr{P}$ *iff $\mathscr{P}$ is an antichain.*

(Clearly, these assertions hold for more general categories, too.)

2.29. Definition. (a) Let $\lambda \in \mathrm{Ord} - \{0\}$ be arbitrary. Then we put

$$(f, \lambda) - \mathscr{M} = \{A \in (f, o) - \mathscr{M};\ \sup\{\vartheta T; T \in \Theta A\} = \lambda\}.$$

(b) Let $\mathscr{N}$ be the category defined in 2.12. Then we denote by $\mathscr{A}(\mathscr{N})$ the thin category of all non empty antichains in $\mathscr{N}$ such that, for each $P, Q \in \mathscr{A}(\mathscr{N})$, $[P, Q]_{\mathscr{A}(\mathscr{N})} \neq \emptyset$ iff, for each $m \in P$, there is $n \in Q$ such that $[m, n]_{\mathscr{N}} \neq \emptyset$.

(c) Let $P \in \mathscr{A}(\mathscr{N})$ be arbitrary. Then we put

$(\bar{f}, P) - \mathscr{M} = \{A \in (\bar{f}, \infty_2) - \mathscr{M}; \max\{RT \in \mathscr{N}; T \in \Theta A \cap (\bar{f}, \infty_2) - \mathscr{M}^c\} = P\}$.

(d) Let $\mathscr{O}$ be the category defined in 2.12 and let $[\infty_1]$, $[o]$ be thin categories defined on $\{\infty_1\}$, $\{o\}$. Then we put

$$\bar{\mathscr{O}} = \mathscr{O} \oplus [\infty_1], \qquad \mathscr{A}^*(\mathscr{N}) = [o] \oplus [\infty_1] \oplus \mathscr{A}(\mathscr{N}).$$

2.30. Lemma. *The following simple assertions hold*:

(a) $\mathscr{A}(\mathscr{N})$ *is antisymmetric.*

(b) *For each* $A \in (\bar{f}, \infty_2) - \mathscr{M}$, *there is* $P \in \mathscr{A}(\mathscr{N})$ *such that* $\max\{RT \in \mathscr{N};\ T \in \Theta A \cap (\bar{f}, \infty_2) - \mathscr{M}^c\} = P$.

Indeed, (a) is easy to prove and (b) holds by 2.21 and 2.23 because $\Theta A \cap (\bar{f}, \infty_2) - \mathscr{M}^c \neq \emptyset$.

2.31. Lemma. *The following assertions hold*:

(a) *Let* $\mathcal{O}$ *be the category defined in 2.12. Then* $(f, o) - \mathscr{M}(b) \cong \sum^o_{\lambda \in \mathcal{O}} (f, \lambda) - \mathscr{M}(b)$.

(b) *Let* $\mathscr{A}(\mathscr{N})$ *be the category defined in* 2.29 (b). *Then* $(\bar{f}, \infty_2) - \mathscr{M}(b) \cong \sum^l_{P \in \mathscr{A}(\mathscr{N})} (\bar{f}, P) - \mathscr{M}(b)$.

Proof. The assertion (a) follows from 2.26 (a) (the case (1)) and (b) follows from 2.26 (b) (the case (6)).

2.32. Lemma. *Let* $\lambda \in \mathcal{O}$, $P \in \mathscr{A}(\mathscr{N})$ *be arbitrary. Then* $(f, \lambda) - \mathscr{M}(b)$, $(\bar{f}, P) - \mathscr{M}(b)$ *are categories with non empty homs.*

Indeed, the assertion follows directly from 2.29 and 2.26 (a) (the case (1)) and 2.26 (b) (the case (6)).

2.33. Theorem. *The following assertions hold*:

(a) *Let* $\bar{\mathcal{O}}$ *be the category defined in* 2.29 (d). *Then* $f - \mathscr{M}(b) \cong \sum^o_{\lambda \in \bar{\mathcal{O}}} (f, \lambda) - \mathscr{M}(b)$ where $(f, \lambda) - \mathscr{M}(b)$ *is with non empty homs for each* $\lambda \in \bar{\mathcal{O}}$.

(b) *Let* $\mathscr{A}^*(\mathscr{N})$ *be the category defined in* 2.29 (d). *Then* $\bar{f} - \mathscr{M}(b) \cong \sum^l_{P \in \mathscr{A}^*(\mathscr{N})} (\bar{f}, P) - \mathscr{M}(b)$ *where* $(\bar{f}, P) - \mathscr{M}(b)$ *is with non empty homs for each* $P \in \mathscr{A}^*(\mathscr{N}) - [o]$.

Indeed, the assertions are consequences of 2.29 (d), 2.31, 2.32 and 2.26 (a) (the case (3)) and 2.26 (b) (the case (3)).

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