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IDEALS OF WEAKLY ASSOCIATIVE LATTICES AND PSEUDO-ORDERED SETS

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The concept of weakly associative lattice was introduced by E. Fried in [2] as a generalization of the lattice. Many of lattice-theoretical concepts can be transferred in the theory of weakly associative lattices as it is shown in [3], [4]. Also some of properties of ideals on partially ordered sets (introduced in [1], [7] and by another way in [6]) can be investigated for the so-called pseudo-ordered sets .In this contribution, there are studied ideals on weakly associative lattices and pseudo-ordered sets and their connections with congruence relations on WA-lattices.

1. Preliminaries. Let A be a non-void set, $\leqq$ a reflexive and antisymmetric binary relation on A. The pair $\langle A, \leqq \rangle$ will be called a *pseudo-ordered set*, the relation $\leqq$ a *pseudo-ordering* (In [3], $\langle A, \leqq \rangle$ is called a *partial tournament*). If there exist for each pair $a, b \in A$ the l.u.b. of $\{a, b\}$ in $\langle A, \leqq \rangle$ and the g.l.b. of $\{a, b\}$ in $\langle A, \leqq \rangle$, then $\langle A, \leqq \rangle$ will be called a *weakly associative lattice* (*WA-lattice*). As it was shown in [2], we can introduce operations $\wedge$ and $\vee$ on the WA-lattice $\langle A, \leqq \rangle$ by the prescription: $a \wedge b =$ g.l.b. of $\{a, b\}$, $a \vee b =$ l.u.b. of $\{a, b\}$. Then the following identities are satisfied in the algebra $\langle A, \wedge, \vee \rangle$:

(i) $a \wedge a = a$	(i′) $a \vee a = a$
(ii) $a \wedge b = b \wedge a$	(ii′) $a \vee b = b \vee a$
(iii) $((a \wedge c) \vee (b \wedge c)) \vee c = c$	(iii′) $((a \vee c) \wedge (b \vee c)) \wedge c = c$
(iv) $(a \vee b) \wedge a = a$	(iv′) $(a \wedge b) \vee a = a$.

In [2] it was proved that also if in the algebra $\langle A, \wedge, \vee \rangle$ the preceding identities are satisfied, then there exists a pseudo-ordering $\leqq$ such that $\langle A, \leqq \rangle$ is a WA-lattice and $a \wedge b =$ g.l.b. of $\{a, b\}$, $a \vee b =$ l.u.b. of $\{a, b\}$.

By a *compatible relation* on the WA-lattice $\langle A, \leqq \rangle$, is meant a binary relation R

compatible on the algebra $\langle A, \wedge, \vee\rangle$, i.e. if aRb, cRd, then $(a \wedge c)\, R(b \wedge d)$ and $(a \vee c)\, R(b \vee d)$.

Further, we will write $b < c$ instead of $b \leqq c$, $b \neq c$.

Definition 1. Let W be a WA-lattice, I a non-void subset of W. I will be called an *ideal* of W, if the following hold:

$$i, j \in I \Rightarrow i \vee j \in I \tag{1}$$

$$i \in I, a \in W \Rightarrow a \wedge i \in I. \tag{2}$$

Thus, the concept of ideal can be transferred into the theory of WA-lattices. In the case of pseudo-ordered sets, the situation is similar. Let $\langle A, \leqq\rangle$ be a pseudo-ordered set and $a, b \in A$. Denote by $U(a, b)$ the set of all upper bounds of $\{a, b\}$ in $\langle A, \leqq\rangle$, by $L(a, b)$ the set of all lower bounds of $\{a, b\}$ in $\langle A, \leqq\rangle$. The psuedo-ordered set $\langle A, \leqq\rangle$ will be called *pu-directed* (*pl-directed*) if for each $a, b \in A$ we have $U(a\ \ b) \neq \emptyset$, ($L(a, b) \neq \emptyset$, respectively). If $\langle A, \leqq\rangle$ is both pu-directed and pl-directed, then it will be called p-directed. The following definition is a generalization of the one in [6], introducing the concept of o-ideal for partially ordered sets.

Definition 2. Let $\langle A, \leqq\rangle$ be a pseudo-ordered set, I be a non-void subset of A. I will be called a *p-ideal* of A, if

$$i \in I, a \in A,\ a \leqq i \Rightarrow a \in I \tag{3}$$

$$i, j \in I \Rightarrow U(i, j) \cap I \neq \emptyset. \tag{4}$$

Lemma 1. *The condition* (2) *in Definition* 1 *can be replaced by* (3).

Proof. Suppose (2) be true. Let $i \in I$, $a \in A$, $a \leqq i$. Then $a = a \wedge i \in I$. Conversely, let (3) hold, If $i \in I$, $a \in A$, then $a \wedge i \leqq i$, thus $a \wedge i \in I$.

d **Lemma 2.** *Let $W = \langle A, \leqq\rangle$ be a WA-lattice. The ideals of W are exactly the - ideals of W.*

Proof. Clearly $a \vee b \in U(a, b)$, thus, by Lemma 1, (1), (2) $\Rightarrow$ (1), (3) $\Rightarrow$ (3), (4). Conversely, if I is a p-ideal, then $U(a, b) \cap I \neq \emptyset$, thus, by (3), $a \vee b \in U(a, b) \cap I$, i.e. $a \vee b \in I$.

Definition 3. Let I be a p-ideal of the pseudo-ordered set $\langle A, \leqq\rangle$. I will be called a *maximal p-ideal of* $\langle A, \leqq\rangle$, if for each other p-ideal J such that $I \subseteq J$, $I \neq J$ ve have $J = A$.
I will be called a *p-prime ideal*, if

$$a, b \in A,\ \emptyset \neq L(a, b) \subseteq I \text{ imply } a \in I \text{ or } b \in I. \tag{5}$$

All above mentioned concepts can be dualized.

Definition 4. Let $\langle A, \leqq \rangle$ be a pseudo-ordered set, $B \subseteq A$. B is called a *convex subset* of $\langle A, \leqq \rangle$, if $b, c \in B$, $a \in A$, $b \leqq a \leqq c$ imply $a \in B$.

2. Ideals of pseudo-ordered sets.

Proposition 1. *Every p-ideal I of a pseudo-ordered set $\langle A, \leqq \rangle$ is a convex pu-directed subset of A. If $\langle A, \leqq \rangle$ is also pl-directed, then I is p-directed.*

Proof. By (3), I is a convex subset of $\langle A, \leqq \rangle$. From (4) it follows that I is also pu-directed. Let $\langle A, \leqq \rangle$ be pl-directed. Then for each pair $a, b \in A$ we have $L(a, b) \neq \emptyset$. If $a, b \in I$, then $L(a, b) \subseteq I$ and so I is also pl-directed. Summary, I is p-directed.

Proposition 2. *Let $\langle A, \leqq \rangle$ be a pseudo-ordered set, $\{I_\gamma, \gamma \in \Gamma\}$ a chain of its p-ideals (i.e. for each $\gamma, \delta \in \Gamma$ $I_\gamma \subseteq I_\delta$ or $I_\delta \subseteq I_\gamma$). Then* $I = \bigcup_{\gamma \in \Gamma} I_\gamma$ *is also a p-ideal of* $\langle A, \leqq \rangle$.

Proof. Let $a, b \in I$, $x \in A$. Then there exist $\gamma_1, \gamma_2 \in \Gamma$ such that $a \in I_{\gamma_1}$, $b \in I_{\gamma_2}$. Without loss of generality, suppose $I_{\gamma_1} \subseteq I_{\gamma_2}$. Then $a, b \in I_{\gamma_2}$. Thus, by (4), $U(a, b) \cap I_{\gamma_2} \neq \emptyset$, hence $U(a, b) \cap I \neq \emptyset$. If $x \leqq a$, then $x \in I_{\gamma_2} \subseteq I$, thus I is a p-ideal of $\langle A, \leqq \rangle$.

Corollary. *Every proper p-ideal of a pseudo-ordered set $\langle A, \leqq \rangle$ is contained in some maximal p-ideal.*

Proposition 3. *Let $\langle A, \leqq \rangle$ be a pl-directed pseudo-ordered set, I be a p-prime ideal of $\langle A, \leqq \rangle$. If $F = A - I$ is non-void, then F is a dual p-prime ideal of $\langle A, \leqq \rangle$.*

Proof. Suppose $F \neq \emptyset$.

(a) Let $c, d \in F$ and $L(c, d) \cap F = \emptyset$. Then $L(c, d) \subseteq I$, however, A is pl-directed, thus $L(c, d) \neq \emptyset$. As I is a p-prime ideal, we have either $c \in I$ or $d \in I$, which is a contradiction. Thus $L(c, d) \cap F \neq \emptyset$.

(b) Let $c \in F$, $u \geqq c$ and $u \notin F$. Then $u \in I$, I is a p-ideal, thus $c \in I$, also a contradiction. Hence $u \in F$.

(c) Let $u, v \in F$, $U(u, v) \subseteq F$ and $u \in I$, $v \in I$. Then $U(u, v) \cap I \neq \emptyset$, because I is a p-prime ideal, which is a contradiction. Hence $u \in F$ or $v \in F$. In the summary, F is a dual p-prime ideal of $\langle A, \leqq \rangle$.

For partially ordered sets it holds (cf. [6]):

Every convex subset of a partially ordered set is the intersection of an o-ideal and a dual o-ideal. It can be easy shown that this result is not rue for the pseudo-ordered sets and p-ideals in the general case.

Example. Let P be a pseudo-ordered set with a diagram on Fig. 1. Put $C = \{a, b, c, g, f\}$, then C is clearly a convex subset of P, however, $I = P = D$ for each p-ideal I and each dual p-ideal D of P. Thus $I \cap D = P \neq C$ for each I, D.

3. Ideals of WA-lattices. By Lemma 2, all the results of the preceding Propositions remain valid also for ideals of WA-lattices.

Proposition 4. *Let W be a WA-lattice. The set $\mathscr{I}(W)$ of all ideals of W ordered by the set inclusion forms a conditionally meet-complete lattice. The operation meet in $\mathscr{I}(W)$ coincides with the set-theoretical intersection.*

Proof. (a) Clearly W is the greatest element of $\mathscr{I}(W)$. Let $I_\gamma \in \mathscr{I}(W)$ for $\gamma \in \Gamma$ $I = \bigcap_{\gamma \in \Gamma} I_\gamma$. Let $I \neq \emptyset$. Clearly I fulfils (1), (2), thus I is an ideal of W.

(b) For $I_1, I_2 \in \mathscr{I}(W)$ clearly $I_1 \cap I_2 \neq \emptyset$, because $a \in I_1$; $b \in I_2$ imply $a \wedge b \in I_1 \cap I_2$. By (a), $I_1 \cap I_2 \in \mathscr{I}(W)$.

(c) Let $I_1, I_2 \in \mathscr{I}(W)$. Evidently, $W \in \{I_\beta \in \mathscr{I}(W);\ I_\beta \supseteq I_1,\ I_\beta \supseteq I_2\}$ thus, by (a), $I = \bigcap I_\beta \neq \emptyset$ is an ideal of W. Clearly, I is the supremum od I_1, I_2 in $\mathscr{I}(W)$. The proof is finished.

Let W be a WA-lattice, $a \in W$. Denote by $I(a)$ the intersection of all ideals of W containing a. Proposition 4, $I(a)$ is an ideal of W.

Definition 5. The ideal $I(a)$ of W for $a \in W$ is called the *principle ideal generated by a*.

Remark. For the case of lattices $I(a) = \{x \in W,\ x \leqq a\}$. For the general case of WA-lattices it is not true.

Lemma 3. *Let W be a WA-lattice, $c, d \in W$, $c \leqq d$. Then $I(c) \subseteq I(d)$.*

Proof. By Lemma 1, $c \in I(d)$. Hence we obtain the assertion.

Remark. Contrary to lattices, for WA-lattices the inclusion in Lemma 3 cannot be replaced by the strict inclusion for the case of the strict inequality (see e.g. Example).

Corollary. *For every WA-lattice W the relationship $a \to I(a)$ is an isotone mapping of W into $\mathscr{I}(W)$.*

Lemma 4. *Let W be a WA-lattice, I be an ideal of W and J and ideal of I, Then J is an ideal of W.*

The proof is clear.

Proposition 5. *Let W be a WA-lattice, I be an ideal of W and $a \in W - I$. Then there exists an ideal J such that $I \subseteq J$, $a \notin J$, which is maximal of this property.*

Proof. Let $\mathscr{I}_1$ denote the subset of $\mathscr{I}(W)$ consisting of all ideals of W non-containing the element a. Let $\{I_\gamma, \gamma \in \Gamma\}$ be a chain in $\mathscr{I}_1$. Then, by Proposition 2, $\bigcup_{\gamma \in \Gamma} I_\gamma$ is again an ideal of W non-containing a, thus, by the Kuratowski-Zorn lemma, $\mathscr{I}_1$ contains a maximal element.

4. Relation between ideals and congruences.

Proposition 6. *Let Θ be a congruence relation on the WA-lattice W. If* O *is the least element of W, then set $I_\Theta = \{x \in W, x\Theta\ \mathrm{O}\}$ is an ideal of W.*

Proof. Let $a, b \in I_\Theta$, then $a\ \Theta\ \mathrm{O}$, $b\ \Theta\ \mathrm{O}$, thus, from the compatibility of Θ we have $(a \vee b)\ \Theta\ \mathrm{O}$, thus $(a \vee b) \in I_\Theta$. If $x \in W$, then $(a \wedge x)\ \Theta\ (\mathrm{O} \wedge x) = \mathrm{O}$, thus $(a \wedge x) \in I_\Theta$.

Proposition 7. *Let W_1, W_2 be WA-lattices and let W_2 have the least element* O. *If φ is a homomorphism of W_1 onto W_2, then $I = \{y \in W_1, \varphi(y) = \mathrm{O}\}$ forms an ideal of W_1.*
The proof is evident. These propositions can be also dualized for WA-lattices with the greatest element and for dual ideals.

Proposition 8. *Let I be an ideal of a WA-lattice W and T_I a binary relation on W defined by the rule:*

(*) *$a\ T_I b$ if and only if there exist $u \in W$ and $i, j \in I$ such that $a = u \vee i$, $b = u \vee j$* (i.e. $a, b \in u \vee I$).

If T_I is compatible on W, then it is a congruence relation on W.

Proof. Evidently, T_I is reflexive and symmetric relation. It remains to prove the transitivity only. Let $a, b, c \in W$, aT_Ib, bT_Ic. Then, by (*), there exist $u, v \in W$, $i, j, k, l \in I$ such that $a = u \vee i$, $b = u \vee j = v \vee k$, $c = v \vee l$. As $i, l \in I$, then

(1°) $$iT_Il.$$

As $u \in u \vee I$, $a \in u \vee I$, we obtain uT_Ia. Analogously, we can prove uT_Ib, vT_Ib, vT_Ic. From the compatibility of T_I it follows

(2°) $$uT_Ib,\ bT_Iv \Rightarrow (u \wedge b)\ T_I(b \wedge v).$$

Further, $b = u \vee j$ implies $b \leqq u$, $b = v \vee k$ implies $b \leqq v$. Thus $u \wedge b = u$, $b \wedge v = v$, and, by (2°), consequently

(3°) $$uT_Iv.$$

By the compatibility of T_I, (1°) and (3°) give

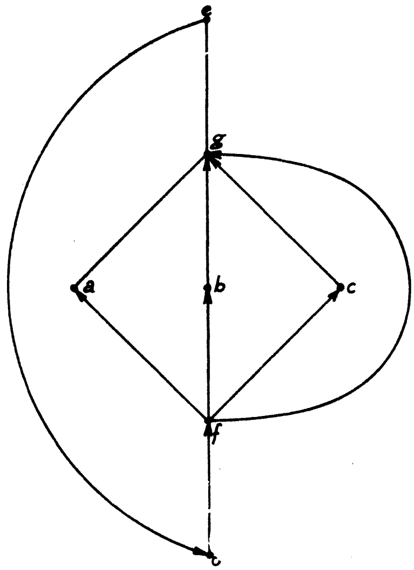
$$a = (u \vee i)\ T_I(v \vee l) = c,$$

hence T_I is transitive.

Remark. As a matter of interest, if W is a modular lattice, then also the converse statement of Proposition 8 is true, namely, for the case of modular lattices:

(I) T_I *is a compatible relation*
(II) T_I *is an equivalence relation*

are equivalent propositions. It can be proved by a rather tedious coputation by the using of Theorems from [5].



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