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ON DISTRIBUTION OF ZEROS OF SOLUTIONS OF THE DIFFERENTIAL EQUATION $(p(t)y')' + f(t, y, y') = 0$

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1. Consider a differential equation

$$(1)\quad \begin{cases} (p(t)\,y')' + f(t, y, y') = 0 \\ \text{where } p \in C^1[t_0, \infty),\ p(t) > 0,\ t \in [t_0, \infty),\ f \text{ is continuous in} \\ D = \{(t, y, v) : t \in [t_0, \infty),\ -\infty < y < \infty,\ -\infty < v < \infty\}, \\ f(t, y, v)\,y > 0 \text{ for } y \neq 0 \text{ in } D. \end{cases}$$

It is evident that for an arbitrary point $M = (t_0, y_0, y_0') \in D$ there exists an interval J such that the equation (1) has a continuous solution y on J, $y(t_0) = y_0$, $y'(t_0) = y_0'$. But we do not suppose that this solution is univocally determined by M. We shall omit the trivial solution $y \equiv 0$ from our considerations.

A solution y of (1) is called oscilatory if there exists the sequence of numbers $\{t_k\}_1^\infty$ such that $t_0 \leqq t_k < t_{k+1}$, $y(t_k) = 0$, $y(t) \neq 0$ on (t_k, t_{k+1}), $k = 1, 2, 3, \ldots$

We shall need the following Lemma the proof of which can be found in [1].

Lemma 1. *Let y be a solution of* (1) *and $t_1 < t_2$ its consecutive zeros ($y(t) \neq 0$, $t \in (t_1, t_2)$). Then there exists exactly one number τ, $t_1 < \tau < t_2$ such that $y'(\tau) = 0$ holds and the function py'* sgn y' *is decreasing on (t_1, t_2).*

If y is an oscillatory solution, then according to Lemma 1 there exists exactly one sequence $\{\tau_k\}_1^\infty$ such that $t_k < \tau_k < t_{k+1}$, $y'(\tau_k) = 0$, $k = 1, 2, 3, \ldots$ hold. $\{|\,y(\tau_k)\,|\}_1^\infty$ is the sequence of the absolute values of all local extremes of y.

In the present paper we shall consider only oscillatory solutions of (1) with the property $\lim_{k\to\infty} t_k = \infty$ and we shall deal with the relations among $y(\tau_k)$, $y'(t_k)$ and $\Delta_k = t_{k+1} - t_k$. Note that with respect to (1) $\{|\,y'(t_k)\,|\}_1^\infty$ is the sequence of the absolute values of all local extremes of py'. The symbols t_k, τ_k, Δ_k have the above mentioned meaning in all the paper.

2. **Lemma 2.** *Let y be an oscillatory solution of* (1) *and let $I = [t_k, \tau_k]$, $I_1 = (\tau_k, \tau_k]$, $I_2 = [t_k, t_{k+1}]$, $I_3 = (t_k, t_{k+1})$, $I_4 = [\tau_k, t_{k+1}]$. Then*

(i) $$\inf_{t\in I_1} \frac{f(t, |y(t)|, y'(t))}{|y(t)|^s} \min_{t\in I} p(t) \leqq \frac{s+1}{2} \frac{[p(t_k) y'(t_k)]^2}{|y(\tau_k)|^{s+1}} \leqq$$
$$\leqq \sup_{t\in I_1} \frac{f(t, |y(t)|, y'(t))}{|y(t)|^s} \max_{t\in I} p(t), \qquad k = 1, 2, \ldots, s > -1.$$

(ii) $$\Delta_k^2 \geqq 8 \min_{t\in I_2} p(t) |y(\tau_k)|^{1-s} \inf_{t\in I_3} \frac{|y(t)|^s}{f(t, |y(t)|, y'(t))},$$
$$\Delta_k^2 \leqq 4(s+2)\left(\frac{\max_{t\in I_2} p(t)}{\min_{t\in I_2} p(t)}\right)^s \max_{t\in I_2} p(t) |y(\tau_k)|^{1-s} \sup_{t\in I_3} \frac{|y(t)|^s}{f(t, |y(t)|, y'(t))},$$
$$k = 1, 2, 3, \ldots, s > 0.$$

Proof (i). By multiplicating the equation (1) by $-2py'$ and by integration in the limits from t_k to τ_k we obtain the validity of the following estimation

$$[p(t_k) y'(t_k)]^2 = -2\int_{t_k}^{\tau_k} (p(t) y'(t))' p(t) y'(t) \, dt = 2\int_{t_k}^{\tau_k} p(t) f(t, y(t), y'(t)) y'(t) \, dt =$$
$$= 2\int_{t_k}^{\tau_k} p(t) \frac{f(t, |y(t)|, y'(t))}{|y(t)|^s} |y(t)|^s |y'(t)| \, dt \geqq$$
$$\geqq \frac{2}{s+1} |y(\tau_k)|^{s+1} \min_{t\in I} p(t) \inf_{t\in I_1} \frac{f(t, |y(t)|, y'(t))}{|y(t)|^s}.$$

So the first part of the statement is proved. The rest can be proved similarly.
(ii) By successive integration of (1) we have

$$p(t) y'(t) = \int_{\tau_k}^{t} [p(t) y'(t)]' \, dt = -\int_{\tau_k}^{t} f(t, y(t), y'(t)) \, dt,$$

(2) $$y(t) - y(\tau_k) = -\int_{\tau_k}^{t} \frac{1}{p(x)} \int_{\tau_k}^{x} f(z, y(z), y'(z)) \, dz \, dx.$$

It follows from Lemma 1 that the arch of the curve

$$\left|\int_{t_k}^{t} p(t) y'(t) \, dt\right|, \qquad t \in [t_k, \tau_k] \left(\left|\int_{t}^{t_{k+1}} p(t) y'(t) \, dt\right|, t \in [\tau_k, t_{k+1}]\right)$$

does not lay under the line segment connecting the points $[t_k, 0]$, $[\tau_k, |\int_{t_k}^{\tau_k} p(t) y'(t) \, dt|]$ ($[\tau_k, |\int_{\tau_k}^{t_{k+1}} p(t) y'(t) \, dt|]$, $[t_{k+1}, 0]$). Thus (according to the mean value theorem)

(3) $$p(\xi) \mid y(t) \mid \geqq p(\xi_1) \mid y(\tau_k) \mid \frac{t_{k+1} - t}{t_{k+1} - \tau_k}, \qquad t \in [\tau_k, t_{k+1}],$$

$$\xi \in (t, t_{k+1}), \ \xi_1 \in (\tau_k, t_{k+1}),$$

(4) $$p(\xi_2) \mid y(t) \mid \geqq p(\xi_3) \mid y(\tau_k) \mid \frac{t - t_k}{\tau_k - t_k}, \qquad t \in [t_k, \tau_k],$$

$$\xi_2 \in (t_k, t), \ \xi_3 \in (t_k, \tau_k).$$

With respect to (3) and (2) for $t = t_{k+1}$ we have

$$\mid y(\tau_k) \mid = \int_{\tau_k}^{t_{k+1}} \frac{1}{p(t)} \int_{\tau_k}^{x} \frac{f(z, \mid y(z) \mid, y'(z))}{\mid y(z) \mid^s} \mid y(z) \mid^s \mathrm{d}z \, \mathrm{d}x \geqq$$

$$\geqq \left(\frac{\min_{t \in I_2} p(t)}{\max_{t \in I_2} p(t)} \right)^s \frac{1}{\max_{t \in I_2} p(t)} \inf_{t \in I_4} \frac{f(t, \mid y(t) \mid, y'(t))}{\mid y(t) \mid^s} \int_{\tau_k}^{t_{k+1}} \int_{\tau_k}^{x} \left[\mid y(\tau_k) \mid \frac{t_{k+1} - z}{t_{k+1} - \tau_k} \right]^s \mathrm{d}z \, \mathrm{d}x =$$

$$= \frac{\mid y(\tau_k) \mid^s}{s + 2} (t_{k+1} - \tau_k)^2 \left(\frac{\min_{t \in I_2} p(t)}{\max_{t \in I_2} p(t)} \right)^s \frac{1}{\max_{t \in I_2} p(t)} \inf_{t \in I_4} \frac{f(t, \mid y(t) \mid, y'(t))}{\mid y(t) \mid^s}.$$

Thus

$$(t_{k+1} - \tau_k)^2 \leqq (s + 2) \mid y(\tau_k) \mid^{1-s} \left(\frac{\max_{t \in I_2} p(t,}{\min_{t \in I_2} p(t)} \right)^s \max_{t \in I_2} p(t) \sup_{t \in I_4} \frac{\mid y(t) \mid^s}{f(t, \mid y(t) \mid, y'(t))}.$$

The following estimation can be proved analogously (we must use (4) instead of (3)):

$$(\tau_k - t_k)^2 \leqq (s + 2) \mid y(\tau_k) \mid^{1-s} \left(\frac{\max_{t \in I_2} p(t)}{\min_{t \in I_2} p(t)} \right)^s \max_{t \in I_2} p(t) \sup_{t \in I_1} \frac{\mid y(t) \mid^s}{f(t, \mid y(t) \mid, y'(t))}.$$

So we can see that the second part of the statement of lemma is a consequence of these two inequalities. The rest can be proved similarly, we must use $\mid y(t) \mid \leqq \mid y(\tau_k) \mid$, $t \in [t_k, t_{k+1}]$ instead of (3) and (4).
Lemma is proved.

The following two theorems are simple consequences of Lemma 2. They discover some relations among the sequences $\{\Delta_k\}_1^\infty$, $\{\mid y(\tau_k) \mid\}_1^\infty$, $\{\mid y'(t_k) \mid\}_1^\infty$.

Theorem 1. *Let y be an oscillatory solution of* (1) *and let* $\mid f(t, y, v) \mid \geqq \mathrm{d}(t) \mid y \mid^s g(v)$, $(t, y, v) \in D$ *hold where* $\mathrm{d} \in C^0[t_0, \infty)$, $\mathrm{d}(t) > 0$, $g(v) \in C^0(-\infty, \infty)$, $s \geqq 0$,

(5) $$g(v) \geqq M = \text{const.} > 0, \qquad v \in (-\infty, \infty).$$

Then

$$(6)\qquad \frac{[p(t_k)\,y'(t_k)]^2}{|\,y(\tau_k)\,|^{s+1}} \geqq \frac{2M}{s+1} \min_{t\in[t_k,\tau_k]} \mathrm{d}(t) \min_{t\in[t_k,\tau_k]} p(t)$$

$$(7)\qquad \Delta_k^2 \leqq \frac{4(s+2)}{M}\,|\,y(\tau_k)\,|^{1-s}\,\frac{\max\limits_{t\in I} p(t)}{\min\limits_{t\in I} \mathrm{d}(t)}\left(\frac{\max\limits_{t\in I} p(t)}{\min\limits_{t\in I} p(t)}\right)^s,$$

$$I = [t_k, t_{k+1}].$$

Theorem 2. *Let y be an oscillatory solution of* (1) *and let* $|\,f(t, y, v)\,| \leqq \mathrm{d}(t)\,|\,y\,|^s g(v)$, $(t, y, v) \in D$ *hold where* $\mathrm{d}(t) \in C_0[t_0, \infty)$, $g(v) \in C_0(-\infty, \infty)$, $s \geqq 0$,

$$(8)\qquad 0 < g(v) \leqq M = \text{const.} < \infty, \qquad v \in (-\infty, \infty).$$

Then

$$(9)\qquad \frac{[p(t_k)\,y'(t_k)]^2}{|\,y(\tau_k)\,|^{s+1}} \leqq \frac{2M}{s+1} \max_{t\in[tk,\,sk]} \mathrm{d}(t) \max_{t\in[tk,\,sk]} p(t),$$

$$(10)\qquad \Delta_k^2 \geqq \frac{8}{M}\,|\,y(\tau_k)\,|^{1-s}\,\frac{\min\limits_{t\in I} p(t)}{\max\limits_{t\in I} \mathrm{d}(t)}, \qquad I = [t_k, t_{k+1}].$$

Remark 1. If y' is bounded on $[t_0, \infty)$, $0 \leqq |\,y'(t)\,| \leqq N < \infty$, $t \in [t_0, \infty)$, then Theorems 1 and 2 hold, too, even if we suppose $g(v) > 0$, $v \in (-\infty, \infty)$ instead of (5) and (8). In this case we must take $M = \min\limits_{v\in[-N,N]} g(v) > 0$ or $M = \max\limits_{v\in[-N,N]} g(v)$, respectively.

Corollary 1. Let the assumptions of Theorem 1 be valid. Let $\lim\limits_{t\to\infty} \mathrm{d}(t) = \infty$.

(i) If $p(t) \geqq M = \text{const.} > 0$, $t \in [t_0, \infty)$, then at least one of the following assertions is valid

1^0 $\lim\limits_{t\to\infty} y(t) = 0$

2^0 the function py' is unbounded on $[t_0, \infty)$.

(ii) If $0 < M = \text{const.} \leqq p(t) \leqq M_1 = \text{const.} < \infty$, $t \in [t_0, \infty)$, $s = 1$, then $\lim\limits_{k\to\infty} \Delta_k = 0$ holds.

Corollary 2. Let the assumptions of Theorem 2 be valid and let $\lim\limits_{t\to\infty} \mathrm{d}(t) = 0$.

(i) If $p(t) \leqq M = \text{const.} < \infty$, $t \in [t_0, \infty)$, then at least one of the following assertions is valid

1^0 the solution y is unbounded on $[t_0, \infty)$

2^0 $\lim\limits_{t\to\infty} p(t)\,y'(t) = 0$.

(ii) If $p(t) \geqq M = \text{const.} > 0$, $t \in [t_0, \infty)$, $s = 1$, then $\lim \Delta_k = \infty$ holds.

Corollary 3. Let y be an oscillatory solution of (1) and let constants M_1, M_2, M_3, M_4 exist such that

$$0 \leqq M_1 |y|^s \leqq |f(t, y, v)| \leqq M_2 |y|^s < \infty, \qquad M_1 > 0, M_2 < \infty,$$
$$0 < M_3 \leqq p(t) \leqq M_4 < \infty, \qquad (t, y, v) \in D, s \geqq 0.$$

Then (i)

$$\frac{2M_1M_3}{(s+1)M_4^2} \leqq \frac{y'^2(t_k)}{|y(\tau_k)|^{s+1}} \leqq \frac{2M_2M_4}{(s+1)M_3^2}, \qquad k = 1, 2, 3, \ldots$$

(ii) for $s = 1$ the inequalities

$$\frac{8M_3}{M_2} \leqq \Delta_k^2 \leqq \frac{12M_4^2}{M_1M_3}, \qquad k = 1, 2, 3, \ldots$$

hold.

Remark 2. We can see from the Corollary 3 that the functions y, y' behave in the same way—both are either bounded or unbounded on $[t_0, \infty)$. If one of them converges to zero for $t \to \infty$, then the other has the same property.

Remark 3. The reached results are generalizations of some well known results for the linear differential equation of second order, see [6], [4].

Remark 4. The problems of Theorems 1 and 2 are studied in [1], [3], [5], [7]. The statements of Theorems 1 and 2 are proved in [1] for $s = 1$.

Foltýńska [3] proved the inequalities (6), (9) for the very special type of (1)

$$y'' + \mathrm{d}(t) |y|^s \operatorname{sgn} y = 0,$$

where $\mathrm{d}(t) > 0$, d monotone, s is a quotient of two odd integers, $s > 1$.

Katranov [5] deals with the equation

$$y'' + f(t, y) g(y') = 0 \tag{11}$$

and he supposes that

(i) f, g are continuous in D

(ii) $\operatorname{sgn} f(t, y) = \operatorname{sgn} y$, $\frac{\partial}{\partial t} |f(t, y)| > 0$

(iii) there exist a non-decreasing function $a(t) \in C^0[t_0, \infty)$ and a number $n = (2m+1)/(2l+1)$, m, l integers, such that

$$\lim_{\substack{t \to \infty \\ y \to 0}} \frac{f(t, y)}{a(t) y^n} = 1$$

holds.

(iv) there exist positive constants g_1, g_2 such that $g_1 \leqq g(v) \leqq g_2$, $v \in (-\infty, \infty)$.

Under the validity of these assumptions he proved the limit case of (6), (7), (9), (10).

Authors of [7] obtain an upper estimation of Δ_k for the equation (1), $p \equiv 1$, $|f(t, y, v)| \geqq \mathrm{d}(t)|y|$, $\mathrm{d}(t) \geqq \text{const.} > 0$, $(t, y, v) \in D$, the solution exists on $[t_0, \infty)$. The statement of Corallary 1 (ii) follows from this estimation.

3. Correction. Proving Theorem 2 in [2] I made a mistake, the estimations (3) (in [2]) do not hold in common case. The above mentioned theorem holds (under small changes), its proof is the same, but we must use the inequalities (3), (4) in the present paper instead of (3) in [2]. The correct formulation is as follows.

Theorem 2. *Let y be an oscillatory solution of* (1) *and let $f^*(t, y)$ exist such that f^* is continuous on $D_1 = \{(t, y) : t \in [a, \infty),\ 0 \leqq y < \infty\}$, f^* is non-decreasing with respect to y in D_1, $|f(t, y, v)| \geqq f^*(t, |y|) \geqq 0$, $(t, y, v) \in D$, $\lim_{t \to \infty} f^*(t, M) = \infty$ for an arbitrary constant M, $0 < M < \infty$. Further, let $0 < M_3 = \text{const.} \leqq |y(\tau_k)| \leqq M_1 = \text{const.} < \infty, k = 1, 2, 3, \ldots$ and $0 < M_4 = \text{const.} \leqq p(t) \leqq M_2 = \text{const.} < \infty$, $t \in [a, \infty)$. Then the function y' is unbounded on $[a, \infty)$ and $\lim_{t \to \infty} \Delta_k = 0$.*

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