Archivum Mathematicum

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Archivum Mathematicum, Vol. 13 (1977), No. 4, 195--199

Persistent URL: http://dml.cz/dmlcz/106979

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ARCH. MATH. SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XIII: 195—200, 1977

MONOMORPHISMS IN THE CATEGORY OF SMALL CONNECTED CATEGORIES WITH SURJECTIVE FUNCTORS

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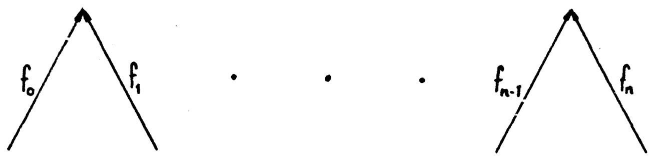
(Received August 1, 1976)

Definition. A category is said to be *connected* if for every pair of its morphisms f, f' there exist morphisms f_i, $i = 0, 1, \ldots, n$, n an odd natural number, such that $f_0 = f, f_n = f'$, $\operatorname{dom} f_i = \operatorname{dom} f_{i-1}$ for 2, 4, ..., $n - 1$ (even) and $\operatorname{cod} f_{i-1} = \operatorname{cod} f_i$ for $i = 1, 3, \ldots, n$ (odd).

Such an (n + 1)-tuple of morphisms will be called the *path* from f to f'.

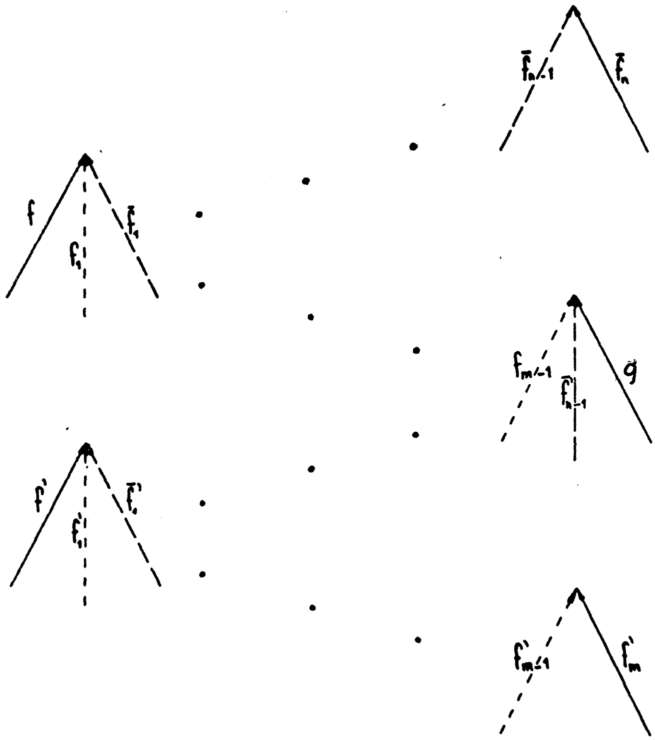


Let $\mathscr{K}$ denote the category the objects of which are all small connected categories and morphisms all surjective covariant functirs between them.

Monomorphisms in $\mathscr{K}$ are to be found.

Definition. Let $\mathbf{F} : \mathbf{B} \to \mathbf{A}$ be a morphism in $\mathscr{K}$. Two paths $(f_0, \ldots, f_m)$, $(f'_0, \ldots, f'_n)$ will be called **F**-*isomorphic* if $m = n$ and $\mathbf{F}f_i = \mathbf{F}f'_i$ for $i = 0, \ldots, n$.

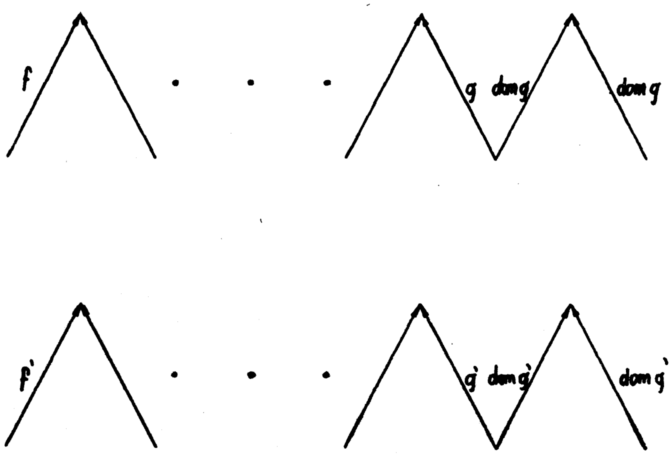
Morphisms $f, f' \in \mathbf{B}$, $f \neq f'$ will be called **F**-*symmetrical* if for each morphism $g \in \mathbf{B}$ there are **F**-isomorphic paths $(f = f_0, \ldots, f_m = g)$, $(f' = f'_0, \ldots, f'_m)$ and **F**-isomorphic paths $(f = \bar{f}_0, \ldots, \bar{f}_n)$, $(f' = \bar{f}'_0, \ldots, \bar{f}'_n = g)$.



Lemma. *Let* $\mathbf{F} : \mathbf{B} \rightarrow \mathbf{A}$ *be an element of Mor* $\mathscr{K}$ *and* f, f' *F-symmetrical. Let* $\mathbf{C}$ *denote the set of all ordered pairs* $[g, g']$, $g, g' \in \mathbf{B}$ *such that there exist* $\mathbf{F}$*-isomorphic paths* $(f = f_0, \ldots, f_n = g)$, $(f' = f_0, \ldots, f_n = g')$. *Then* $\mathbf{C}$ *is a connected subcategory of the category* $\mathbf{B} \times \mathbf{B}$ *and the restrictions of both projections are surjective.*

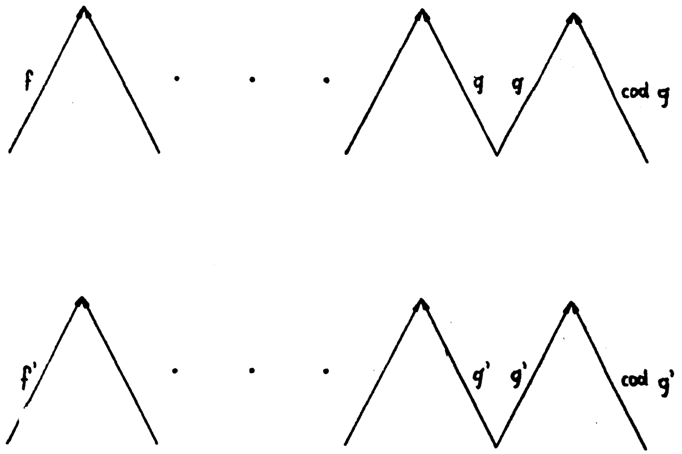
Proof. 1. Identities:

$$[g, g'] \in \mathbf{C} \Rightarrow [\operatorname{dom} g, \operatorname{dom} g'] \in \mathbf{C}:$$



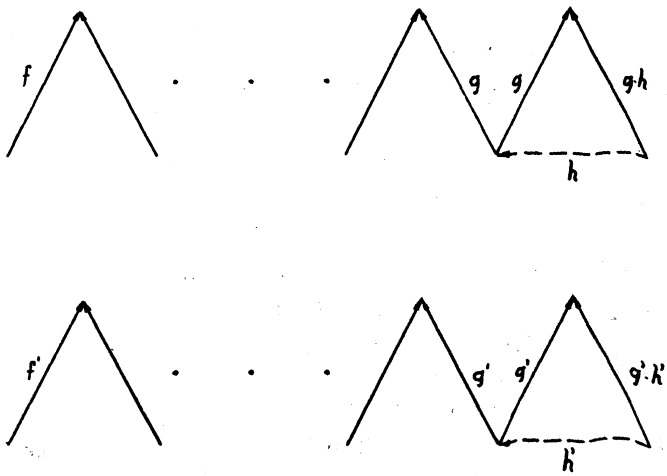
Let $(f = f_0, \ldots, f_n = g)$, $(f' = f'_0, \ldots, f'_n = g')$ be **F**-isomorphic. Then $(f = f_0, \ldots, f_n = g,\ f_{n+1} = \operatorname{dom} g,\ f_{n+2} = \operatorname{dom} g)$, $(f' = f'_0, \ldots, f'_n = g',\ f'_{n+1} = \operatorname{dom} g',\ f'_{n+2} = \operatorname{dom} g')$ are **F**-isomorphic, too.

$$[g, g'] \in \mathbf{C} \Rightarrow [\operatorname{cod} g, \operatorname{cod} g'] \in \mathbf{C}:$$



Let $(f = f_0, \ldots, f_n = g)$, $(f' = f'_0, \ldots, f'_n = g')$ be **F**-isomorphic. Then $(f = f_0, \ldots, f_n = g,\ f_{n+1} = g,\ f_{n+2} = \operatorname{cod} g)$, $(f' = f'_0, \ldots, f'_n = g',\ f'_{n+1} = g',\ f'_{n+2} = \operatorname{cod} g')$ are also **F**-isomorphic.

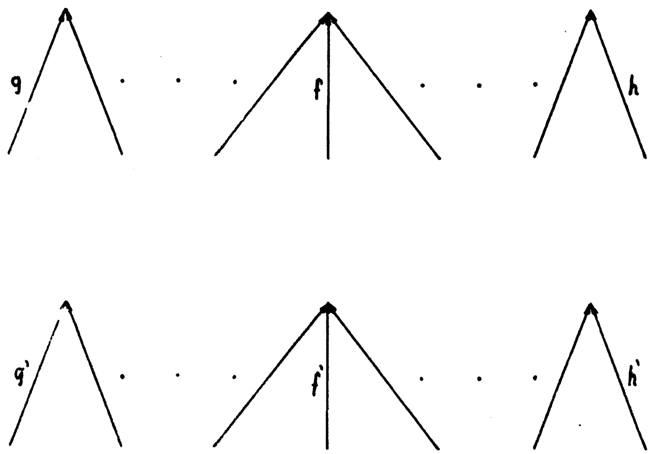
2. Composition of morphisms:



Let $[g, g'] \cdot [h, h']$ exist in $\mathbf{B} \times \mathbf{B}$, i.e. $g . h$, $g' . h'$ exist. If $[g, g'] \in \mathbf{C}$ and $[h, h'] \in \mathbf{C}$ then $[g . h, g' . h'] \in \mathbf{C}$: Let $(f = f_0, \ldots, f_n = g)$, $(f' = f'_0, \ldots, f'_n = g')$ be **F**-isomorphic. Then $(f = f_0, \ldots, f_n = g, f_{n+1} = \operatorname{dom} g \ (= \operatorname{cod} h), f_{n+2} = h)$, $(f' = f'_0, \ldots f'_n = g', f'_{n+1} = \operatorname{dom} g' \ (= \operatorname{cod} h'), f'_{n+2} = h')$ are **F**-isomorphic and $(f = f_0, \ldots f_n = g, f_{n+1} = g, f_{n+2} = g . h)$ $(f' = f'_0, \ldots, f'_n = g', f'_{n+1} = g', f'_{n+2} = g' . h')$ are **F**-isomorphic.

3. Connectedness:

Let $[g, g'], [h, h'] \in \mathbf{C}$, $(f = g_0, \ldots, g_n = g)$, $(f' = g'_0, \ldots, g'_n = g')$ **F**-isomorphic, $(f = h_0, \ldots, h_n = h)$, $(f' = h'_0, \ldots, h'_n = h')$ **F**-isomorphic.



Then $([g, g'] = [g_m, g'_m], \ldots, [g_1, g'_1], [h_1, h'_1], \ldots, [h_n, h'_n] = [h, h'])$ is evidently a path from $[g, g']$ to $[h, h']$ because both $[g_i, g'_i]$ and $[h_j, h'_j] \in \mathbf{C}$. The surjectivity of projections follows immediately from the construction.

Theorem. *For a functor* $\mathbf{F} : \mathbf{B} \to \mathbf{A}$, $\mathbf{F} \in Mor\ \mathscr{K}$, *holds:* **F** *is monic if and only if there exists no pair of* **F**-*symmetrical morphisms in* **B**.

Proof. $\Rightarrow$: Suppose that $f \neq f' \in \mathbf{B}$ are **F**-symmetrical. Then, by the lemma, it is easy to see that there exist a small connected category **C** and a pair of different surjective functors $\mathbf{G}, \mathbf{H} : \mathbf{C} \to \mathbf{B}$ such that $\mathbf{F} . \mathbf{G} = \mathbf{F} . \mathbf{H}$. Hence **F** is not a monomorphism.
$\Leftarrow$: Let $\mathbf{F} : \mathbf{B} \to \mathbf{A}$ be a morphism that is not monic.

Then there exist a small connected category **C** and a pair of different surjective functors $\mathbf{G}, \mathbf{H} : \mathbf{C} \to \mathbf{B}$ such that $\mathbf{F} . \mathbf{G} = \mathbf{F} . \mathbf{H}$. Let, for instance, e be one of those morphisms in **C**, for which $\mathbf{H}e \neq \mathbf{G}e$. It will be shown that $\mathbf{H}e$, $\mathbf{G}e$ are **F**-symmetrical. Clearly $\mathbf{FH}e = \mathbf{FG}e$. Let g be an arbitrary morphism in **B**. In **C** there must exist morphisms g_G, g_H such that $\mathbf{G}g_G = g$, $\mathbf{H}g_H = g$. Let $(e = g_0, \ldots, g_n = g_G)$ be a path from e to g_G and $(e = h_0, \ldots, h_n = g_H)$ a path from e to g_H. It is obvious that the paths $(\mathbf{G}e = \mathbf{G}g_0, \ldots, \mathbf{G}g_n = g)$, $(\mathbf{H}e = \mathbf{H}g_0, \ldots, \mathbf{H}g_n)$ are then **F**-isomorphic and so are the paths $(\mathbf{H}e = \mathbf{H}h_0, \ldots, \mathbf{H}h_n = g)$, $(\mathbf{G}e = \mathbf{G}h_0, \ldots, \mathbf{G}h_m)$.

Note. The preceding construction is possible and the theorem is valid whenever $\mathscr{K}$ is a full subcategory of the category of all small connected categories such that $\mathscr{K}$ contains, together with every small connected category **B**, at least one isomorphic copy of each of its connected subdirect power $\mathbf{C} = \mathbf{B} \times_s \mathbf{B}$. Especially, the category regarded in [1] is of this type.

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