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Archivum Mathematicum, Vol. 14 (1978), No. 1, 21--44

Persistent URL: `http://dml.cz/dmlcz/106988`

ARCH. MATH. 1, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XIV: 21—44, 1978

ON THE OSCILLATORY AND MONOTONE SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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(Received August 29, 1977)

During the last twenty years a great number of papers dealing with non-autonomous ordinary differential equations have appeared in which various classifications of equations according to the oscillatory properties of their solutions are proposed, the existence or absence of singular, proper, oscillatory and monotone solutions of various types are established and asymptotic properties of such solutions are studied.

In the present report an attempt is made to illustrate the main results obtained in the above-mentioned directions (mainly in the non-linear case) for the differential equations

$$u^{(n)}(t) = a(t) \mid u(t) \mid^{\lambda} \operatorname{sign} u(t) \tag{0.1}$$

and

$$u^{(n)}(t) = a(t) \mid u(\tau(t)) \mid^{\lambda} \operatorname{sign} u(\tau(t)) \tag{0.2}$$

and to formulate some unsolved problems.

In what follows it will be assumed that $n \geqq 2$, $\lambda > 0$, the function $a: [t_0, \infty[\to R$ is summable on each finite segment and the function $\tau : [t_0, +\infty[\to R$ is continuous and satisfies the conditions

$$\tau(t) \leqq t \qquad \text{for } t \geqq t_0, \ \lim_{t \uparrow +\infty} \tau(t) = +\infty.$$

We shall use the following definitions.

The solution u of the equation (0.1) or (0.2) which is defined in a certain neighbourhood of $+\infty$ is called *proper*, if for all sufficiently large values of t

$$\sup \{\mid u(s) \mid : t \leqq s < +\infty\} > 0;$$

if, in addition, u has (has not) a sequence of zeroes converging to $+\infty$, then it is called *a proper oscillatory* (*non-oscillatory*) *solution.*

The solution u which is defined in a certain interval $]t_*, +\infty[$ is called *a singular solution of the first kind* if there exists $t^* \in]t_*, +\infty[$ such that

$$\max\{|u(s)| : t \leqq s \leqq t^*\} > 0 \quad \text{for } t_* < t < t^* \qquad \text{and} \qquad u(t) = 0 \quad \text{for } t \geqq t^*;$$

if, in addition, u has (has not) a sequence of zeroes in $]t_*, t^*[$ converging to t^*, then it is called *an oscillatory (non-oscillatory) singular solution of the first kind.*

The solution u which is defined in a certain finite interval $]t_*, t^*[$ is called *a singular solution of the second kind if*

$$\limsup_{t \uparrow t^*} |u(t)| = +\infty;$$

if, in addition, u has (has not) a sequence of zeroes converging to t^*, then it is called *an oscillatory (non-oscillatory) singular solution of the second kind.*

§ 1 of this survey deals with the equation (0.1) in the case when $n = 2$ and $\lambda \neq 1$, [1]) in § 2 the same equation is considered when $n \geqq 3$ and in § 3 the equation (0.2) is studied.

§ 1. THE EQUATION OF EMDEN – FOWLER TYPE

As it was already noted above in this paragraph we shall consider the equation

$$u''(t) = a(t) \, |u(t)|^\lambda \operatorname{sign} u(t) \tag{1.1}$$

where $\lambda \neq 1$, which is known in the literature as the equation of Emden-Fowler type.

The equation of this type for the first time attracted attention at the close of the XIX century in connection with the astrophysical investigations of R. Emden, and in the thirties it appeared in the papers of E. Fermi and L. H. Thomas devoted to the distribution of electrons in the heavy atom.

A detailed study of the asymptotic behaviour of proper solutions of the equation (1.1) in a rather special but significant for application case when $a(t) = \pm t^\sigma$ and $\lambda > 1$ is carried out in the well-known monograph by R. Bellman [3]. The results to be discussed below appeared after this monograph had been published.

1.1. Theorems on oscillation and non-oscillation of proper solutions. F. V. Atkinson [2] for $\lambda > 1$ and Š. Belohorec [4a] for $0 < \lambda < 1$ proved the following.

Theorem 1.1. *If a is non-positive then the condition*

$$\int^{+\infty} t^\mu a(t) \, dt = -\infty, \tag{1.2}$$

[1]) In the case $n = 2$, $\lambda = 1$, quite full information about oscillating and monotone solutions of equation (0.1) may be found in investigations of M. Ráb [52].

where $\mu = \min\{1, \lambda\}$, *is necessary and sufficient for oscillation of all proper solutions of the equation* (1.1).

The sufficient conditions of oscillation of all proper solutions without the assumption of non-positivity of the coefficient were first established by P. Waltman [50]. More general conditions are given in [22i] (for $\lambda > 1$) and in [4d] (for $0 < \lambda < 1$). In [22i, 4d], in particular, the following theorem is proved.

Theorem 1.2. *If for a certain non-negative constant* $\mu \leqq \min\{1, \lambda\}$ (1.2) *is fulfilled, then each proper solution of the equation* (1.1) *is oscillatory.*

This theorem allows to generalize Theorem 1.1 in the following way

Theorem 1.1′. *Suppose* $a(t) = a_0(t) + \alpha(t)$, a_0 *is non-positive and* α *satisfies the condition*

$$\int^{+\infty} t^{\mu} \, |\alpha(t)| \, \mathrm{d}t < +\infty,$$

where $\mu = \min\{1, \lambda\}$. *Then for oscillation of all proper solutions of the equation* (1.1) *it is necessary and sufficient that*

$$\int^{+\infty} t^{\mu} a_0(t) \, \mathrm{d}t = -\infty.$$

Recently G. J. Butler [6] has discussed the rather interesting case of strong oscillation of the coefficient a, which was not covered by Theorem 1.2. Here we shall bring only one result of [6].

Theorem 1.3. *Let a be different from zero on the set of positive measure and let it be periodic with* $\omega > 0$ *period. Then for oscillation of all proper solutions of the equation* (1.1) *it is sufficient and when* $\lambda > 1$ *it is necessary as well that*

$$\int_0^{\omega} a(t) \, \mathrm{d}t \leqq 0.$$

M. Jasny and J. Kurzweil [19, 32b] and Kuo-liang Chiou [31] investigated the question of the existence of at least one proper oscillatory solution of the equation (1.1) in the cases when $\lambda > 1$ and $0 < \lambda < 1$, respectively. They proved the following

Theorem 1.4. *If a is negative and* $t^{\frac{\lambda+3}{2}} \, |a(t)|$ *is non-decreasing, then the equation* (1.1) *possesses at least one proper oscillatory solution.*

In [22a] the following theorem on non-oscillation of all proper solutions was suggested.

Theorem 1.5. *Suppose that a is negative and that for certain sufficiently small* $\varepsilon > 0$, $t^{\frac{\lambda+3}{2}+\varepsilon} \, |a(t)|$ *is non-increasing. Then in the case when* $\lambda > 1$ *all proper solutions of the equation* (1.1) *are non-oscillatory.*

Z. Nehary [45] has somewhat generalized this result requiring that $(t \ln t)^{\frac{\lambda+3}{2}} \times$ $\times |a(t)|$ be non-increasing.

Up to now the following problem remains unsolved.

Problem 1.1. Does Theorem 1.5 remain valid in the case when $0 < \lambda < 1$?

The theorem of Š. Belohorec ([4c], Theorem 6) gives the positive answer to this question only in a very particular case, when

$$\liminf_{t\uparrow+\infty}\left[t^{\frac{\lambda+3}{2}+\varepsilon}\,|a(t)|\right] > 0\,.$$

Theorems 1.4 and 1.5 (even if the latter is assumed to be valid when $\lambda \in]0, 1[$) do not give an exhaustive answer to the question of the existence of at least one oscillatory proper solution. The considerable advance in this direction would be the solution of the following.

Problem 1.2. Let a be a negative function of bounded variation on each finite segment.

a) Does the condition

$$\int^{+\infty} t^{\frac{\lambda+1}{2}} a(t)\,\mathrm{d}t = -\infty$$

provide the existence of at least one oscillatory proper solution of the equation (1.1)?

b) Do the conditions

$$\int^{+\infty} t^{\frac{\lambda+1}{2}} a(t)\,\mathrm{d}t > -\infty \tag{1.3}$$

and

$$\int^{+\infty} |\mathrm{d}a(t)| < +\infty$$

provide the non-oscillation of all proper solutions of the equation (1.1)?

The fulfilment of the condition (1.3) is not sufficient for non-oscillation of all proper solutions of the equation (1.1) because, as it is shown in [13, 7, 4d, 14a], there exists the continuous function $a : [0, +\infty[\to]-\infty, 0[$ of bounded variation on each finite segment such that

$$\lim_{t\uparrow+\infty}\,[t^{\lambda+3}a(t)] > -\infty$$

and the equation (1.1) possesses at least one oscillatory proper solution.

In connection with this example there arises the following

Problem 1.3. Does there exist for arbitrarily fixed $\mu > 0$ the continuous function $a : [0, +\infty[\to]-\infty, 0[$ such that

$$\lim_{t\uparrow+\infty}\,[t^{\mu}a(t)] > -\infty$$

and that the equation (1.1) possesses at least one oscillatory solution?

In conclusion of this section it should be pointed out that the above theorems allow generalizations in various directions and this was carried out by many authors.

We should mention first all the papers by D. V. Izjumova [17a, b, d, e] in which the theorems of 1.1, 1.2, 1.4 and 1.5 type are proved for the differential equation

$$u''(t) = f(t, u(t)). \tag{1.4}$$

The results of the similar character are contained in the papers by Š. Belohorec [4e, f], Z. Nehary [45], C. V. Coffman and J. S. Wong [8], J. W. Heidel and I. T. Kiguradze [15], I. V. Kamenev [20], L. H. Erbe and J. S. Muldowney [9].

The recent investigations of J. D. Mirzov [41a–d] concerning the two-dimensional differential systems

$$x'(t) = f_i(t, x_1(t), x_2(t)) \qquad (i = 1, 2)$$

are also related with this series of papers.

1.2. On the existence of proper and singular solutions. It is obvious that when $\lambda > 1$ $(0 < \lambda < 1)$ the equation (1.1) has not a singular solution of the first (second) kind. If a is a non-positive (non-negative) function, then the equation (1.1) has not a non-oscillatory (oscillatory) singular solution. In [7; 22i; 18] the following theorem is proved.

Theorem 1.6. *If a is a negative function of boundary variation in each finite segment, then the equation* (1.1) *has not a singular solution, i.e. every non-trivial maximally continued to the right solution of this equation is proper.*

The assumption on the boundedness of variation in this theorem is rather important. In the case $\lambda > 1$ $(0 < \lambda < 1)$, S. P. Hastings [13], C. V. Coffman and D. F. Ullrich [7], (Š. Belohorec [4d] and J. W. Heidel [14a]) have constructed an example of continuous function $a : [0, +\infty[\to]-\infty, 0[$ such that the equation (1.1) possesses an oscillatory singular solution of the second (first) kind.[1])

The following theorem holds.

Theorem 1.7. *Let $a(t) = a_0(t) + \alpha(t)$, where a_0 is negative and absolutely continuous on each finite segment and α satisfies the condition*

$$\int^{+\infty} |\alpha(t)|\, |a_0(t)|^{-\frac{1+3\lambda}{4(1+\lambda)}} \exp\left[\frac{|\lambda - 1|}{4(\lambda + 1)} \int_{t_0}^{t} \frac{|a_0'(s)|}{|a_0(s)|} \mathrm{d}s\right] \mathrm{d}t < +\infty.$$

Then the equation (1.1) *has proper solutions.*

In conditions of this theorem the equation (1.1) may possess singular solutions as well as proper ones, which is demonstrated by

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[1]) See in this connection the paper by A. D. Myshkis [43].

Theorem 1.8. *Let there be an interval* $]t_1, t_2[\subset [t_0, +\infty[$ *such that* a *is non-negative in* $]t_1, t_2[$ *and is different from zero on a certain set of positive measure from this interval. Then when* $0 < \lambda < 1$ $(\lambda > 1)$, *the equation* (1.1) *possesses non-oscillatory singular solutions of the first (second) kind.*

Theorems 1.1 – 1.3 are concerned with the oscillation of all proper solutions when the existence of such solutions is a priori assumed. This assumption is justified for the coefficients which satisfy the conditions of Theorem 1.7. In the general case this question remains open.

Problem 1.4. Does the equation (1.1) possess at least one proper solution in conditions of any Theorem 1.1, 1.2 or 1.3 (without additional restrictions on a)?

1.3. The asymptotic properties of oscillatory solutions. Throughout this section it will be assumed that a is negative and has bounded variation on each finite segment. Below we shall consider the maximally continued to the right non-trivial solutions which according to Theorem 1.6 are proper.

Let

$$a_1(t) = \frac{1}{2}\Big[\int_{t_0}^{t} |\mathrm{d}a(\tau)| - a(t)\Big], \qquad a_2(t) = \frac{1}{2}\Big[\int_{t_0}^{t} |\mathrm{d}a(\tau)| + a(t)\Big].$$

Theorem 1.9. *If*

$$\int^{+\infty} \frac{\mathrm{d}a_2(t)}{|a(t)|} < +\infty$$

then for any solution u *of the equation* (1.1) *there exists the finite limit*

$$\lim_{t \uparrow +\infty} \left[\frac{u'^2(t)}{|a(t)|} + \frac{2}{\lambda + 1} |u(t)|^{\lambda+1}\right]. \tag{1.5}$$

If $\lambda = 1$ and $a_1(t) \uparrow +\infty$ when $t \uparrow +\infty$, then by the theorem by H. M. Milloux [40] the equation (1.1) possesses a non-trivial solution, for which the limit (1.5) is equal to zero. With the same assumption the equation (1.1) may also possess solutions for which (1.5) is not equal to zero [11, 23]. There are some statements as the well-known theorem by Armellini – Tonelli – Sansone containing conditions under which the limit (1.5) is equal to zero for any solution of the equation (1.1) (in the case when $\lambda = 1$). Note for example the theorem by J. Kurzweil [32a] and P. Hartman [12]. In [18] the following non-linear analogue of theorem by J. Kurzweil for the equation (1.1) is proved.

Theorem 1.10. *Let*

$$\int_H \mathrm{d} \ln a_1(t) = +\infty$$

for every open set $H \subset]t_0, +\infty[$ *which satisfies the condition*
$\mathrm{mes}\,(H \cap]t, t+1[) \to 1$ *when* $t \uparrow +\infty$.
Then for any solution u *of the equation* (1.1) *the limit* (1.5) *is equal to zero.*

T. A. Chanturia [51c] has proved a somewhat more general statement than Theorem 1.10. Besides he specially considered the case when

$$\lim_{t \uparrow +\infty} a(t) = 0$$

and showed that under certain additional restrictions on the function a the limit (1.5) is equal to $+\infty$ for any oscillatory solution u of the equation (1.1).

Problem 1.5. Does the condition
$a(t) \downarrow -\infty$ when $t \uparrow +\infty$ ($a(t) \uparrow 0$ when $t \uparrow +\infty$)
provide the existence of at least one non-trivial solution of (1.1) which tends to zero (which is not bounded) when $t \uparrow +\infty$.

The sufficient conditions of boundedness and tending to zero when $t \uparrow +\infty$ for oscillatory solutions of the equations (1.1) and (1.4) as well as asymptotic formulae for such solutions are given in [16; 17c; 18; 22c, g; 25; 51b, c].

In [22c] the following theorem is proved.

Theorem 1.11. *Let $\lambda > 1$. Suppose that the function $a : [t_0, +\infty[\to]-\infty, 0[$ is absolutely continuous on each finite segment and*

$$a'(t) \leqq 0 \qquad \text{when } t \geqq t_0 \qquad \text{and} \int^{+\infty} \left| \mathrm{d} \frac{a'(t)}{|a(t)|^{1+\frac{1}{\lambda+3}}} \right| < +\infty.$$

Then any non-trivial solution u of the equation (1.1) *may be represented in a neighbourhood of $+\infty$ in the form*

$$\begin{aligned} u(t) &= |a(t)|^{-\frac{1}{\lambda+3}} \varrho(t)\, W\Big(\eta(t) \int_{t_0}^{t} |a(s)|^{\frac{2}{\lambda+3}} \mathrm{d}s\Big), \\ u'(t) &= |a(t)|^{-\frac{1}{\lambda+3}} [\varrho(t)]^{\frac{\lambda+1}{2}} W'\Big(\eta(t) \int_{t_0}^{t} |a(s)|^{\frac{2}{\lambda+3}} \mathrm{d}s\Big), \end{aligned} \tag{1.6}$$

where η and ϱ are continuous functions satisfying the following conditions

$$\lim_{t \uparrow +\infty} \varrho(t) = \varrho_0, \qquad \lim_{t \uparrow +\infty} \eta(t) = \varrho_0^{\frac{\lambda+1}{2}}, \qquad 0 < \varrho_0 < +\infty$$

and w is the solution of the problem

$$w'' = -|w|^{\lambda} \operatorname{sign} w; \qquad w(0) = 0, \qquad w'(0) = 1.$$

In [51b] it is shown that if $0 < \lambda < 1$ and the function a satisfies the conditions of Theorem 1.11, then the equation (1.1) possesses solutions which may be represented as (1.6). But the question of the correctness of such asymptotic representation for all solutions still remains open.

1.4. Asymptotics of non-oscillatory solutions.; F. V. Atkinson [2] and Š. Belohorec [4b] in the cases $\lambda > 1$ and $0 < \lambda < 1$ respectively, have proved the following

Theorem 1.12. *Let* $a : [t_0, +\infty[\to]-\infty, 0[$ *be non-decreasing and*

$$\int^{+\infty} t^{\nu} |a(t)| \, dt < +\infty, \tag{1.7}$$

where $\nu = \max\{1, \lambda\}$. *Then any proper solution of the equation* (1.1) *when* $t \uparrow +\infty$ *is of the form*

$$u(t) \sim c_0 + c_1 t \qquad (|c_0| + |c_1| \neq 0).$$

In the paper [22d] asymptotic formulae are given for non-oscillatory proper solutions of the equation (1.1) in the case when $\lambda > 1$ and a is a negative function (which does not satisfy the condition (1.7)).

In [22h] the asymptotic formulae for the proper solutions are established in the case when

$$a(t) = a_0(t) + \alpha(t),$$

where $a_0 : [t_0, +\infty[\to]0, +\infty[$ is a twice continuously differentiable function and $\alpha : [t_0, +\infty[\to R$ is small in a certain sense.

Assume

$$b(t) = [a_0(t)]^{-\frac{1}{\lambda+3}}, \qquad b_1(t) = \frac{2(\lambda+1)}{(\lambda-1)^2} + b''(t)\, b(t) \left[\int_t^{+\infty} b^{-2}(\tau)\, d\tau\right]^2,$$

$$b_2(t) = \frac{2(\lambda+1)}{(\lambda-1)^2} + b''(t)\, b(t) \left[\int_{t_0}^{t} b^{-2}(\tau)\, d\tau\right]^2.$$

In [22h] the following theorem is proved.

Theorem 1.13. *Let* $\lambda > 1$

$$\lim_{t \uparrow +\infty} \frac{\alpha(t)}{a_0(t)} = 0, \qquad \int^{+\infty} b^{-2}(\tau)\, d\tau < +\infty$$

and let there exist the finite limit $b_1(+\infty) > 0$. *Then for any proper solution* u *of the equation* (1.1) *when* $t \uparrow +\infty$ *we have*

$$u(t) \sim c_0 + c_1 t, \qquad \text{where } |c_0| + |c_1| \neq 0,$$

or

$$u(t) \sim \pm [b_1(+\infty)]^{\frac{1}{\lambda-1}}\, b(t) \left[\int_t^{+\infty} b^{-2}(\tau)\, d\tau\right]^{\frac{2}{1-\lambda}}.$$

Theorem 1.14. *Let* $\lambda > 1$,

$$\lim_{t \uparrow +\infty} \frac{\alpha(t)}{a_0(t)} = 0, \qquad \int^{+\infty} b^{-2}(\tau)\, d\tau = +\infty$$

and let there exist the finite limit $b_2(+\infty) > 0$. *Then for each proper solution* u_1 *of the equation* (1.1) *we have*

$$u(t) \sim \pm[b_2(+\infty)]^{\frac{1}{\lambda-1}} b(t) \Big[\int_{t_0}^{t} b^{-2}(\tau)\, d\tau\Big]^{\frac{2}{1-\lambda}}.$$

As it is shown in [51a] Theorem 1.13 (Theorem 1.14) under additional conditions

$$b_1(+\infty) < \frac{(\lambda+3)^3}{4(\lambda-1)^2} \qquad \text{and} \qquad \int^{+\infty} |db_1(t)| < +\infty \Big(\int^{+\infty} |db_2(t)| < +\infty \Big)$$

remains valid also in the case when $0 < \lambda < 1$.

In [22h, 51a] the cases when $b_1(+\infty) = 0$ and $b_2(+\infty) = 0$ are investigated as well.

From the further investigations which contain asymptotic representations for non-oscillatory solutions of the equations of the type (1.1) we shall note here the papers by A. V. Kostin [30], L. B. Klebanov [24] and M. M. Aripov [1a, b].

2. THE EQUATION (0.1)

2.1. On A_k and B_k (A_k^* and B_k^*) properties. In fact the question of finding conditions under which there is a certain similarity between the equation (0.1) and one of the equations $u^{(n)}(t) = -u(t)$ or $u^{(n)}(t) = u(t)$ in the sense of oscillation of their solutions was raised by A. Kneser [26]. By this similarity, is usually meant that the equation (0.1) has the so called A_0 or B_0 properties.

It is said that the equation (0.1) has A_0 *property* (B_0 *property*) if each proper solution of this equation is oscillatory (is either oscillatory or satisfies at least one of the following conditions

$$|u^{(i-1)}(t)| \downarrow 0 \quad \text{when } t \uparrow +\infty \quad (i = 1, \ldots, n) \tag{2.1}$$

or

$$|u^{(i-1)}(t)| \uparrow +\infty \quad \text{when } t \uparrow +\infty \quad (i = 1, \ldots, n) \tag{2.2}$$

when n is even and is either oscillatory or satisfies the condition (2.1) (the condition (2.2)), when n is odd.

A. Kneser [26] has proved that if

$$\limsup_{t\uparrow+\infty} a(t) < 0, \tag{2.3}$$

then for $\lambda = 1$ the equation (0.1), i.e. the equation

$$u^{(n)}(t) = a(t)\, u(t), \tag{2.4}$$

has A_0 property. W. B. Fite [10] substituted (2.3) for the condition

$$\limsup_{t \uparrow +\infty} \left[t^{n-\varepsilon} a(t)\right] < 0, \tag{2.3'}$$

where $\varepsilon > 0$ is an arbitrarily small number. In the case of even [39] J. G. Mikusinski generalized the theorem of W. B. Fite assuming that instead of (2.3′) the conditions

$$a(t) \leqq 0 \quad \text{for } t \geqq t_0 \quad \text{and} \quad \int^{+\infty} t^{n-1-\varepsilon} a(t)\,\mathrm{d}t = -\infty .$$

are fulfilled. In [22e] the following theorem is proved.

Theorem 2.1. *Let a be non-positive (non-negative) and let there exist a continuous non-decreasing function $\omega : [t_0, +\infty[\to]0, +\infty[$ such that*

$$\int^{+\infty} \frac{\mathrm{d}t}{t\omega(t)} < +\infty \quad \textit{and} \quad \int^{+\infty} \frac{|a(t)|\, t^{n-1}}{\omega(t)} \mathrm{d}t = +\infty. \tag{2.5}$$

Then the equation (2.4) *has A_0 property (B_0 property).*

On the basis of the comparison theorems which will be stated in Section 2.2, V. A. Kondratjev [27b] (for negative a) and T. A. Chanturia [51e] (for positive a) have proved the following theorem.

Theorem 2.2. *Let*

$$a(t) \leqq \frac{\mu_n - \varepsilon}{t^n} \left(a(t) \geqq \frac{\nu_n + \varepsilon}{t^n}\right) \qquad \text{for } t \geqq t_0, \tag{2.5'}$$

where μ_n is the least (ν_n is the largest) of the local minima (maxima) of the polynomial $x(x-1)\dots(x-n+1)$. Then the equation (0.1) *has A_0 property (B_0 property).*

According to the theorem of I. M. Sobol [47] the condition

$$\int^{+\infty} t^{n-1}\, |a(t)|\, \mathrm{d}t = +\infty$$

is necessary for the equation (0.1) to have A_0 or B_0 properties, but it is not sufficient since the equation

$$u^{(n)}(t) = \frac{a_0}{t^n} u(t) \qquad (a_0 = \text{const} \neq 0)$$

has not oscillatory solutions provided all the roots of the algebraic equation

$$x(x-1)\dots(x-n+1) = a_0$$

are real.

Thus if one is confined to the case when a is a function of constant signs the conditions (2.5) and (2.5′) should not be consider as rough. In the case when a is a function with alternating signs the question whether the equation (2.4) has A_0 or B_0 properties is not, in fact, studied.

Problem 2.1. Is the condition

$$\int^{+\infty}\left[a(t)-\frac{\mu_n}{t^n}\right]t^{n-1}\,dt=-\infty\left(\int^{+\infty}\left[a(t)-\frac{\nu_n}{t^n}\right]t^{n-1}\,dt=+\infty\right),$$

where the number $\mu_n(\nu_n)$ is the same as in Theorem 2.2, sufficient for the equation (2.4) to have A_0 property (B_0 property)[1])?

Theorem 2.3. *Let $\lambda \neq 1$ and let a be a non-positive (non-negative) function. Then the condition*

$$\int^{+\infty} t^{(n-1)\mu}\,|\,a(t)\,|\,dt=+\infty, \tag{2.6}$$

where $\mu = \min\{1, \lambda\}$ is necessary and sufficient for the equation (0.1) *to have A_0 property (B_0 property).*

In the case when $\lambda > 1$ and a is non-positive this theorem was independently proved by the author [22b] and by I. Ličko and M. Švec [37], in the case when $0 < \lambda < 1$ and a is still non-positive it was proved in [37] and in all other cases — in [22e, f].

For further statements we need the following definitions.

Let $k \in \{1, \dots, n-1\}$. It is said that the equation (0.1) has A_k *property* (B_k *property*) if each proper solution u of this equation is either oscillatory or satisfies the condition

$$|\,u^{(i-1)}(t)\,|\downarrow 0 \qquad \text{when } t\uparrow +\infty \quad (i=k+1,\dots,n) \tag{2.7}$$

(is either oscillatory or satisfies at least one of the conditions (2.2) or (2.7)).

It is said that the equation (0.1) has A_k^* *property* (B_k^* *property*) if each proper solution u of this equation is either oscillatory or satisfies the condition

$$\lim_{t\uparrow +\infty}|\,u^{(n-k)}(t)\,|>0 \tag{2.8}$$

(is either oscillatory or satisfies at least one of the conditions (2.1) or (2.8)) when n is even and is either oscillatory or satisfies at least one of the conditions (2.1) or (2.8) (is either oscillatory or satisfies the condition (2.8)) when n is odd.

From the general oscillation theorems of the author [22k] concerning the equation

$$u^{(n)}(t)=f(t, u(t), u'(t), \dots, u^{(n-1)}(t)) \tag{2.9}$$

the following statements may be obtained.

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[1]) In the case when $n \in \{3, 4\}$ and $a(t) \leq \frac{\mu_n}{t^n}$ or $a(t) \geq \frac{\nu_n}{t^n}$ the theorems of V. A. Kondratjev [27a] give the positive answer to this question.

Theorem 2.4. *Let $\lambda > 1$ and let a be a non-positive (non-negative) function. Then the condition*

$$\int^{+\infty} t^{n-1+(\lambda-1)k} \, |a(t)| \, dt = +\infty,$$

where $k \in \{0, 1, \ldots, n-1\}$, is necessary and sufficient for the equation (0.1) *to have A_k property (B_k property).*

Theorem 2.5. *Let $0 < \lambda < 1$ and let a be a non-positive (non-negative) function. Then the condition*

$$\int^{+\infty} t^{(n-1)\lambda+(1-\lambda)k} \, |a(t)| \, dt = +\infty,$$

where $k \in \{1, 2, \ldots, n-1\}$ is an even (odd) number, is necessary and sufficient for the equation (0.1) *to have A_k^* property (B_k^* property).*

2.2. Comparison theorems. When studying oscillation of solutions of ordinary differential equations, the comparison theorems by V. A. Kondratjev's type [27b] become of special importance.

V. A. Kondratjev [27b] (for A_0 property) and T. A. Chanturia [51e] (for B_0 property) proved the following

Theorem 2.6. *If the equation*

$$u^{(n)}(t) = b(t)\, u(t) \tag{2.11}$$

has A_0 property (B_0 property) and

$$a(t) \leqq b(t) \leqq 0 \qquad (a(t) \geqq b(t) \geqq 0) \qquad \text{for } t \geqq t_0 \tag{2.12}$$

then the equation (2.4) *also has A_0 property (B_0 property).*

Problem 2.2. Let the equation (2.11) have at least one non-trivial oscillatory solution and suppose that the condition (2.12) is fulfilled. Has the equation (2.4) at least one non-trivial oscillatory solution?

In the case when $n \in \{3, 4\}$ and $b(t) \leqq 0$ the positive answer to this question immediately follows from the theorems of M. Švec [48] and Theorem 2.6.

For non-linear ordinary differential equations the theorem of 2.6 type was first proved by G. Kartsatos [21]. A little later T. A. Chanturia [29] established the comparison theorem for rather general functional-differential equations.[1])

Theorems 2.7 and 2.8 stated below from the theorem of G. Kartsatos when $k = 0$ and $g(t, y_1, \ldots, y_n)\, y_1 \leqq 0$; in the general case they are proved by T. A. Chanturia.

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[1]) Here the paper by B. Půža [46] containing the comparison theorem for two-dimensional systems should also be mentioned.

Theorem 2.7. *Let $k \in \{0, 1, \ldots, n-1\}$, let the equation*

$$(2.13) \qquad u^{(n)}(t) = g(t, u(t), \ldots, u^{(n-1)}(t))$$

have A_k property (B_k property) and suppose that the condition

$$(2.12') \qquad \begin{gathered} f(t, x_1, \ldots, x^n) \operatorname{sign} x_1 \leqq g(t, y_1, \ldots, y_n) \operatorname{sign} y_1 \leqq 0 \\ (f(t, x_1, \ldots, x_n) \operatorname{sign} x_1 \geqq g(t, y_1, \ldots, y_n) \operatorname{sign} y_1 \geqq 0) \qquad \text{for } x_1 y_1 \geqq 0 \\ |y_i| \leqq |x_i| \qquad (i = 1, \ldots, n) \end{gathered}$$

is fulfilled. Then the equation (2.9) *has also A_k property (B_k property).*

Theorem 2.8. *Let $k \in \{1, 2, \ldots, n-1\}$ be an even (odd) number. Suppose that the equation* (2.13) *has A_k^* property (B_k^* property) and that the condition* (2.12') *is fulfilled. Then the equation* (2.9) *has also A_k^* property (B_k^* property).*

2.3. On the solutions of (2.1) type. Naturally there arises the question of existence of solutions of all those types which are in the definitions of A_k and B_k (A_k^* and B_k^*) properties. We shall begin the discussion about this question with the solutions of (2.1) type.

From the statements of the author [22j] concerning the solvability of Kneser's problem for the equation (2.9) the following theorem is obtained.

Theorem 2.9. *Let*

$$(2.14) \qquad (-1)^n a(t) \geqq 0 \qquad \text{for } t \geqq t_0 .$$

Then: a) *for every $u_0 \neq 0$ the equation* (0.1) *has at least one solution which satisfies the conditions*

$$(2.15) \qquad u(t_0) = u_0, \quad (-1)^i u^{(i)}(t)\, u_0 \geqq 0 \qquad \text{for } t \geqq t_0 \quad (i = 0, \ldots, n-1);$$

b) *the condition*

$$(2.16) \qquad \int^{+\infty} t^{n-1} |a(t)|\, dt = +\infty$$

is necessary and sufficient for each solution of the problem (0.1), (2.15) *to satisfy the condition* (2.1) *whatever $u_0 \neq 0$.*

Thus if the conditions (2.14) and (2.16) are fulfilled the class of non-trivial solutions of (2.1) type for the equation (0.1) is not empty. For arbitrary element u of this class it is easy to verify that if $(-1)^i u^{(i)}(t_1)\, u(t_1) > 0$ $(i = 0, \ldots, n-1)$, where $t_1 > t_0$, then on $[t_0, t_1]$ the condition

$$(2.17) \qquad (-1)^i u^{(i)}(t)\, u(t) > 0 \qquad (i = 0, 1, \ldots, n-1)$$

is fulfilled; and if u is a proper solution, then the inequalities (2.17) are fulfilled in the whole interval $[t_0, +\infty[$. In order that all the solutions of the above-mentioned class may be proper it is sufficient that $\lambda \geqq 1$. But if $0 < \lambda < 1$ this does not remain valid. In [22l] the following theorem is proved.

Theorem 2.10. *Let* $0 < \lambda < 1$ *and let the function* a *be different from zero on a certain set of positive measure and satisfy the condition* (2.14). *Then the equation* (0.1) *has singular solutions of the first kind, which in every point where they are different from zero satisfy the inequalities* (2.17); *and if*

$$(2.18) \qquad \liminf_{t \uparrow +\infty} [(-t)^n a(t)] > 0$$

then each solution of (2.1) *type of the equation* (0.1) *is a singular solution of the first kind.*[1])

From Theorems 2.3 and 2.10 the following theorem may be obtained.

Theorem 2.3'. *Let* $0 < \lambda < 1$ *and let* (2.18) *be valid. Then the condition*

$$(2.19) \qquad \int^{+\infty} t^{(n-1)\lambda} \left| a(t) \right| \mathrm{d}t = +\infty$$

is necessary and sufficient for each proper solution of the equation (0.1) *to be oscillatory when* n *is odd and to be either oscillatory or to satisfy the condition* (2.2) *when* n *is even.*

Recently G. G. Kvinikadze [34] has shown the correctness of the following

Theorem 2.11. *Let* $0 < \lambda < 1$. *Suppose that the function* a *satisfying the condition* (2.14), *is not identical zero in every neighbourhood of* $+\infty$ *and*

$$(2.20) \qquad \int^{+\infty} t^{n-1} \left| a(t) \right| \mathrm{d}t < +\infty .$$

Then the equation (0.1) *has at least one proper solution of* (2.1) *type.*

In connection with Theorems 2.10 and 2.11 there arises the following problem.

Problem 2.3. *Let* $0 < \lambda < 1$. Suppose that the function a satisfies the condition (2.14) and is not identical zero in every neighbourhood of $+\infty$. Is the condition (2.20) necessary for the equation (0.1) to have at least one proper solution of (2.1) type?

2.4. On the solutions of (2.2) type. The following theorem results from the theorem of I. M. Sobol [47].

Theorem 2.12. *Let* $0 < \lambda \leqq 1$ *and let the function* a *be non-negative. Then the maximally continued solutions of the equation* (0.1) *which for a certain* $t_1 \in [t_0, +\infty]$ *satisfy the inequalities*

$$u^{(i)}(t_1)\, u(t_1) > 0 \qquad (i = 1, \ldots, n-1)$$

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[1]) The analogous results for non-linear differential equations of the third order are contained in the paper by D. Bobrowski [5] and for differential equations systems in the paper by T. A. Chanturia [51d].

are proper; moreover each of these solutions satisfies the condition (2.2) *if and only if*

$$\int^{+\infty} t^{(n-1)\lambda}\, a(t)\, \mathrm{d}t = +\infty.$$

When $\lambda > 1$, we have another picture. The following theorem is proved in [221].

Theorem 2.13. *Let $\lambda > 1$. Suppose that the function a is non-negative and is different from zero on a certain set of positive measure. Then the equation* (0.1) *possesses non-oscillatory singular solutions of the second kind; and if*

$$\liminf_{t \uparrow +\infty} \left[t^{1+(n-1)\lambda} a(t)\right] > 0\,, \tag{2.21}$$

then this equation has not proper solutions of (2.2) *type.*[1])

From Theorems 2.3 and 2.13 we obtain the following

Theorem 2.3″. *Let $\lambda > 1$ and let* (2.21) *be valid. Then the condition*

$$\int^{+\infty} t^{n-1} a(t)\, \mathrm{d}t = +\infty$$

is necessary and sufficient for each proper solution of the equation (0.1) *to be oscillatory when n is odd and to be either oscillatory or to satisfy the condition* (2.1) *when n is even.*

The following theorem belongs to G. G. Kvinikadze and to the author.

Theorem 2.14. *Let $\lambda > 1$. Suppose that the non-negative function a is not identical zero in every neighbourhood of $+\infty$ and*

$$\int^{+\infty} t^{(n-1)\lambda} a(t)\, \mathrm{d}t < +\infty. \tag{2.22}$$

Then the equation (0.1) *has at least one proper solution of* (2.2) *type.*

Problem 2.4. Let $\lambda > 1$. Suppose that the non-negative function a is not identical zero in every neighbourhood of $+\infty$. Is the condition (2.22) necessary for the equation (0.1) to have at least one proper solution of (2.2) type?

2.5. On proper oscillatory solutions. As it was shown in [27b] if a is non-positive (non-negative) and the equation (2.4) has A_0 property (B_0 property), then this equation possesses proper oscillatory solutions.)[2]

Below we shall deal with the equation (0.1) in the case, when $\lambda \neq 1$ and $n \geqq 3$.

J. W. Heidel [14b] has shown that if $0 < \lambda < 1$ and a is non-positive, then each maximally continued to the right solution u of the equation (0.1), satisfying the conditions

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[1]) The analogous results for the systems of ordinary differential equations was established by T. A. Chanturia [51f].

[2]) See also [22k], Theorems 2.1 and 2.2.

$$u(t_1) = 0, \qquad u^{(i)}(t_1)\, u'(t_1) > 0 \qquad (i = 1, \dots, n)$$

for a certain $t_1 \in [t_0, +\infty[$ is proper.

With regard to this fact we obtain the following theorem from Theorem 2.3.

Theorem 2.15. *Suppose that* $0 < \lambda < 1$ *and that the function* a *is non-positive and satisfies the condition* (2.19). *Then the equation* (0.1) *possesses proper oscillatory solutions.*

In the case when $\lambda > 1$ and a is non-positive the conditions under which the equation (0.1) would possess at least one proper oscillatory solution are unknown.

Theorem 2.16. *Let* $\lambda > 1$ $(0 < \lambda < 1)$. *Suppose that the function* a *is non-negative and*

$$\int^{+\infty} t^{\mu} a(t) \mathrm{d}t = +\infty, \qquad \mu = n - 1 + \frac{1 + (-1)^n}{2}(\lambda - 1) \Big(\int^{+\infty} t^{(n-2)\lambda + 1} a(t)\, \mathrm{d}t = +\infty \Big).$$

If, in addition, the equation (0.1) has not oscillatory singular solutions of the second (first) kind, then it has proper oscillatory solutions.

Corollary. Let one of the following three conditions be valid:

1. $n = 3$, $\lambda > 1$, a is non-negative and

$$\int^{+\infty} t^2 a(t)\, \mathrm{d}t = +\infty;$$

2. $n = 4$, $\lambda > 1$, a is positive, of bounded variation on each finite segment and

$$\int^{+\infty} t^{2+\lambda} a(t)\, \mathrm{d}t = +\infty;$$

3. $n = 4$, $0 < \lambda < 1$, a is positive, of bounded variation on each finite segment and

$$\int^{+\infty} t^{2\lambda + 1} a(t)\, \mathrm{d}t = +\infty.$$

In the cases not appearing in this corollary we know nothing about conditions under which the equation (0.1) with a non-negative coefficient a has proper oscillatory solutions.

Problem 2.5. Let the conditions of either Theorem 2.4 or Theorem 2.5 be valid. Does the equation (0.1) possess at least one proper oscillatory solution?

In conclusion let us note that in the case when $n \geqq 3$ the problem of asymptotic behaviour of the proper oscillatory solutions of the equation (0.1), in fact, remains unstudied. We know only some similar facts. As an example we shall bring one theorem concerning the third order equation.

Theorem 2.17. *If*

$$\liminf_{t \uparrow +\infty} a(t) > 0,$$

then all proper oscillatory solutions[1]) *of the equation*

$$u'''(t) = a(t) \mid u(t) \mid^{\lambda} \operatorname{sign} u(t)$$

tend to zero when $t \uparrow +\infty$.

3. THE EQUATION (0.2)

In investigations of oscillatory properties of differential equations with the retarded argument two directions may be distinguished; the first one consists in finding the analogues to the theorems given in Section 2.1, and the second one in finding the specific features which are caused by the delay.

In the first direction a great deal of research work has been carried out and quite a lot of results which often cover each other has been obtained. Here we shall bring only those results (Theorems 3.1 – 3.4) which to a certain extent describe the present state of the question.

N. V. Varekh and V. N. Shevelo [49] have proved the following

Theorem 3.1. *Let* $0 < \lambda < 1$ *and let the function a be non-positive* (*non-negative*). *Then the condition*

$$\int^{+\infty} [\tau(t)]^{(n-1)\lambda} \left| a(t) \right| dt = +\infty \tag{3.1}$$

is necessary and sufficient for the equation (0.2) *to have* A_0 *property* (B_0 *property*).

In the case when $\lambda \geqq 1$ in [49] as well as in other papers the authors in formulating oscillation theorems tried to keep formal similarity with Theorems 2.1 and 2.3 by substitution of t^{n-1} and $\dfrac{t^{n-1}}{\omega(t)}$ in the conditions (2.5) and (2.6) for $[\tau(t)]^{n-1}$ and $\dfrac{[\tau(t)]^{n-1}}{\omega(\tau(t))}$. As it comes out from the recent paper by Chanturia [51g] such an approach is not always valid.

In [51g] the following theorems are proved.

Theorem 3.2.[2]) *Suppose* $\lambda = 1$. *Let the function a be non-positive* (*non-negative*) *and let there exist* $\varepsilon > 0$ *such that*

$$\int^{+\infty} [\tau(t)]^{n-1-\varepsilon} a(t)\, dt = -\infty \left(\int^{+\infty} t[\tau(t)]^{n-2-\varepsilon} a(t)\, dt + \int^{+\infty} [\tau(t)]^{n-1} a(t)\, dt = +\infty \right).$$

Then the equation (0.2) *has* A_0 *property* (B_0 *property*).

---

[1]) According to Theorem 2.3″, when $\lambda > 1$, the equation to be considered has not non-oscillatory proper solutions.

[2]) This theorem is new for non-negative a. In the case when a is non-positive it was first proved by N. V. Varekh and V. N. Shevelo [49].

Theorem 3.2′. *Suppose $\lambda = 1$. Let the function a be non-negative and let the function τ satisfy the condition*

$$\limsup_{t \uparrow +\infty} \frac{\tau(t)}{t^{\nu}} < +\infty\,, \qquad \text{where } 0 < \nu < 1\,.$$

Then the condition

$$\int^{+\infty} [\tau(t)]^{n-1}\, a(t)\, \mathrm{d}t = +\infty$$

is necessary and sufficient for the equation (0.2) *to have B_0 property.*

Theorem 3.3. *Let $\lambda > 1$ and let n be an even* (*odd*) *number. Suppose that the function a is non-positive and*

$$\int^{+\infty} [\tau(t)]^{n-1}\, a(t)\, \mathrm{d}t = -\infty$$

($\int^{+\infty} [\tau(t)]^{n-2+\lambda-\varepsilon}\, a(t)\, \mathrm{d}t = \int^{+\infty} t^{n-1} a(t)\, \mathrm{d}t = -\infty$, $\varepsilon > 0$ *is arbitrarily small*). *Then the equation* (0.2) *has A_0 property.*

Theorem 3.3′. *Let $\lambda > 1$ and let n be an even* (*odd*) *number. Suppose that the function a is non-positive and that the function τ satisfies the condition*

$$\liminf_{t \uparrow +\infty} \left[\frac{\tau(t)}{t}\right] > 0 \left(\liminf_{t \uparrow +\infty} \frac{\tau(t)}{t^{\nu}} > 0, \text{ where } \nu > \frac{n-1}{n-2+\lambda}\right).$$

Then the condition

$$\int^{+\infty} t^{n-1} a(t)\, \mathrm{d}t = -\infty$$

is necessary and sufficient for the equation (0.2) *to have A_0 property.*

Theorem 3.4. *Let $\lambda > 1$ and let $n \geqq 3$ be an even* (*odd*) *number. Suppose that the function a is non-negative and*

$$\int^{+\infty} [\tau(t)]^{n-2+\lambda}\, a(t)\, \mathrm{d}t = \int^{+\infty} t^{n-1} a(t)\, \mathrm{d}t = +\infty \left(\int^{+\infty} [\tau(t)]^{n-1}\, a(t)\, \mathrm{d}t = +\infty \right).$$

Then the equation (0.2) *has B_0 property.*

Theorem 3.4′. *Let $\lambda > 1$ and let $n \geqq 3$ be an even* (*odd*) *number. Suppose that the function a is non-positive and that the function τ satisfies the condition*

$$\liminf_{t \uparrow +\infty} \left[t^{-\frac{n-1}{n-2+\lambda}} \tau(t)\right] > 0 \left(\liminf_{t \uparrow +\infty} \left[\frac{\tau(t)}{t}\right] > 0\right).$$

Then the condition

$$\int^{+\infty} t^{n-1} a(t)\, \mathrm{d}t = +\infty$$

is necessary and sufficient for the equation (0.2) *to have B_0 property.*

The list of papers containing oscillatory theorems for more general equations than (0.2) can be found in the survey by I. A. Mitropolsky and V. N. Shevelo [42] and in the paper by R. G. Koplatadze and T. A. Chanturia [29].[1])

The oscillatory theorems stating the specific properties of differential equations with retarded argument appeared first in the papers by G. Ladas, G. Ladde and J. S. Papadakis [35] and R. G. Koplatadze [28a, b] concerning the linear equations of the second order and the non-linear equations of the first and second order respectively. A little later G. Ladas, V. Lakshmikantham and J. S. Papadakis [36], R. G. Koplatadze [28c, d], T. Kusano and H. Onose [33], M. Naito [44] and T. A. Chanturia [29] have proved such theorems for the equations of an arbitrary order.

Theorem 3.5 stated below belongs to R. G. Koplatadze [29] and Theorems 3.6 and 3.6′ – to T. A. Chanturia [29].

Theorem 3.5. *Let* $\lambda = 1$,

$$(3.2) \qquad (-1)^n a(t) \geqq 0 \qquad \text{for } t \geqq t_0$$

and let the function τ *be non-decreasing. Suppose that for a certain* $k \in \{0, 1, \dots, n-1\}$ *and for arbitrarily small* $\varepsilon > 0$ *there holds*

$$\limsup_{t \uparrow +\infty} \frac{1}{k!(n-k-1)!} \int_{\tau(t)}^{t} [s - \tau(t)]^{n-k-1} [\tau(t) - \tau(s)]^k \, | a(s) | \, ds > 1$$

and

$$\int^{+\infty} [\tau(t)]^{n-1-\varepsilon} \, | a(t) | \, dt = +\infty.$$

Then each proper solution of the equation (0.2) *is oscillatory when n is odd, and is either oscillatory or satisfies the condition*

$$(3.3) \qquad | u^{(i)}(t) | \uparrow +\infty \qquad \text{when } t \uparrow +\infty \quad (i = 0, 1, \dots, n-1)$$

when n is even.

Theorem 3.6. *Let* $0 < \lambda < 1$ *and let the conditions* (3.2) *and*

$$\int^{+\infty} [\tau(t)]^{(n-1)\lambda} \, | a(t) | \, dt = \int^{+\infty} t^{(n-1)(1-\lambda)} [t - \tau(t)]^{(n-1)\lambda} \, | a(t) | \, dt = +\infty.$$

be fulfilled. Then each proper solution of the equation (0.2) *is oscillatory when n is odd and is either oscillatory or satisfies the condition* (3.3) *when n is even.*

Theorem 3.6′. *Let* $0 < \lambda < 1$ *and let the conditions* (3.1) *and*

$$\liminf_{t \uparrow +\infty} \frac{[t - \tau(t)] \, t^{\frac{1-\lambda}{\lambda}}}{\tau(t)} > 0.$$

[1]) Here as well the paper by P. Marušiak [38] should be mentioned where the theorems of 2.4 and 2.5 types are proved for the differential equations with retarded argument.

be fulfilled. Then the condition

$$\int^{+\infty} [\tau(t)]^{(n-1)\lambda} \, |\, a(t) \,|\, dt = +\infty$$

is necessary and sufficient for each proper solution of the equation (0.2) *to be oscillatory when n is odd and to be either oscillatory or to satisfy the condition* (3.3) *when n is even.*

R. G. Koplatadze [29] has shown that the equation (0.2) possesses proper oscillatory solutions if the conditions of one of the above theorems 3.1 – 3.6 are fulfilled and, in addition,

$$a(t) > 0 \quad (a(t) < 0) \qquad \text{and} \qquad \tau(t) < t \qquad \text{for } t \geqq t_0.$$

Problem 3.1. Let $\lambda > 1$ and let the condition (3.2) be fulfilled. Under what additional restrictions each proper solution of the equation (0.2) is oscillatory when n is odd and is either oscillatory or satisfies the condition (3.3) when n is even?

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