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ON THE STRUCTURE OF SECOND-ORDER PERIODIC DIFFERENTIAL EQUATIONS WITH GIVEN CHARACTERISTIC MULTIPLIERS

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I. Problem

Consider a second-order differential equation

$$(q) \qquad y'' = q(t)\, y$$

with a periodic coefficient $q \in C^0(\mathbf{R})$, $q(t + \pi) = q(t)$ for all $t \in \mathbf{R} = (-\infty, \infty)$. According to Floquet Theory, (q) admits independent solutions u and v that satisfy

either

$$(1) \qquad u(t + \pi) = \varrho \,.\, u(t), \qquad v(t + \pi) = 1/\varrho \,.\, v(t), \qquad \varrho \neq 0$$

or

$$(2) \qquad u(t + \pi) = \varrho \,.\, u(t) + v(t), \qquad v(t + \pi) = \varrho \,.\, v(t), \qquad \varrho^2 = 1.$$

(Generally complex) numbers ϱ and $1/\varrho$ are called characteristic (or Floquet's) multipliers of (q).

The purpose of this paper is to give a description of the structure of classes of those oscillatory equations (q) that admit the same characteristic multipliers.

II. Basic notions and relations

When differential equations (q) have the same characteristic multipliers, their solutions still may behave in a different way with respect to the number of their zeros (on an interval). Following O. Borůvka [5] we say that (q) is of category $(1, k)$, k being a positive integer, if (q) is both side oscillatory (i.e., both for $t \to -\infty$ and for $t \to +\infty$), it has real characteristic multipliers, and it admits a solution y, $y(t_0) = 0$, for which $t_0 + \pi$ is the k-th zero on the right of t_0. In the case of complex characteristic multipliers $e^{\pm a\pi i}$, $a \in (0, 1)$, (and only then) the general solution y of (q) can be written as

$$(3) \qquad y(t) = c_1 \frac{\sin [P(t) + (2k + a)t + c_2]}{|P'(t) + 2k + a|^{\frac{1}{2}}}, \qquad P(t + \pi) = P(t) \in C^3(\mathbf{R}),$$

see [8]. Then (q) is of category $(2, k)$.

We say that differential equations (q_1) and (q_2) are of the same behavior if

1. they have the same characteristic multipliers, and
2. they are of the same category, and
3. if the relation (2) holds for a suitable pair of solutions of one equation, then it holds also for a suitable pair of solutions of the second equation, wronskians of the both pairs being of the same sign.

The condition 3. is in a close relation to the problem of "the coexistence of periodic solutions" (see e.g. [2], [7]), since, in particular, if all solutions of (q_1) are periodic, then the same is true for (q_2).

In accordance with [3], define a phase $\alpha : \mathbf{R} \to \mathbf{R}$ of a pair r, s of independent solutions of (q) as a continuous function on $\mathbf{R}$ satisfying $\tan \alpha(t) = r(t)/s(t)$ on $\mathbf{R} - \{t \in \mathbf{R};\ s(t) = 0\}$. Then $\alpha \in C^3(\mathbf{R})$, $\alpha'(t) \neq 0$ on $\mathbf{R}$, and the general solution y of (q) can be written in the form

$$(4) \qquad y(t) = c_1 \frac{\sin (\alpha(t) + c_2)}{|\alpha'(t)|^{\frac{1}{2}}}, \qquad c_1 \text{ and } c_2 \text{ being constants.}$$

If (q) is both-side oscillatory, then and only then $\alpha(\mathbf{R}) = \mathbf{R}$.

All bijections $f: \mathbf{R} \to \mathbf{R}$, $f(\mathbf{R}) = \mathbf{R}$, $f \in C^3(\mathbf{R})$, $\mathrm{d}f(t)/\mathrm{d}t \neq 0$ on $\mathbf{R}$, together with the composition rule form the group $\mathfrak{G}$. The set of all phases of the equation $y'' = -y$ on $\mathbf{R}$ is a subgroup $\mathfrak{E}$ of $\mathfrak{G}$. If α is a phase of a both-side oscillatory (q), then all phases of (q) form the set $\mathfrak{E}\alpha = \{\varepsilon\alpha;\ \varepsilon \in \mathfrak{E}\}$ that is an element of the right decomposition of $\mathfrak{G}$ with respect to $\mathfrak{E}$, $\mathfrak{G}/_r\mathfrak{E}$.

The elements of $\mathfrak{G}/_r\mathfrak{E}$ are in $1 - 1$ correspondence with both-side oscillatoric equations (q) on $\mathbf{R}$, since for $\alpha \in \mathfrak{G}$, $q_\alpha(t) := -\frac{1}{2}(\alpha''(t)/\alpha'(t))' + \frac{1}{4}(\alpha''(t)/\alpha'(t))^2 - \alpha'^2(t)$, the function α is a phase of the differential equation $y'' = q_\alpha(t)\, y$ on $\mathbf{R}$.

The set $\mathfrak{H} := \{f \in \mathfrak{G};\ f(t + \pi) = f(t) + \pi \,.\, \operatorname{sign} f' \text{ for } t \in \mathbf{R}\}$ is a subgroup of $\mathfrak{G}$ and is called the group of elementary phases. For more details see [3].

In [4] O. Borůvka introduced a "block" of phases as an element of the least common covering of the right and left decompositions of $\mathfrak{G}$ with respect to $\mathfrak{E}$, i.e. for a given $\alpha \in \mathfrak{G}$ a block is the set $\{\varepsilon_1\alpha\varepsilon_2;\ \varepsilon_1 \in \mathfrak{E},\ \varepsilon_2 \in \mathfrak{E}\} = \mathfrak{E}\alpha\mathfrak{E}$. There he also proved that all the differential equations (q), whose phases are in the same block, are of the same behavior.

A natural question arose then, which blocks correspond to differential equations of a given behavior. The problem is solved in the theorem of the paper.

III. Preparatory lemmas

Let (q) be a π-periodic both-side oscillatory differential equation and α be a phase of (q). Then due to Floquet Theory and [5]

$$\frac{\sin \alpha(t+\pi)}{|\alpha'(t+\pi)|^{\frac{1}{2}}} = c_{11}\frac{\sin \alpha(t)}{|\alpha'(t)|^{\frac{1}{2}}} + c_{12}\frac{\cos \alpha(t)}{|\alpha'(t)|^{\frac{1}{2}}}$$

$$\frac{\cos \alpha(t+\pi)}{|\alpha'(t+\pi)|^{\frac{1}{2}}} = c_{21}\frac{\sin \alpha(t)}{|\alpha'(t)|^{\frac{1}{2}}} + c_{22}\frac{\cos \alpha(t)}{|\alpha'(t)|^{\frac{1}{2}}},$$

or

(5) $$\alpha(t+\pi) = \varepsilon\alpha(t) \qquad \text{for all } t \in \mathbf{R},$$

where

(6) $$\varepsilon(x) = \operatorname{arctg}\frac{c_{11}\operatorname{tg} x + c_{12}}{c_{21}\operatorname{tg} x + c_{22}} \in \mathfrak{E},$$

arctg denoting a suitable branch of the function such that $\alpha \in C^3(\mathbf{R})$ (that is possible, since $\varepsilon(t) = \alpha(\alpha^{-1}(t) + \pi)$). And also conversely: If $\alpha \in \mathfrak{G}$ and (5) is satisfied for some $\varepsilon \in \mathfrak{E}$, then (q) with the phase α is a both-side oscillatory differential equation with π-periodic coefficient; see [5].

The constant 2×2 matrix C formed by c_{ij} $(i, j = 1, 2)$ from (6) is unimodular. It is evident for the special pair y_1 and y_2 of independent solutions of (q) determined by the conditions $y_1(0) = 0$, $y_1'(0) = 1$, $y_2(0) = 1$, $y_2'(0) = 0$, where

$$\det \bar{C} = \det\begin{pmatrix} y_1'(\pi) & y_1(\pi) \\ y_2'(\pi) & y_2(\pi) \end{pmatrix} = \text{wronskian of } (y_2, y_1) = 1.$$

And for other pairs of independent solutions, the corresponding matrix C is similar to $\bar{C}$.

In such a situation it holds

Lemma 1. *If (q) is of category $(1, k)$, then for each phase α of (q) there exists $x_0 \in \mathbf{R}$ such that*

(7) $$\varepsilon(x_0) = x_0 + k\pi \,.\, \operatorname{sign} \alpha'$$

where ε is determined by (5).

Proof. Let t_0 be a zero of a solution y of (q) and $t_0 + \pi$ be the k-th zero of y on the right of t_0. Then due to (4)

$$|\alpha(t_0+\pi) - \alpha(t_0)| = k\pi,$$

or

$$\alpha(t_0+\pi) = \alpha(t_0) + k\pi \,.\, \operatorname{sign} \alpha'.$$

From (5) we get $\varepsilon\alpha(t_0) = \alpha(t_0) + k\pi \,.\, \text{sign}\, \alpha'$, and $x_0 := \alpha(t_0)$ completes the proof. ∎

Lemma 2. *If the relation* (5) *is satisfied for some phase* $\alpha \in \mathfrak{G}$ *of* (q) *and some* $\varepsilon \in \mathfrak{E}$, *and if* (7) *holds for an* $x_0 \in \mathbf{R}$, *then* (q) *is* π*-periodic both-side oscillatory equation of category* $(1, k)$.

Proof. It is sufficient to show that (q) is of category $(1, k)$. For $t_0 := \alpha^{-1}(x_0)$ we have

$$\alpha(t_0 + \pi) = \varepsilon\alpha(t_0) = \varepsilon(x_0) = x_0 + k\pi \,\text{sign}\, \alpha' = \alpha(t_0) + k\pi \,\text{sign}\, \alpha'.$$

Hence the solution

$$y(t) := \frac{\sin\left[\alpha(t) - \alpha(t_0)\right]}{|\alpha'(t)|^{\frac{1}{2}}}$$

of (q) vanishes both at t_0 and at $t_0 + \pi$, $t_0 + \pi$ being the k-th zero of the y on the right of t_0, since $|\alpha(t_0 + \pi) - \alpha(t_0)| = k\pi$. Such a solution y with the property cannot be of the form (3), hence (q) has real characteristic multipliers. ∎

Lemma 3. *If* (q) *is of category* $(2, k)$, *then there exists a phase* α *of* (q) *satisfying* (5) *for* $\varepsilon\colon x \mapsto x + (2k + a)\pi$, $a \in (0, 1)$. *And conversely, if* (5) *holds for* $\varepsilon(x) = x + (2k + a)\pi$ *and a phase* α *of* (q), *then* (q) *is of category* $(2, k)$.

Proof. (⇒). Due to the definition of category $(2, k)$, the form (3) of the general solution of (q) shows that $P(t) + (2k + a)t + c_2 =: \alpha$ is a phase of (q). The phase α satisfies

$$\alpha(t + \pi) = \alpha(t) + (2k + a)\pi, \tag{8}$$

and hence (5) gives $\varepsilon(x) = x + (2k + a)\pi$ for $\forall x \in \mathbf{R}$.

(⇐). If (8) holds, then (see [1, p. 67] or [6, p. 163]) the general form for a solution α is $\alpha(t) = (2k + a)t + Q(t)$, $Q(t + \pi) = Q(t)$ for $\forall t \in \mathbf{R}$. Such an α being a phase of (q) implies the category $(2, k)$ for (q), compare (3). ∎

We shall need also

Lemma 4. *Let* C_1 *and* C_2 *be real unimodular similar* 2×2 *matrices.*

A. If their Jordan canonical form is

$$\begin{pmatrix} \varrho & 0 \\ 0 & \varrho^{-1} \end{pmatrix} =: J, \qquad \varrho \neq 0, \qquad \textit{real},$$

then there exist real and regular P *and* Q *such that*

$$PC_1 = C_2P \qquad \textit{and} \qquad QC_1 = C_2Q,$$

det P *and* det Q *being of the opposite signs.*

B. If the Jordan canonical form of C_1 and C_2 is

$$\begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix} =: J^*,$$

then the sign of the determinants of the real regular matrices P such that

$$PC_1 = C_2P$$

is uniquely determined by C_1 and C_2. Moreover, in this case B, C_1 is also similar to C_2^{-1} and for Q, for which

$$QC_1 = C_2^{-1}Q,$$

the sign of det Q *is opposite to* sign det P.

Proof. A. There exist real regular matrices K and L such that

$$KC_1K^{-1} = LC_2L^{-1} = J.$$

For $\bar{E}: = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ we have $\bar{E} = \bar{E}^{-1}$ and $\bar{E}J = J\bar{E}$. Hence also

$$\bar{E}KC_1K^{-1}\bar{E}^{-1} = LC_2L^{-1} = J.$$

For $P: = L^{-1}K$ and $Q: = L^{-1}\bar{E}K$ we get

$$PC_1 = C_2P \qquad \text{and} \qquad QC_1 = C_2Q$$

with sign det $(P \, . \, Q)$ = sign det $\bar{E} = -1$.

B. Let $KC_1K^{-1} = LC_2L^{-1} = J^*$. Suppose $PC_1 = C_2P$ and $\bar{P}C_1 = C_2\bar{P}$, sign det $(P\bar{P}) = -1$. Then

$$LPC_1P^{-1}L^{-1} = J^* \qquad \text{and} \qquad L\bar{P}C_1\bar{P}^{-1}L^{-1} = J^*, \qquad \text{or}$$

$$(LPK^{-1})\, J^*(KP^{-1}L^{-1}) = J^* \qquad \text{and} \qquad (L\bar{P}K^{-1})\, J^*(K\bar{P}^{-1}L^{-1}) = J^*.$$

Hence there exists a real regular 2×2 matrix D (equal to LPK^{-1} or $L\bar{P}K^{-1}$) with det $D < 0$, such that

$$DJ^* = J^*D.$$

That implies $D = \begin{pmatrix} \gamma & \delta \\ 0 & \gamma \end{pmatrix}$, $\gamma \in \mathbf{R}$, $\delta \in \mathbf{R}$. Since det $D = \gamma^2 > 0$, we get a contradiction. To finish the proof of the lemma, denote again $\bar{E} := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and use the relation $\bar{E}J^* = J^{*-1}\bar{E}$, to derive

$$\bar{E}KC_1K^{-1}\bar{E}^{-1} = J^{*-1} = LC_2^{-1}L^{-1}.$$

Hence

$$(L^{-1}K)\, C_1 = C_2(L^{-1}K)$$

and

$$(L^{-1}EK)\,C_1 = C_2^{-1}(L^{-1}EK),$$

or

$$PC_1 = C_2P \qquad \text{and} \qquad QC_1 = C_2^{-1}Q, \text{ sign det } (P\,.\,Q) = -1,$$

for $P := L^{-1}K$ and $Q := L^{-1}EK$. ∎

IV. Main result

Theorem. *Let (q_1) be a both-side oscillatory differential equation on* **R** *with a π-periodic coefficient q_1, and α_1 denote one of its phases.*

Differential equation (q_2) is of the same behavior as (q_1) if and only if a (then every) phase α_2 of (q_2) satisfies

$$\alpha_2 = \varepsilon\alpha_1 h$$

for some $\varepsilon \in \mathfrak{E}$ and $h \in \mathfrak{H}$.

Note. Hence there is a $1 - 1$ correspondence between the decomposition of all π-periodic both-side oscillatory differential equations (q) into classes of equations of the same behavior and the least common covering of the right decomposition of $\mathfrak{G}$ with respect to $\mathfrak{E}$, $\mathfrak{G}/_r\mathfrak{E}$ and the left decomposition of $\mathfrak{G}$ with respect to $\mathfrak{H}$, $\mathfrak{G}_l/\mathfrak{H}$.

Proof of theorem. ($\Rightarrow$). In accordance with the notations introduced in the beginning of the Section III, let C_1 and C_2 stand for 2×2 matrices formed from constants (c_{ij}) of relation (6) considering with respect to (at this moment) arbitrary phases α_1 and α_2 of (q_1) and (q_2), respectively. Then the property 1. of the definition of behavior (Sect. I) implies the same characteristic values of C_1 and C_2, the property 3. gives the same elementary divisors of the both matrices. Hence C_1 and C_2 are similar and there exists a real regular 2×2 matrix C_3 such that

(9) $$C_1C_3 = C_3C_2.$$

Denote by ε_3 a function from $\mathfrak{E}$ that satisfies (6) with constants taken as elements of C_3. Evidently sign $\alpha_3' =$ sign det C_3. The relation (9) gives

$$\varepsilon_1\varepsilon_3(t) + m\pi = \varepsilon_3\varepsilon_2(t), \qquad m \text{ — an integer,}$$

or

(10) $$\tau_m\varepsilon_1\varepsilon_3 = \varepsilon_3\varepsilon_2,$$

where τ_m denotes the translation $\tau_m(t) = t + m\pi$.

If (q_1) is of category $(1, k)$ and the case A in Lemma 4 holds for C_1 and C_2, then C_3 in (9) can be chosen such that sign det $C_3 =$ sign $(\alpha_1'\,.\,\alpha_2')$. We have

$$\alpha_1\tau_1 = \varepsilon_1\alpha_1 \qquad \text{and} \qquad \alpha_2\tau_1 = \varepsilon_2\alpha_2, \qquad \text{or from (10),}$$

$$(\varepsilon_3\alpha_2)\,\tau_1 = \varepsilon_3\varepsilon_2\alpha_2 = \tau_m\varepsilon_1(\varepsilon_3\alpha_2).$$

The function $\varepsilon_3\alpha_2$ is also a phase of (q_2), since $\varepsilon_3 \in \mathfrak{E}$ (see Sect. II). According to Lemma 1 there exists $x_0 \in \mathbf{R}$ such that

$$\varepsilon_1(x_0) = x_0 + k\pi \operatorname{sign} \alpha_1',$$

or

$$\tau_m\varepsilon_1(x_0) = x_0 + k\pi \operatorname{sign} \alpha_1' + m\pi = x_0 + k\pi \operatorname{sign} (\varepsilon_3\alpha_2)' + m\pi.$$

Since (q_2) is also of category $(1, k)$, Lemma 2 implies $m = 0$ for the relation (10).

Let (q_1) and (q_2) be of category $(1, k)$ and let the case B from Lemma 4 hold for C_1 and C_2. Due to 3. (see also (2)), we may choose phases α_1 and α_2 corresponding to the special pairs (u_1, v_1) and (u_2, v_2) of solutions of (q_1) and (q_2), resp., i.e.

$$\alpha_1\tau_1 = \varepsilon_1\alpha_1 \qquad \text{and} \qquad \alpha_2\tau_1 = \varepsilon_2\alpha_2, \qquad \alpha_1' \,.\, \alpha_2' > 0,$$

where both α_1 and α_2 satisfy (6) with $C = J^* = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}$. Hence $\varepsilon_2 = \tau_m\varepsilon_1$. According to Lemma 1, $\varepsilon_1(x_0) = x_0 + k\pi \operatorname{sign} \alpha_1'$ for an $x_0 \in \mathbf{R}$. Thus

$$\varepsilon_2(x_0) = \tau_m\varepsilon_1(x_0) = x_0 + k\pi \operatorname{sign} \alpha_1' + m\pi = x_0 + k\pi \operatorname{sign} \alpha_2' + m\pi$$

with Lemma 2 gives $m = 0$, or $\varepsilon_2 = \varepsilon_1$. For the case the relation (10) is satisfied for $\varepsilon_3 = \mathrm{id}$ with $m = 0$.

For (q_1) to be of category $(2, k)$, let α_1 denote the phase of (q_1) that according to Lemma 3 leads to $\varepsilon_1\colon x \mapsto x + (2k + a)\,\pi$. Then

$$\tau_m\varepsilon_1(x) = x + (2k + a + m)\,\pi,$$

or

$$\varepsilon_3\alpha_2(t + \pi) = \varepsilon_3\alpha_2(t) + (2k + a + m)\,\pi.$$

Since (q_2) is of category $(2, k)$, Lemma 3 gives $m = 0$.

Summarizing our considerations, we get $m = 0$ for the relation (10) in all possible cases. Thus

$$\varepsilon_1\varepsilon_3 = \varepsilon_3\varepsilon_2,$$

$$(\alpha_1^{-1}\varepsilon_1\alpha_1)\,\alpha_1^{-1}\varepsilon_3\alpha_2 = \alpha_1^{-1}\varepsilon_3(\varepsilon_2\alpha_2),$$

$$\tau_1\alpha_1^{-1}\varepsilon_3\alpha_2 = \alpha_1^{-1}\varepsilon_3\alpha_2\tau_1,$$

or

$$\alpha_1^{-1}\varepsilon_3\alpha_2 \in \mathfrak{H}$$

and $\alpha_2 = \varepsilon\alpha_1 h$ for suitable $\varepsilon(= \varepsilon_3^{-1}) \in \mathfrak{E}$ and $h \in \mathfrak{H}$.

In the cases when α_1 was not an arbitrary phase of (q_1), but a special one the last relation remains true, because each phase of (q_1) is of the form $\tilde{\varepsilon}\alpha_1$, $\tilde{\varepsilon} \in \mathfrak{E}$, and hence again $\varepsilon\tilde{\varepsilon} \in \mathfrak{E}$. In other words: Once a phase α_2 of (q_2) is of the form $\varepsilon\alpha_1 h(\varepsilon \in \mathfrak{E},\ h \in \mathfrak{H})$, then every phase of (q_2) is of this form, since all phases of (q_2) form the set $\mathfrak{E}\alpha_2$.

($\Leftarrow$). Let (q_1) be a both-side oscillatory π-periodic differential equation, α_1 its phase, ε_1 determined by (5), C_1 being 2×2 matrix of constants c_{ij} from (6). Moreover, let $\varepsilon \in \mathfrak{E}$, $h \in \mathfrak{H}$, $\alpha_2 := \varepsilon\alpha_1 h$, and α_2 be a phase of (q_2). Then $\alpha_2(\mathbf{R}) = \mathbf{R}$ and (q_2) is both-side oscillatory. Since

$$\begin{aligned}\alpha_2\tau_1 &= \varepsilon\alpha_1 h\tau_1 = \varepsilon\alpha_1\tau_{\operatorname{sign} h'} \,.\, h = \varepsilon\varepsilon_1^{\operatorname{sign} h'}\alpha_1 h = \\ &= \varepsilon\varepsilon_1^{\operatorname{sign} h'}\varepsilon^{-1}\varepsilon\alpha_1 h = (\varepsilon\varepsilon_1^{\operatorname{sign} h'}\varepsilon^{-1})\,\alpha_2 = \varepsilon_2\alpha_2\,,\end{aligned}$$

and

$$\varepsilon\varepsilon_1^{\operatorname{sign} h'}\varepsilon^{-1} (= \varepsilon_2) \in \mathfrak{E},$$

q_2 is π-periodic. If we write $\varepsilon\varepsilon_1^{\operatorname{sign} h'}\varepsilon^{-1}$ in the form (6), the corresponding 2×2 constant matrix C_2 is similar to $C_1^{\operatorname{sign} h'}$. Matrices C_1 and C_1^{-1} are similar, since due to (1) and (2), the product of the characteristic values of C_1 is 1. Hence (q_1) and (q_2) have the same characteristic multipliers ($\Rightarrow$ 1.), and the elementary divisors of C_1 and C_2 are the same. For the condition 3. to be satisfied it is sufficient to show that if the differential equation (q_1) admits a pair (u_1, v_1) of solutions satisfying (2), then (q_2) also has a pair (u_2, v_2) satisfying the same relation, wronskians of (u_1, v_1) and (u_2, v_2) being of the same sign. Let α_1 be a phase of (q_1) corresponding to (u_1, v_1). Then

$$\alpha_1\tau_1 = \varepsilon^*\alpha_1\,, \tag{11}$$

where $\varepsilon^* \in \mathfrak{E}$ satisfies (6) with $C = J^*$. Each phase of (q_2) is of the form $\varepsilon\alpha_1 h$, $\varepsilon \in \mathfrak{E}$, $h \in \mathfrak{H}$. The relation (2) holds for (u_2, v_2) if and only if the phase α_2 of the pair (u_2, v_2) satisfies $\alpha_2\tau_1 = \varepsilon^*\alpha_2$. Hence such $\varepsilon \in \mathfrak{E}$ and $h \in \mathfrak{H}$ should exist that

$$\varepsilon\alpha_1 h\tau_1 = \varepsilon^*\varepsilon\alpha_1 h$$

or

$$\varepsilon\alpha_1\tau_1^{\operatorname{sign} h'} h = \varepsilon^*\varepsilon\alpha_1 h. \tag{12}$$

From (11) we get $\alpha_1\tau_1^{\operatorname{sign} h'} = \varepsilon^{*\operatorname{sign} h'}\alpha_1$. Hence (12) gives

$$\varepsilon\varepsilon^{*\operatorname{sign} h'}\varepsilon^{-1} = \varepsilon^*. \tag{13}$$

With respect to the second part of Lemma 4, for $\operatorname{sign} h' = 1$ also $\operatorname{sign} \varepsilon' = 1$, since (13) is then satisfied for $\varepsilon = \mathrm{id}$, $\mathrm{id}' = 1$; and for $\operatorname{sign} h' = -1$, we get $\operatorname{sign} \varepsilon' = -1$ (e.g. for ε in (6) with $C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$). However in both cases $\operatorname{sign}(\varepsilon' \,.\, h') = 1$ and for $\alpha_2 = \varepsilon\alpha_1 h$ we have $\operatorname{sign} \alpha_2' = \operatorname{sign} \alpha_1'$, i.e., the sign of wronskian of (u_2, v_2) is the same as for (u_1, v_1). Hence the condition 3. is satisfied for both (q_1) and (q_2).

If (q_1) is of category $(1, k)$, then $\varepsilon_1(x_0) = x_0 + k\pi \operatorname{sign} \alpha_1'$ (see Lemma 1), and we have

$$\varepsilon_2(\varepsilon(x_0)) = \varepsilon\varepsilon_1^{\operatorname{sign} h'}(x_0) = \varepsilon(x_0 + k\pi \operatorname{sign} \alpha_1' \,.\, \operatorname{sign} h') =$$
$$= \varepsilon(x_0) + k\pi \operatorname{sign} (\alpha_1' \,.\, h' \,.\, \varepsilon')$$
$$= \varepsilon(x_0) + k\pi \operatorname{sign} \alpha_2'.$$

According to Lemma 2, (q_2) is of category $(1, k)$.

If (q_1) is of category $(2, k)$, let $\tilde{\alpha}_1$ denote its phase that satisfies (8). Evidently $\tilde{\alpha}_1\alpha_1^{-1} =: \tilde{\varepsilon} \in \mathfrak{E}$. Then

$$\tilde{\varepsilon}\varepsilon^{-1}\alpha_2(t + \pi) = \tilde{\varepsilon}\alpha_1 h(t + \pi) = \tilde{\alpha}_1 h(t + \pi) = \tag{14}$$
$$= \tilde{\alpha}_1(h(t) + \pi \operatorname{sign} h') =$$
$$= \tilde{\alpha}_1(h(t)) + (2k + a)\pi \operatorname{sign} h' =$$
$$= \tilde{\varepsilon}\varepsilon^{-1}\alpha_2(t) + (2k + a)\pi \,.\, \operatorname{sign} h'.$$

Since

$$\tilde{\varepsilon}_2 := \operatorname{sign} h' \,.\, \tilde{\varepsilon}\varepsilon^{-1} \in \mathfrak{E}, \qquad \tilde{\varepsilon}_2\alpha_2 \text{ is a phase of } (q_2).$$

From (14) we have

$$\tilde{\varepsilon}_2\alpha_2(t + \pi) = \tilde{\varepsilon}_2\alpha_2(t) + (2k + a)\pi,$$

that due to Lemma 3 shows that (q_2) is of category $(2, k)$.

Hence both in real and complex cases (q_1) and (q_2) are of the same category, i.e. also the condition 2. is satisfied. ∎

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