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# ON THE EXISTENCE AND BOUNDEDNESS OF SOLUTIONS OF A NONLINEAR DELAY DIFFERENTIAL SYSTEM

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We consider a perturbated nonlinear delay differential system of the form

$$y'(t) = A(t)\,y(t) + f(t, y(t), y[h(t)]), \tag{1}$$

and an unperturbed system

$$x'(t) = A(t)\,x(t). \tag{2}$$

Here $x, y, f$ are elements of the $n$-dimensional Euclidean space $R^n$ and $A(t)$ is an $n \times n$ matrix. Throughout the paper we assume that $A(t) \in C(J \equiv [t_0, \infty), R^n)$, $f(t, u, v) \in C(D \equiv J \times R^n \times R^n, R^n)$, and $h(t) \in C(J, R)$, $h(t) \leqq t$. The symbol $\|\cdot\|$ denotes some convenient norm of a vector or matrix.

The fundamental initial problem is formulated as follows: Let $E_{t_0} = [\inf_{t \in J} h(t), t_0]$, for $\inf_{t \in J} h(t) > -\infty$ and $E_{t_0} = (-\infty, t_0]$ otherwise, and let $\varphi(t)$ be a vector-function such that $\varphi(t) \in C(E_{t_0})$. It is to find a solution $y(t)$ (vector-function) of (1) on the interval $J$ satisfying the following initial conditions:

$$y(t_0) = \varphi(t_0), \qquad y[h(t)] \equiv \varphi[h(t)], \qquad h(t) < t_0. \tag{3}$$

Let $X(t)$ be a fundamental matrix of (2) such that

$$X(t_0) = I, \tag{4}$$

where $I$ denotes the identity matrix.

If $c$ denotes any constant vector, then the vector-function $x(t) = X(t)\,c$ is a solution of (2).

Define a function $\alpha$ on $E_{t_0} \cup J$ by

$$\alpha(t) = \begin{cases} \| X(t) \|, & t \in J, \\ \| I \|, & t \in E_{t_0}. \end{cases} \tag{5}$$

It is evident that $\alpha$ is continuous on $E_{t_0} \cup J$ and $\alpha(t) > 0$.

Put $c = \varphi(t_0)$.

In Theorem 1, using the methods of [1] and [4], we obtain a generalisation of the result of [3].

**Theorem 1.** *Let there exist a number $\lambda > 0$ such that*

$$\| \varphi(t) \| \leqq \lambda, \qquad t \in E_{t_0}, \qquad \text{and } \| c \| < \lambda. \tag{6}$$

*Suppose that there exists a scalar function $\omega(t, r_1, r_2)$ defined and continuous for $t \in J$ and $0 \leqq r_1, r_2 < \infty$ with the following properties*

(i) *$\omega(t, r_1, r_2)$ is nonnegative and nondecreasing in $r_1, r_2$ for every fixed $t \in J$,*

(ii) *$\| f(t, u, v) \| \leqq \omega(t, \| u \|, \| v \|)$ on $D$,*

(iii)

$$\int_{t_0}^{\infty} \| X^{-1}(t) \| \, \omega(t, \lambda\alpha(t), \lambda\alpha(t)) \, \mathrm{d}t < \lambda - \| c \|, \tag{7}$$

*where $X^{-1}(t)$ is the matrix inverse to $X(t)$.*

*Then every solution $y(t)$ of the initial problem (1), (3) satisfying condition*

$$y(t_0) = \varphi(t_0) = c, \tag{8}$$

*exists on $J$ and the following estimate*

$$\| y(t) - x(t) \| \leqq \lambda\alpha(t) \tag{9}$$

*holds.*

Proof. Let Y be the space of all continuous vector-functions $y$ on $E_{t_0} \cup J$. Let $\{I_k\}_{k=1}^{\infty}$ be a sequence of compact intervals such that $\bigcup_{k=1}^{\infty} I_k = J$, where $I_k = [t_0, t_k]$ and for every $k$ we have $I_k \subset I_{k+1} \subset J$.

Define in the space $Y$ a system of seminorms

$$p_k(y) = \sup_{t \in E_{t_0} \cup I_k} \| y(t) \| .$$

This system of seminorms defines a locally convex topology on $Y$.

Consider the subset

$$F = \{ y \in Y, \ \| y \| \leqq \lambda\alpha(t), \ t \in E_{t_0} \cup J \} \subset Y,$$

where $\alpha(t)$ is defined in (5).

For $y \in F$, define an operator $T$ by

$$(Ty)(t) = \varphi(t), \qquad t \in E_{t_0},$$

$$(Ty)(t) = x(t) + \int_{t_0}^{t} X(t) X^{-1}(s) f(s, y(s), y[h(s)]) \, \mathrm{d}s, \qquad t \in J, \tag{10}$$

where $x(t)$ is a solution of (2).

It is evident that $F$ is a convex closed set.

We show that $TF \subset F$.

If $t \in E_{t_0}$, then

$$\| (Ty)(t) \| = \| \varphi(t) \| \leqq \lambda \leqq \lambda \| I \| = \lambda\alpha(t)$$

by (6).

If $t \in J$, then

$$\| (Ty)(t) \| \leqq \| x(t) \| + \| X(t) \| \int_{t_0}^{t} \| X^{-1}(s) \| \; \| f(s, y(s), y[h(s)]) \| \, ds \leqq$$

$$\leqq \| X(t) \| \; \| c \| + \| X(t) \| \int_{t_0}^{t} \| X^{-1}(s) \| \, \omega(s, \| y(s) \|, \| y[h(s)] \|) \, ds \leqq$$

$$\leqq \alpha(t) [\| c \| + \int_{t_0}^{\infty} \| X^{-1}(t) \| \, \omega(t, \lambda\alpha(t), \lambda\alpha(t)) \, dt] \leqq$$

$$\leqq \alpha(t) [\| c \| + \lambda - \| c \|] = \lambda\alpha(t).$$

Further we show that $T$ is continuous on $F$. Let $\{y_n\}_{n=1}^{\infty}$, $y_n \in F$, be a sequence converging uniformly to $y \in F$ on every compact subinterval $I_k \subset J$. Let $\varepsilon > 0$ be given. We show that for $t \in I_k$ we have $(Ty_n)(t) \rightrightarrows (Ty)(t)$. Denote $A = \max_{t \in [t_0, t_k]} \alpha(t)$

Since $f$ is continuous and $y_n(t) \rightrightarrows y(t)$ on each compact interval $I_k$, there exists a constant $N > 0$ such that for $n \geqq N$ we have

(11) $$\| X^{-1}(t) \| \; \| f(t, y_n(t), y_n[h(t)]) - f(t, y(t), y[h(t)]) \| < \frac{\varepsilon}{A(t_k - t_0)}.$$

Using (10) and (11), for $t \in I_k$ and $n \geqq N$, we obtain

$$\| (Ty_n)(t) - (Ty)(t) \| \leqq \| X(t) \| \int_{t_0}^{t} \| X^{-1}(s) \| \; \| f(s, y_n(s), y_n[h(s)]) -$$

$$- f(s, y(s), y[h(s)]) \| \, ds < \frac{A \cdot \varepsilon}{A(t_k - t_0)} \int_{t_0}^{t} ds < \frac{\varepsilon(t - t_0)}{(t_k - t_0)} \leqq \frac{\varepsilon(t_k - t_0)}{(t_k - t_0)} = \varepsilon.$$

For $t \in E_{t_0}$, $(Ty)(t) = \varphi(t)$ is continuous.

We show that $\overline{TF}$ is a compact set. From (10) we obtain the following estimate

$$\| (Ty)'(t) \| \leqq \| x'(t) \| + \| X'(t) \| \int_{t_0}^{t} X^{-1}(s) \, \omega(s, \lambda\alpha(s), \lambda\alpha(s)) \, ds +$$

$$+ I\omega(t, \lambda\alpha(t), \lambda\alpha(t)), \qquad t \in J.$$

From the last estimate there follows the uniform boundedness of $(Ty)'(t)$ and $(Ty)(t)$ for $t \in I_k$ and also the equicontinuity $(Ty)(t)$ on $E_{t_0} \cup I_k$. Therefore $\overline{TF}$ is a compact set.

By Schauder–Tychonoff fixed point theorem, the operator $T$ has a fixed point $\bar{y} \in F$ and

(12) $$(T\bar{y})(t) = \bar{y}(t)$$

holds.

Assertion (9) follows from (10) and (12). This completes the proof.

From this point on we will assume that

$$\lim_{t\to\infty} h(t) = \infty. \tag{13}$$

From (13) it follows that $E_{t_0} = [\inf_{t\in J} h(t), t_0]$.

In [6, p. 33] the author defined a function $\gamma^*(t)$ by

$$\gamma^*(t) = \sup\{z, t_0 \leqq z, h(z) < t, t \in J\}$$

and proved that if $\lim_{t\to\infty} h(t) = \infty$, then $\gamma^*(t)$ is bounded on each finite subinterval of $J$.

In the following lemma it is used the procedure of [3] and [7].

**Lemma 1.** *Let $a(t), g(t), F(t), p(t), q(t) \in C([t_0, b), [0, \infty))$ and $r(t) \in C(E_{t_0}, [0, \infty))$ Furthermore, let $\omega(z) \in C([0, \infty), (0, \infty))$ be a nondecreasing function.*

*Denote*

$$\Omega(z) = \int_{z_0}^{z} \frac{1}{\omega(s)} \, ds, \qquad z_0 > 0, \qquad z \geqq 0. \tag{14}$$

Let $z(t) \in C([t_0, b), [0, \infty))$ *be such that*

$$z(t) \leqq g(t) + a(t) \int_{t_0}^{t} F(s) \{p(s)\, \omega[z(s)] + q(s)\, \omega(z[h(s)])\} \, ds, \tag{15}$$

$$z(t) \equiv r(t), \qquad t \in E_{t_0}. \tag{16}$$

*Then it is*

$$z(t) \leqq \Omega^{-1}\{\Omega[H(t)] + A(t) \int_{t_0}^{t} F(s)\, [p(s) + q(s)] \, ds\}, \tag{17}$$

*where $\Omega^{-1}$ is the inverse function to* (14), $H(t) = G(t) + A(t) \int_{t_0}^{\gamma^*(t_0)} F(t) q(t) \omega(z[h(t)]) \, dt$ *and* $G(t) = \max_{t_0 \leqq s \leqq t} g(s)$, $A(t) = \max_{t_0 \leqq s \leqq t} a(s)$, $t \in [t_0, b)$. *The inequality* (17) *remains valid for every $t \in [t_0, b)$ for which the right hand side is defined.*

Proof. We define the function $Z(t)$ by

$$Z(t) = \begin{cases} \max_{t_0 \leqq s \leqq t} z(s), & t \in [t_0, b), \\ r(t), & t \in E_{t_0}. \end{cases}$$

It is evident that $Z(t)$ is a continuous, nonnegative function and, further, also nondecreasing for $t \in [t_0, b)$.

Since the function $\omega(z)$ is monotone, from (15) we get

$$z(t) \leqq G(t) + A(t) \int_{t_0}^{t} F(s) \{p(s)\, \omega(Z(s)) + q(s)\, \omega(Z[h(s)])\} \, ds, \qquad t \in [t_0, b).$$

Let $\bar{t} \in [t_0, t]$ be a number in which the function $z(t)$ reaches its greatest value on $[t_0, t]$. Then

$$(18) \qquad Z(t) = z(\bar{t}) \leqq G(\bar{t}) + A(\bar{t}) \int_{t_0}^{\bar{t}} F(s) \{p(s)\, \omega[Z(s)] + q(s)\, \omega(Z[h(s)])\} \,\mathrm{d}s \leqq$$

$$\leqq G(\bar{t}) + A(\bar{t}) \int_{t_0}^{t} F(s) \{p(s)\, \omega[Z(s)] + q(s)\, \omega(Z[h(s)])\} \,\mathrm{d}s = \text{def } U(t),$$

or simply

$$(19) \qquad \begin{aligned} Z(t) &\leqq U(t), \qquad t \in [t_0, b), \\ Z(t) &\equiv r(t), \qquad t \in E_{t_0}. \end{aligned}$$

From (18) with regard to assumption of Lemma 1 it is evident that $U(t)$ is nonnegative and nondecreasing on $[t_0, b)$, and $U(t_0) = G(\bar{t})$.

Differentiating the function $U(t)$ we get

$$U'(t) = A(\bar{t})\, F(t) \{p(t)\, \omega[Z(t)] + q(t)\, \omega(Z[h(t)])\} \geqq 0, \qquad t \in [t_0, b),$$

from which, with respect to the function $\omega$ and to (19), we have

$$(20) \qquad U'(t) \leqq A(\bar{t})\, F(t) \{p(t)\, \omega[U(t)] + q(t)\, \omega(Z[h(t)])\},$$

where

$$(21) \qquad \omega(Z[h(t)]) = \omega(r[h(t)]) \qquad \text{for } h(t) < t_0,$$

and

$$(22) \qquad \omega(Z[h(t)]) = \omega[U(t)] \qquad \text{for } t \geqq t_0.$$

Integrating the inequality (20) from $t_0$ to $t$ and using (21), (22), we get

$$U(t) \leqq G(\bar{t}) + A(\bar{t}) \int_{t_0}^{\gamma^*(t_0)} F(t)\, \omega(r[h(t)]) \,\mathrm{d}t + A(\bar{t}) \int_{t_0}^{t} F(s) \{p(s) + q(s)\}\, \omega[U(s)] \,\mathrm{d}s =$$

$$(23) \qquad = H(\bar{t}) + A(\bar{t}) \int_{t_0}^{t} F(s) \{p(s) + q(s)\}\, \omega[U(s)] \,\mathrm{d}s.$$

Applying Bihari's lemma to (23) we get the inequality

$$(24) \qquad U(t) \leqq \Omega^{-1}\{\Omega[H(\bar{t})] + A(\bar{t}) \int_{t_0}^{t} F(s) [p(s) + q(s)] \,\mathrm{d}s\}.$$

Since (19) holds and $G(t) \geqq G(\bar{t})$, $A(t) \geqq A(\bar{t})$, $t \in [t_0, b)$, from (24) we get

$$Z(t) \leqq \Omega^{-1}\{\Omega[H(t)] + A(t) \int_{t_0}^{t} F(s) [p(s) + q(s)] \,\mathrm{d}s\}.$$

With respect to $z(t) \leqq Z(t)$, (17) holds. The proof is complete.

**Remark 1.** Putting $q(t) \equiv 0$ in Lemma 1, we get the assertion of Lemma 2 in [3]. The just proved Lemma 1 has the following corollaries.

**Corollary 1.** Assume that the hypotheses of Lemma 1 are satisfied. Furthermore, suppose that $\omega(z) \equiv z$. Then from the inequality

$$z(t) \leqq g(t) + a(t) \int_{t_0}^{t} F(s) \{p(s)\, z(s) + q(s)\, z h(s)]\} \, \mathrm{d}s \,, \qquad t \in [t_0, b) \,,$$

$$z(t) \equiv r(t) \,, \qquad t \in E_{t_0} \,,$$

it follows

$$z(t) \leqq H(t) \exp \{A(t) \int_{t_0}^{t} F(s) [p(s) + q(s)] \, \mathrm{d}s\}, \qquad t \in [t_0, b) \,,$$

where $H(t) = G(t) + A(t) \int_{t_0}^{\gamma^*(t_0)} F(t)\, q(t)\, z[h(t)] \, \mathrm{d}t$.

**Corollary 2.** Assume that the hypotheses of Lemma 1 are satisfied and let $g(t) \equiv C_1 \geqq 0$, $a(t) \equiv C_2 \geqq 0$, where $C_1, C_2$ are arbitrary constants. Then from the inequality

$$z(t) \leqq C_1 + C_2 \int_{t_0}^{t} F(s) \{p(s)\, \omega[z(s)] + q(s)\, \omega(z[h(s)])\} \, \mathrm{d}s, \qquad t \in [t_0, b),$$

$$z(t) \equiv r(t) \,, \qquad t \in E_{t_0} \,,$$

it follows

$$z(t) \leqq \Omega^{-1}\{\Omega(H) + C_2 \int_{t_0}^{t} F(s) [p(s) + q(s)] \, \mathrm{d}s\} \,,$$

where $H = C_1 + C_2 \int_{t_0}^{\gamma^*(t_0)} F(t)\, q(t)\, \omega[r(t)] \, \mathrm{d}t$.

**Corollary 3.** Let the assumptions of Corollary 2 hold and let $\omega(z) \equiv z$. Then from the inequality

$$z(t) \leqq C_1 + C_2 \int_{t_0}^{t} F(s) \{p(s)\, z(s) + q(s)\, z[h(s)]\} \, \mathrm{d}s \,, \qquad t \in [t_0, b) \,,$$

$$z(t) \equiv r(t) \,, \qquad t \in E_{t_0} \,,$$

it follows

$$z(t) \leqq H \exp \{C_2 \int_{t_0}^{t} F(s) [p(s) + q(s)] \, \mathrm{d}s\} \,,$$

where $H = C_1 + C_2 \int_{t_0}^{\gamma^*(t_0)} F(t)\, q(t)\, r(t) \, \mathrm{d}t$.

**Remark 2.** If we put $F(t) \equiv 1$ in Corollary 3, we get the assertion of Lemma 2 in [5].

**Lemma 2.** *Let $[t_0, T)$ be the maximal interval of a solution $y(t)$ of the initial problem* (1) (3), *and let the function $y(t)$ be bounded on $[t_0, T)$. Suppose that $\varphi(t)$ is bounded on $E_{t_0}$. Then $T = \infty$.*

The proof is similar to that of Lemma 1 in [2].

**Theorem 2.** *Let*

(i) $\psi_1(t), \psi_2(t) \in C[J, [0, \infty)]$,

(ii) $\omega(z) \in C[[0, \infty), (0, \infty)]$ *be a nondecreasing such that*

$$\int_{t_0}^{\infty} \frac{ds}{\omega(s)} = \infty ,$$

(iii) $\| f(t, u, v) \| \leqq \psi_1(t)\, \omega(\| u \|) + \psi_2(t)\, \omega(\| v \|)$, for $(t, u, v) \in D$.

*Then every solution $y(t)$ of the initial problem* (1), (3) *with $y(t_0) = x(t_0)$ has the following properties: it exists on $J$ and satisfies the inequality*

$$\| y(t) \| \leqq \Omega^{-1}\{\Omega[H(t)] + A(t) \int_{t_0}^{t} \| X^{-1}(s) \| [\psi_1(s) + \psi_2(s)] \, ds\}, \tag{25}$$

*where $\Omega$, $\Omega^{-1}$ has the same meaning as in Lemma 1,*

$$H(t) = G(t) + A(t) \int_{t_0}^{y^*(t_0)} \| X^{-1}(t) \| \, \psi_2(t)\, \omega(y[h(t)]) \, dt,$$

$$G(t) = \max_{t_0 \leqq s \leqq t} \| x(s) \| , \quad A(t) = \max_{t_0 \leqq s \leqq t} \alpha(s) ,$$

*and $\alpha(t)$ is defined in* (5).

Proof. Using the variation of constants formula, we can represent any solution $y(t)$ of the initial problem (1), (3) by the integral equation

$$y(t) = x(t) + X(t) \int_{t_0}^{t} X^{-1}(s)\, f(s, y(s), y[h(s)]) \, ds , \tag{26}$$

where $X(t)$ is a fundamental matrix and $x(t)$ is a solution of (2).

Denote

$$G(t) = \max_{t_0 \leqq s \leqq t} \| x(s) \| \quad \text{and} \quad A(t) = \max_{t_0 \leqq s \leqq t} \alpha(s) ,$$

where $\alpha(s)$ is defined in (5).

With respect to the assumptions of the theorem, from (26) we get

(27)

$$\| y(t) \| \leqq G(t) + A(t) \int_{t_0}^{t} \| X^{-1}(s) \| \{\psi_1(s)\, \omega(\| y(s) \|) + \psi_2(s)\, \omega(\| y[h(s)] \|)\} \, ds , \quad t \in J,$$

and

$$\| y(t) \| = \| \varphi(t) \| , \qquad t \in E_{t_0} .$$

Let $[t_0, T)$ be the interval of existence of a solution $y(t)$ of (1), (3). Applying

Lemma 1 to the inequality (27), for $t \in [t_0, T)$, we obtain the inequality (25). Furthermore, if $T < \infty$, then from (27) there follows the boundedness of $y(t)$ on $[t_0, T)$. Lemma 2 implies that the solution $y(t)$ of the initial problem (1), (3) exists for each $t \in J$ and (25) holds. This completes the proof.

From Theorem 2 and Corollary 1 we obtain.

**Corollary 4.** Suppose that

(i) $\psi_1(t), \psi_2(t) \in C(J, [0, \infty))$,

(ii) $\|f(t, u, v)\| \leqq \psi_1(t)\|u\| + \psi_2(t)\|v\|$, for $(t, u, v) \in D$.

Then every bounded solution $y(t)$ of the initial problem (1), (3) exists on $J$ and satisfies the following inequality

$$\| y(t) \| \leqq H(t) \exp \{A(t) \int_{t_0}^{t} \| X^{-1}(s) \| [\psi_1(s) + \psi_2(s)] \, ds\} ,$$

where $H(t) = G(t) + A(t) \int_{t_0}^{\gamma^*(t_0)} \| X^{-1}(t) \| \psi_2(t) \| \varphi(t) \| \, dt$, and $G(t)$, $A(t)$ has the meaning as in Theorem 2.

**Remark 3.** Assertions similar to Corollary 4 can be obtained from Theorem 2 by using Corollaries 2 and 3.

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