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ON VARIETIES OF NON-INDEXED ALGEBRAS

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In many algebraical considerations we are interested in all polynomials of a given algebra rather than in basic operations of it. Therefore the notion of non-indexed algebra is a fruitfull one. Morphisms of non-indexed algebras are weak homomorphisms. There is the great collection of papers devoted to the theory of non-indexed algebras (see e.g. [1]). The most of them deal with the characterization of weak isomorphisms. The categorical point is studied in [2].

The purpose of this paper is to give definitions of non-indexed variety and of weak equivalence of varieties and to study some basic properties of them. This notion of non-indexed variety is another than that given in [3].

Preliminaries

The reader is assumed to be familiar with the basic notions of universal algebra. We shall use the following notation:

$\mathbf{N}$ is the set of all non-negative integers,
$\mathbf{O}^n A$ is the set of all n-ary operations on the set A, and
$\mathbf{O}A$ is the set of all finitary operations on the set A.

By a *type* we mean a system $\Delta = (n_i)_{i \in I}$ of non-negative integers. An ***algebra*** of type Δ is an ordered pair $\mathscr{A} = (A, (f_i)_{i \in I})$, where $f_i \in \mathbf{O}^{n_i} A$ for any $i \in I$.

Homomorphisms between algebras of type Δ are sometimes called *Δ-homomorphisms*, and if $\mathscr{V}$ is a class of such algebras, $\mathscr{A} \in \mathscr{V}$, we speak about *$\mathscr{V}$-algebra* $\mathscr{A}$ or about Δ-algebra $\mathscr{A}$.

For $F \subseteq \mathbf{O}A$ we put $F^n = F \cap \mathbf{O}^n A$.
$F \subseteq \mathbf{O}A$ is called a *clone* on the set A if

(i) for any $n \in \mathbf{N} \setminus \{0\}$, $i \in \{1, \ldots, n\}$ the trivial operation p_i^n belongs to F^n, and
(ii) for any $m, n \in N$, $f \in F^m$, $f_1, \ldots, f_m \in F^n$ the composition $f(f_1, \ldots, f_m)$ belongs to F^n.

(For any $a_1, \ldots, a_n \in A$, $p_i^n(a_1, \ldots, a_n) = a_i$, and
$$(f(f_1, \ldots, f_m))\,(a_1, \ldots, a_n) = f(f_1(a_1, \ldots, a_n), \ldots, f_m(a_1, \ldots, a_n)).)$$

The intersection of any family of clones on a set A is a clone on A again. Therefore, for any $\Phi \subseteq \mathbf{O}A$ there exists the smallest clone on A including Φ; we shall denote it by $[\Phi]_A$.

A *non-indexed algebra* is an ordered pair $\mathfrak{A} = (A, F)$, where F is any clone on the set A. If $\mathscr{A} = (A, (f_i)_{i \in I})$ is an algebra of type $\Delta = (n_i)_{i \in I}$, we write $\overline{\mathscr{A}} = (A, [\{f_i \mid i \in I\}]_A)$. Finally, for a class $\mathscr{V}$ of Δ-algebras we use the notation $\overline{\mathscr{V}} = \{\overline{\mathscr{A}} \mid \mathscr{A} \in \mathscr{V}\}$.

Let $\alpha : A \to B$ be any mapping. For $n \in \mathbf{N}, f \in \mathbf{O}^n A$, $g \in \mathbf{O}^n B$ we write $(f, g) \in \mathbf{R}_\alpha$ if for all $a_1, \ldots, a_n \in A$, $\alpha f(a_1, \ldots, a_n) = g(\alpha a_1, \ldots, \alpha a_n)$ holds.

Let $\mathfrak{A} = (A, F)$ and $\mathfrak{B} = (B, G)$ be non-indexed algebras. A mapping $\alpha : A \to B$ is called a *weak homomorphism* of $\mathfrak{A}$ into $\mathfrak{B}$ if

(i) for any $n \in \mathbf{N}, f \in F^n$ there exists $g \in G^n$ such that $(f, g) \in \mathbf{R}_\alpha$, and
(ii) for any $n \in \mathbf{N}$, $g \in G^n$ there exists $f \in F^n$ such that $(f, g) \in \mathbf{R}_\alpha$.

If $\alpha : \mathfrak{A} \to \mathfrak{B}$ is a surjective weak homomorphism and $f \in F^n$, then there exists exactly one $g \in G^n$ such that $(f, g) \in \mathbf{R}_\alpha$; we shall write $g = \alpha^* f$.

Any bijective weak homomorphism is called a *weak isomorphism.*

Let $\mathscr{V}$ be a class of algebras of fixed type Δ. Let the operators $\mathbf{E}$, $\mathbf{S}$, $\mathbf{H}$ have the following meaning: $\mathbf{E}\mathscr{V}(\mathbf{S}\mathscr{V}, \mathbf{H}\mathscr{V})$ is the class of all Δ-algebras which are powers (subalgebras, homomorphic images, respectively) of $\mathscr{V}$-algebras.

For any variety $\mathscr{V}$ we have $\mathscr{V} = \mathbf{HSE}F_{\mathscr{V}}$, where $F_{\mathscr{V}}$ is a free algebra in $\mathscr{V}$ with a countable infinite set of free generators.

Varities of non-indexed algebras

Let $\mathfrak{N}$ be the class of all non-indexed algebras. For $\mathfrak{V} \subseteq \mathfrak{N}$ we define:

$\overline{\mathbf{E}}\mathfrak{V} = \{(A, F)^I \mid (A, F) \in \mathfrak{V}, I \text{ is a set}\}$, where $(A, F)^I = (A^I, F^I)$, $F^I = \{f^I \mid f \in F\}$, and $f^I((a_{1i})_{i \in I}, \ldots, (a_{ni})_{i \in I}) = (f(a_{1i}, \ldots, a_{ni}))_{i \in I}$ for all $a_{ji} \in A$.

$\overline{\mathbf{S}}\mathfrak{V} = \{\mathfrak{B} \in \mathfrak{N} \mid \text{there exists } \mathfrak{A} \in \mathfrak{V} \text{ such that } \mathfrak{B} \text{ is a subalgebra of } \mathfrak{A}\}$,

$\overline{\mathbf{H}}\mathfrak{V} = \{\mathfrak{B} \in \mathfrak{N} \mid \text{there exists } \mathfrak{A} \in \mathfrak{V} \text{ such that } \mathfrak{B} \text{ is a weak homomorphic image of } \mathfrak{A}\}$,

$\overline{\mathbf{I}}\mathfrak{V} = \{\mathfrak{B} \in \mathfrak{N} \mid \text{there exists } \mathfrak{A} \in \mathfrak{V} \text{ such that } \mathfrak{A} \text{ and } \mathfrak{B} \text{ are weakly isomorphic}\}$.

Lemma 1. *Let $\mathscr{V}$ be a class of Δ-algebras. Then*

(i) $\overline{\mathbf{E}\mathscr{V}} = \overline{\mathbf{E}}\overline{\mathscr{V}}$, (ii) $\overline{\mathbf{S}\mathscr{V}} = \overline{\mathbf{S}}\overline{\mathscr{V}}$, *and* (iii) $\overline{\mathbf{H}\mathscr{V}} = \overline{\mathbf{H}}\overline{\mathscr{V}}$.

Proof. The statements (i), (ii), and $\overline{\mathbf{H}\mathscr{V}} \subseteq \overline{\mathbf{H}}\overline{\mathscr{V}}$ are obvious. If $\mathfrak{A} \in \overline{\mathbf{H}}\overline{\mathscr{V}}$, there exist $\mathscr{A} \in \mathscr{V}$ and a surjective weak homomorphism $\alpha : \overline{\mathscr{A}} \to \mathfrak{A}$. Designating the operations in $\mathfrak{A}$ as it is determined by α^*, we get a Δ-algebra $\mathscr{B}$ such that $\overline{\mathscr{B}} = \mathfrak{A}$ and $\alpha : \mathscr{A} \to \mathscr{B}$ is a surjective Δ-homomorphism. Therefore $\mathfrak{A} \in \overline{\mathbf{H}\mathscr{V}}$.

Lemma 2. *Let $\mathcal{V}$ be a variety of Δ-algebras. Then* $\overline{\mathcal{V}} = \overline{\mathbf{H}}\overline{\mathbf{S}}\overline{\mathbf{E}}\overline{F_{\mathcal{V}}}$.

Proof. Following Lemma 1 we have $\overline{\mathcal{V}} = \overline{\mathbf{HSE}F_{\mathcal{V}}} = \overline{\mathbf{H}}\overline{\mathbf{SE}F_{\mathcal{V}}} = \overline{\mathbf{H}}\overline{\mathbf{S}}\overline{\mathbf{E}F_{\mathcal{V}}} =$
$= \overline{\mathbf{H}}\overline{\mathbf{S}}\overline{\mathbf{E}}\overline{F_{\mathcal{V}}}$.

A class of non-indexed algebras of the form $\overline{\mathcal{V}}$, where $\mathcal{V}$ is a variety of algebras of some type, is called a *non-indexed variety*.

Proposition 3. *The following conditions are equivalent for arbitrary class $\mathfrak{B}$ of non-indexed algebras:*

(i) *$\mathfrak{B}$ is a non-indexed variety,*
(ii) *is closed under $\overline{\mathbf{H}}$, $\overline{\mathbf{S}}$, $\overline{\mathbf{E}}$ and posseses some single generator $\mathfrak{A} \in \mathfrak{B}$,*
(iii) *there exists $\mathfrak{A} \in \mathfrak{B}$ such that $\mathfrak{B} = \overline{\mathbf{H}}\overline{\mathbf{S}}\overline{\mathbf{E}}\mathfrak{A}$.*

Proof. The equivalence of (ii) and (iii) follows from the following statements:

$$\overline{\mathbf{H}}\overline{\mathbf{H}} = \overline{\mathbf{H}}, \qquad \overline{\mathbf{S}}\overline{\mathbf{S}} = \overline{\mathbf{S}}, \qquad \mathbf{I}\overline{\mathbf{E}}\overline{\mathbf{E}} = \mathbf{I}\overline{\mathbf{E}}, \quad \text{and}$$
$$\overline{\mathbf{S}}\overline{\mathbf{H}} \leqq \overline{\mathbf{H}}\overline{\mathbf{S}}, \qquad \overline{\mathbf{E}}\overline{\mathbf{H}} \leqq \overline{\mathbf{H}}\overline{\mathbf{E}}, \qquad \overline{\mathbf{E}}\overline{\mathbf{S}} \leqq \overline{\mathbf{S}}\overline{\mathbf{E}}.$$

(Here $\mathbf{X} \leqq \mathbf{Y}$ means that for any $\mathfrak{W} \subseteq \mathfrak{N}$, $\mathbf{X}\mathfrak{W} \subseteq \mathbf{Y}\mathfrak{W}$ is satisfied.) The proofs are the same as in the case of algebras and therefore they are omitted.

The implication "(i) ⇒ (iii)" follows from Lemma 2.

"(iii) ⇒ (i)": Let $\mathfrak{B} = \overline{\mathbf{H}}\overline{\mathbf{S}}\overline{\mathbf{E}}\mathfrak{A}$ and $\mathfrak{A} = (A, F)$. Let Δ be the type, where n-ary operations are indexed by elements of F^n. Then $\mathfrak{A}$ may be treated as a Δ-algebra. If we define $\mathcal{V} = \mathbf{HSE}\mathfrak{A}$, we have $\overline{\mathcal{V}} = \mathfrak{B}$.

The class of all non-indexed varieties ordered by inclusion will be denoted by Λ.

Proposition 4. *The family $(\mathfrak{B}_i)_{i \in I}$ of non-indexed varieties has an infimum in Λ if $\bigcap_{i \in I} \mathfrak{B}_i$ is a non-indexed variety.*

We omit the proof of this easy statement.

Example 5. (i) *Let $(\mathcal{V}_i)_{i \in I}$ be a family of varieties of abelian groups. Then* $\overline{\bigcap_{i \in I} \mathcal{V}_i} = \bigcap_{i \in I} \overline{\mathcal{V}_i}$ *is an infimum of the family $(\overline{\mathcal{V}_i})_{i \in I}$ of non-indexed varieties in Λ.*

Proof. The only weak homomorphisms of abelian group are their homomorphisms (see [1]), and therefore for arbitrary variety $\mathcal{V}$ of abelian groups the functor $^-: \mathcal{V} \to \mathfrak{N}$ is a full embedding.

(ii) *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of 2-unary algebras with operations f and g. Let $\mathcal{V}$ be defined by the identity $f^2x = fx$, and let $\mathcal{W}$ be defined by $f^2x = x$. Then the non-indexed varieties $\overline{\mathcal{V}}$ and $\overline{\mathcal{W}}$ have not an infimum in Λ.*

The proof will be given after proposition 10.

Proposition 6. *Let $\mathscr{V}$ be a variety of algebras of type Δ and let $\mathfrak{W}$ be a non-indexed variety satisfying $\mathfrak{W} \subseteq \overline{\mathscr{V}}$. Then there exists a subvariety $\mathscr{W}$ of $\mathscr{V}$ such that $\overline{\mathscr{W}} = \mathfrak{W}$.*

Proof. There exists $\mathfrak{A} \in \overline{\mathscr{V}}$ such that $\mathfrak{W} = \overline{\mathbf{HSE}}\mathfrak{A}$ (Proposition 3). There exists $\mathscr{A} \in \mathscr{V}$ for which $\overline{\mathscr{A}} = \mathfrak{A}$. If we define $\mathscr{W} = \mathbf{HSE}\mathfrak{A}$, we have $\overline{\mathscr{W}} = \mathfrak{W}$ (Lemma 1).

Let us denote: $\mathfrak{O} = \{(\emptyset, \{\emptyset\})\}$,
$\mathfrak{F}_0 = \bar{\mathbf{I}}\,\{(\{*\}, \{p_1^n \mid n = 1, 2, \ldots\} \cup \{*\})\}$,
$\mathfrak{F}_1 = \bar{\mathbf{I}}\,\{(\{*\}, \{p_1^n \mid n = 1, 2, \ldots\})\} \cup \{\mathfrak{O}\}$, where $*$ is some element, and $\Lambda_0 = \{\mathfrak{V} \in \Lambda \mid \mathfrak{F}_0 \subseteq \mathfrak{V}\}$, $\Lambda_1 = \{\mathfrak{V} \in \Lambda \mid \mathfrak{F}_1 \subseteq \mathfrak{V}\}$.

Then $\{\mathfrak{O}\}$ and $\mathfrak{F}_0$ are the only minimal elements of Λ, and $\mathfrak{F}_1$ is the only element in Λ which covers $\mathfrak{O}$ in Λ.

Proposition 6 has an immediately corollary:

Proposition 7. *The ordered classes Λ_0 and Λ_1 are atomic. Moreover their atoms are of the form $\overline{\mathscr{V}}$, where $\mathscr{V}$ is an atom in the lattice of all varieties of some type.*

Weak equivalence of varieties

Let $\mathfrak{A} = (A, F)$ and $\mathfrak{B} = (B, G)$ be non-indexed algebras. A mapping $\xi\colon F \to G$ s called a clone homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ if

(i) $f \in F^n$ implies than $\xi f \in G^n$,
(i) for any $n \in \mathbf{N} - \{0\}$, $i \in \{1, \ldots, n\}$ $\xi p_i^n = p_i^n$, and
(iii) for any $m, n \in \mathbf{N}$, $f \in F^n$, $f_1, \ldots, f_m \in F^n$

$$\xi(f(f_1, \ldots, f_m)) = (\xi f)\,(\xi f_1, \ldots, \xi f_m).$$

Proposition 8. *For arbitrary varieties $\mathscr{V}$ and $\mathscr{W}$ the inclusion $\overline{\mathscr{W}} \subseteq \overline{\mathscr{V}}$ ho'ds if there exists surjective clone homomorphism of $\overline{\boldsymbol{F}}_{\mathscr{V}}$ into $\overline{\boldsymbol{F}}_{\mathscr{W}}$.*

Proof. Let $\mathfrak{A} = (A, F)$ be a non-indexed algebra, I a set, $\mathfrak{B} = (B, G)$ a subalgebra of $\mathfrak{A}$, and $\alpha : \mathfrak{A} \to \mathfrak{C}$ a surjective weak homomorphism. Then $f \mapsto f^I$, $f \mapsto f \mid B$, and α^* are surjective clone homomorphisms of $\mathfrak{A}$ into $\mathfrak{A}^I$, $\mathfrak{B}$, and $\mathfrak{C}$, respectively.

The necessity follows from the fact that $\overline{\mathscr{W}} \subseteq \overline{\mathscr{V}}$ implies $\overline{\boldsymbol{F}}_{\mathscr{W}} \in \overline{\mathbf{HSE}}\overline{\boldsymbol{F}}_{\mathscr{V}}$.

Now let ξ be a surjective clone homomorphism of $\overline{\boldsymbol{F}}_{\mathscr{V}}$ into $\overline{\boldsymbol{F}}_{\mathscr{W}}$. Let $\mathscr{V}$ be of type $\Delta = (n_t)_{t \in T}$, $\boldsymbol{F}_{\mathscr{V}} = (A, (f_t)_{t \in T})$ and let B be the support of $\boldsymbol{F}_{\mathscr{W}}$. The algebra $(B, (\xi f_t)_{t \in T})$ belongs to $\mathbf{HSE}\boldsymbol{F}_{\mathscr{V}}$. (It is of type Δ and it satisfies all identities holding in $\boldsymbol{F}_{\mathscr{V}}$.) We have $\overline{\boldsymbol{F}}_{\mathscr{W}} = \overline{(B, (\xi f_t)_{t \in T})} \in \overline{\mathbf{HSE}\boldsymbol{F}_{\mathscr{V}}} = \overline{\mathscr{V}}$. Therefore $\overline{\mathscr{W}} = \overline{\mathbf{HSE}\boldsymbol{F}_{\mathscr{W}}} \subseteq \overline{\mathscr{V}}$.

The well-known notion of equivalence of varieties may be defined in several ways. For our purposes the following definition will be convenient: The varieties

$\mathcal{V}$ and $\mathcal{W}$ are called *equivalent* if the non-indexed algebras $\overline{\boldsymbol{F}_{\mathcal{V}}}$ and $\overline{\boldsymbol{F}_{\mathcal{W}}}$ are weakly isomorphic.

The varieties $\mathcal{V}$ and $\mathcal{W}$ are called *weakly equivalent* if $\overline{\mathcal{V}} = \overline{\mathcal{W}}$.

Proposition 9. *The varieties $\mathcal{V}$ and $\mathcal{W}$ are weakly equivalent if there exist surjective clone homomorphisms* $\overline{\boldsymbol{F}_{\mathcal{V}}} \underset{\eta}{\overset{\xi}{\leftrightarrows}} \overline{\boldsymbol{F}_{\mathcal{W}}}$.

The proof is a direct consequence of Proposition 8.

Proposition 10. *If the varieties $\mathcal{V}$ and $\mathcal{W}$ are equivalent, then they are also weakly equivalent.*

Proof. If α is a weak isomorphism, then α^* is a bijective clone homomorphism.

The proof of example 5 (ii). The variety $\mathcal{V} \cap \mathcal{W}$ is defined by $fx = x$. Let $\mathcal{A} = (\{a, b, c\}, (f_{\mathcal{A}}, g_{\mathcal{A}}))$ be determined by $f_{\mathcal{A}}a = f_{\mathcal{A}}b = g_{\mathcal{A}}b = b$, $g_{\mathcal{A}}a = f_{\mathcal{A}}c = g_{\mathcal{A}}c = c$. Then $\mathcal{A} \in \mathcal{V}$ and $\overline{\mathcal{A}} \notin \overline{\mathcal{W}}$. That is $\overline{\mathcal{A}} \in \overline{\mathcal{V}} - (\overline{\mathcal{V}} \cap \overline{\mathcal{W}})$. Let $\mathcal{B} = (\{a, b\}, (f_{\mathcal{B}}, g_{\mathcal{B}}))$ and $\mathcal{B}' = (\{a, b\}, (f_{\mathcal{B}'}, g_{\mathcal{B}'}))$ be defined by $f_{\mathcal{B}}a = f_{\mathcal{B}}b = g_{\mathcal{B}}a = b$, $g_{\mathcal{B}}b = a$, and $g_{\mathcal{B}'}a = g_{\mathcal{B}'}b = f_{\mathcal{B}'}a = b$, $f_{\mathcal{B}'}b = a$. Then $\mathcal{B} \in \mathcal{V}$, $\mathcal{B}' \in \mathcal{W}$, $\overline{\mathcal{B}} = \overline{\mathcal{B}'}$, and $\mathcal{B} \notin \overline{\mathcal{V} \cap \mathcal{W}}$. That is $\overline{\mathcal{B}} \in (\overline{\mathcal{V}} \cap \overline{\mathcal{W}}) \setminus \overline{\mathcal{V} \cap \mathcal{W}}$. Therefore $\overline{\mathcal{V} \cap \mathcal{W}} \subset \overline{\mathcal{V}} \cap \overline{\mathcal{W}} \subset \overline{\mathcal{V}}$.

Let us suppose that $\overline{\mathcal{V}} \cap \overline{\mathcal{W}}$ is a non-indexed variety. Following Prop. 6 there exist varieties $\mathcal{V}_1$ and $\mathcal{V}_2$ with the properties:

$$\mathcal{V}_2 \subseteq \mathcal{V}_1 \subseteq \mathcal{V}, \qquad \overline{\mathcal{V}_1} = \overline{\mathcal{V}} \cap \overline{\mathcal{W}}, \qquad \overline{\mathcal{V}_2} = \overline{\mathcal{V} \cap \mathcal{W}}.$$

Let $\overline{\boldsymbol{F}_{\mathcal{V} \cap \mathcal{W}}} \underset{\eta}{\overset{\xi}{\rightleftarrows}} \overline{\boldsymbol{F}_{\mathcal{V}_2}}$ be surjective clone homomorphisms from Prop. 9 and let $\overline{\boldsymbol{F}_{\mathcal{V} \cap \mathcal{W}}} = (A, F)$ and $\overline{\boldsymbol{F}_{\mathcal{V}_2}} = (B, G)$. The realizations of the operational symbols f, g in algebras $\boldsymbol{F}_{\mathcal{V} \cap \mathcal{W}}$ and $\boldsymbol{F}_{\mathcal{V}_2}$ will be denoted by f_F, g_F and f_G, g_G, respectively. Then $[\{\xi g_F\}]_B = G$ and therefore there exist $k, l \in \mathbf{N}$ such that $f_G = (\xi g_F)^k$ and $g_F = \eta(\xi g_F)^l$. Now $g_G^{2k} = \eta(f_G^2)^l = \eta(f_G)^l = g_F^k$, which is possible only for $k = 0$. Therefore $\mathcal{V}_2$ satisfies $fx = x$. The operational symbol g in $\mathcal{V}_2$ cannot satisfy any non-trivial identity because the same identity would be valid in $\mathcal{V} \cap \mathcal{W}$. We have $\mathcal{V} \cap \mathcal{W} = \mathcal{V}_2$.

Since $\mathcal{V}$ covers $\mathcal{V} \cap \mathcal{W}$ in the lattice of all varieties of 2-unary algebras, $\mathcal{V}_1 = \mathcal{V}$ or $\mathcal{V}_1 = \mathcal{V} \cap \mathcal{W}$ and each of these possibilities gives a contradiction. ($\mathcal{V}_1 = \mathcal{V} \Rightarrow \overline{\mathcal{V}} \cap \overline{\mathcal{W}} = \overline{\mathcal{V}}$ and $\overline{\mathcal{V}_1} = \overline{\mathcal{V}} \cap \mathcal{W} \Rightarrow \overline{\mathcal{V} \cap \mathcal{W}} = \overline{\mathcal{V}} \cap \overline{\mathcal{W}}$.)

Example 11. *Let $\mathcal{V}$ be a variety of unary algebras with operations $\varphi_1, \varphi_2, \ldots$ defined by identities*

(1) $$\varphi_1 x = \varphi_1 y, \quad \varphi_2^2 x = \varphi_2^2 y, \quad \varphi_3^3 x = \varphi_3^3 y, \ldots$$

and let $\mathscr{W}$ be a variety of unary algebras with operations $\psi_1, \psi_2, \ldots$ defined by identities

$$(2) \qquad \psi_1 x = \psi_1 y, \quad \psi_2 x = \psi_2 y, \quad \psi_3^2 x = \psi_3^2 y, \ldots$$

Then the varieties $\mathscr{V}$ and $\mathscr{W}$ are weakly equivalent, but not equivalent.

Proof. The corespondences $\varphi_n \leftrightarrow \psi_n$, $n = 1, 2, \ldots$ and $\psi_1 \leftrightarrow \varphi_1$, $\psi_n \leftrightarrow \varphi_{n-1}$, $n = 2, 3, \ldots$ define surjective clone homomorphisms between $\overline{F_{\mathscr{V}}}$ and $\overline{F_{\mathscr{W}}}$. Therefore by Proposition 9 we have $\overline{\mathscr{V}} = \overline{\mathscr{W}}$.

Now we assume that $\mathscr{V}$ and $\mathscr{W}$ are equivalent. Then there exists a weak isomorphism $\alpha : \overline{F_{\mathscr{V}}} \to \overline{F_{\mathscr{W}}}$. Let $\overline{F_{\mathscr{V}}} = (A, F)$ and $\overline{F_{\mathscr{W}}} = (B, G)$. The fact $[\{\varphi_1, \varphi_2, \ldots\}]_A = F$ implies that $[\{\alpha^*\varphi_1, \alpha^*\varphi_2, \ldots\}]_B = G$. The only $f \in F^1$ which have not the property:

there exist no $f', f'' \in F^1$, $\quad f', f'' \neq p_1^1 \quad$ such that $\quad f = f'f''$

are $\varphi_1, \varphi_2, \ldots$, and the only $g \in G^1$ which have not the property:

there exist no $g', g'' \in G^1$, $\quad g', g'' \neq p_1^1 \quad$ such that $\quad g = g'g''$

are $\psi_1, \psi_2, \ldots$

According to (1) and (2) we have $\alpha^*\varphi_2 = \psi_3, \alpha^*\varphi_3 = \psi_4, \ldots$ Let x_1 be arbitrary free generator of $F_{\mathscr{W}}$. If $fx_1 = \psi_1 x_1$ for some $f \in F^1$ then $f = \psi_1$ and therefore $[G \setminus \{\psi_1\}]_B \neq G$. Analogously $[G \setminus \{\psi_2\}]_B \neq G$. Therefore for arbitrary $\alpha^*\varphi_1$ we have $[\{\alpha^*\varphi_1, \alpha^*\varphi_2, \ldots\}]_B = [\{\alpha^*\varphi_1, \psi_3, \psi_4, \ldots\}]_B \neq G$, which is a contradiction.

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